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Research Article

Existence for Elliptic Equation Involving Decaying Cylindrical Potentials with Subcritical and Critical Exponent

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We consider the existence of nontrivial solutions to elliptic equations with decaying cylindrical potentials and subcritical exponent. We will obtain a local minimizer by using Ekeland's variational principle.

1. Introduction

In this paper, we study the existence of nontrivial solutions of the following problem:

$$-\Delta u-\mu\left|y\right|^{-2}u=\left|y\right|^{-a\gamma}\left|u\right|^{\gamma-2}u\left(1+\lambda g\left(x\right)\right) \quad \text{in } \mathbb{R}^N,\ y\neq 0,\ u>0, \tag{$\mathscr{P}_{\lambda,\mu}$}$$

where $y\in\mathbb{R}^k$, and let k and N be integers such that $N\geq 3$ and k belongs to $\{1,\ldots,N\}$. $2^*=2N/(N-2)$ is the critical Sobolev exponent, $\gamma\leq 2^*$, $0\leq a<1$, g is a continuous function on $\mathbb{R}^N$, and λ and μ are parameters which we will specify later.

We denote point x in $\mathbb{R}^N$ by the pair $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $\mathscr{D}_0^{1,2}=\mathscr{D}_0^{1,2}((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})$, and $\mathscr{H}_\mu=\mathscr{H}_\mu((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})$, the closure of $C_0^\infty((\mathbb{R}^k\setminus\{0\})\times\mathbb{R}^{N-k})$ with respect to the norms

$$\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2},$$
$$\|u\|_\mu=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\left|y\right|^{-2}|u|^2\right)dx\right)^{1/2} \tag{1}$$

with $\mu<\overline{\mu}_k=((k-2)/2)^2$ for $k\neq 2$.

From the Hardy inequality, it is easy to see that the norm $\|u\|_\mu$ is equivalent to $\|u\|$.

We define the weighted Sobolev space $\mathscr{D}:=\mathscr{H}_\mu\cap L^\gamma(\mathbb{R}^N,|y|^{-b}dx)\cap L^2(\mathbb{R}^N,|y|^{-2}dx)$ with $b=a\gamma$, which is a Banach space with respect to the norm defined by $\mathscr{N}(u):=\|u\|_\mu+(\int_{\mathbb{R}^N}|y|^{-b}|u|^\gamma dx)^{1/\gamma}$.

My motivation of this study is the fact that such equations arise in the search for solitary waves of nonlinear evolution equations of the Schrödinger or Klein-Gordon type (cf. [1–3]). Roughly speaking, a solitary wave is a nonsingular solution which travels as a localized packet in such a way that the physical quantities corresponding to the invariances of the equation are finite and conserved in time. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy, the angular momentum, and the charge, whose finiteness is strictly related to the finiteness of the L^2-norm. Owing to their particle-like behavior, solitary waves can be regarded as a model for extended particles and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, and plasma physics (see, e.g., [4]).

Several existence and nonexistence results are available in the case $k=N$, and we quote, for example, [5–7] and the references therein. When $\mu=0$, $g(x)\equiv 1$; problem ($\mathscr{P}_{\lambda,\mu}$) has been studied in the famous papers by Brézis and Nirenberg [8] and Xuan [9] which consider the existence and nonexistence of nontrivial solutions to quasilinear Brézis-Nirenberg-type problems with singular weights.

Concerning the existence result in the case $k<N$, we cite [10, 11] and the references therein. As noticed in [10], for

$\mu < 0$ and $a = 0$, Badiale and Rolando have considered the problem $(\mathcal{P}_{0,\mu})$. They established the existence of nontrivial nonnegative radial solution when $\beta \in (0,2)$ and $\gamma \in (2_\beta, 2^*)$ or $\beta \in (2,+\infty)$ and $\gamma \in (2^*, 2_\beta)$; in addition, if the function $f(u) = |u|^{\gamma-1}u$ is odd, then $(\mathcal{P}_{0,\mu})$ has infinitely many radial solutions. In [5], Badiale et al. proved the nonexistence of nonzero classical solutions when $k \le N$ and the pair (β,γ) belongs to the light gray region. That is, $(\beta,\gamma) \in \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where

$$\begin{aligned}
\mathcal{A}_1 &:= \left\{(\beta,\gamma) \in \mathbb{R}^2 : \beta \in (0,2),\ \gamma \notin \left(2_\beta, 2^*\right),\ \gamma \ge 2\right\} \setminus \{(2,2^*)\},\\
\mathcal{A}_2 &:= \left\{(\beta,\gamma) \in \mathbb{R}^2 : \beta \in (2,N),\ \gamma \notin \left(2^*, 2_\beta\right),\ \gamma \ge 2\right\},\\
\mathcal{A}_3 &:= \left\{(\beta,\gamma) \in \mathbb{R}^2 : \beta \in [N,+\infty),\ \gamma \in [2,2^*]\right\}.
\end{aligned} \tag{2}$$

Since our approach is variational, we define the functional $I_{\lambda,\mu}$ on $\mathcal{D}$ by

$$\begin{aligned}
I_{\lambda,\mu}(u) &:= \left(\frac{1}{2}\right)\|u\|_\mu^2\\
&\quad - \left(\frac{1}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |u|^\gamma (1 + \lambda g(x))\,dx.
\end{aligned} \tag{3}$$

We say that $u \in \mathcal{D}$ is a weak solution of the problem $(\mathcal{P}_{\lambda,\mu})$ if it is a nontrivial nonnegative function and satisfies

$$\begin{aligned}
\left\langle I'_{\lambda,\mu}(u), v\right\rangle &:= \int_{\mathbb{R}^N} \Big(\nabla u \nabla v - \mu |y|^{-2} uv\\
&\quad - |y|^{-b} |u|^{\gamma-2} uv\,(1 + \lambda g(x))\Big) = 0, \quad \text{for } v \in \mathcal{D}.
\end{aligned} \tag{4}$$

Throughout this work, we consider the following regions $\mathcal{R}_1$, $\mathcal{R}_2$, such that

$$\begin{aligned}
\mathcal{R}_1 &:= \left\{(2,\gamma) \in \mathbb{R}^2 : \gamma \in \left(2_{2-2a}, 2^*\right)\right\},\\
\mathcal{R}_2 &:= \left\{(2,\gamma) \in \mathbb{R}^2 : \gamma \in \left(2, 2_{2-2a}\right)\right\}
\end{aligned} \tag{5}$$

with $2_{2-2a} = 2N/(N-(2-2a))$.

Concerning the perturbation g, we assume

$$\begin{aligned}
&g \in L^\infty\left(\mathbb{R}^N\right),\\
&g(x) > 0 \quad \forall x \in \mathbb{R}^N.
\end{aligned} \tag{G}$$

In our work, we prove the existence of at least one critical point of $I_{\lambda,\mu}$ by Ekeland's variational principle in [12].

We will state our main result.

Theorem 1. *Assume that $2 < k \le N$, $\mu < \overline{\mu}_k$, $0 < a < 1$, and (G) hold.*

If $(2,\gamma) \in \mathcal{R}_1 \cup \mathcal{R}_2$, then there exists $\Lambda^* > 0$ such that the problem $(\mathcal{P}_{\lambda,\mu})$ has at least one nontrivial solution for any $\lambda > \Lambda^*$.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

We list here a few integrals inequalities. The first inequality that we need is the weighted Hardy inequality [13]

$$\overline{\mu}_k \int_{\mathbb{R}^N} |y|^{-2} v^2 dx \le \int_{\mathbb{R}^N} |\nabla v|^2\,dx, \quad \forall v \in \mathcal{H}_\mu. \tag{6}$$

The starting point for studying $(\mathcal{P}_{\lambda,\mu})$ is the Hardy-Sobolev-Maz'ya inequality that is peculiar to the cylindrical case $k < N$ and that was proved by Gazzini and Musina in [14]. It states that there exists positive constant C_γ such that

$$C_\gamma \left(\int_{\mathbb{R}^N} |v|^\gamma\,dx\right)^{2/\gamma} \le \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2\right) dx, \tag{7}$$

for $\mu = 0$; equation of $(\mathcal{P}_{\lambda,\mu})$ is related to a family of inequalities given by Caffarelli et al. [15], for any $v \in C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$. The embedding $\mathcal{H}_\mu \hookrightarrow L^\gamma(\mathbb{R}^N, |y|^{-b}dx)$ is compact, where $b = a\gamma$ and $L^\gamma(\mathbb{R}^N, |y|^{-b}dx)$ is the weighted L^γ space with respect to the norm

$$|u|_{\gamma,b}^2 = \left(\int_{\mathbb{R}^N} |y|^{-b} |v|^\gamma\,dx\right)^{2/\gamma}. \tag{8}$$

Definition 2. Assume $2 \le k < N$, $0 < \mu \le \overline{\mu}_k$, and $2 < \gamma < 2^*$. Then, the infimum $S_{\mu,\gamma}$ defined by

$$S_{\mu,\gamma} = S_{\mu,\gamma}(k,\gamma) := \inf_{v \in \mathcal{D}\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2\right) dx}{\left(\int_{\mathbb{R}^N} |y|^{-b} |v|^\gamma\,dx\right)^{2/\gamma}} \tag{9}$$

is achieved on $\mathcal{H}_\mu$.

Lemma 3. *Let $(u_n) \subset \mathcal{D}$ be a Palais-Smale sequence ($(PS)_\delta$ for short) of $I_{\lambda,\mu}$ such that*

$$\begin{aligned}
&I_{\lambda,\mu}(u_n) \longrightarrow \delta,\\
&I'_{\beta,\lambda,\mu}(u_n) \longrightarrow 0\\
&\qquad\qquad \text{in } \mathcal{D}' \ (\text{dual of } \mathcal{D}) \text{ as } n \longrightarrow \infty,
\end{aligned} \tag{10}$$

for some $\delta \in \mathbb{R}$. Then, $u_n \rightharpoonup u$ in $\mathcal{D}$ and $I'_{\beta,\lambda,\mu}(u) = 0$.

Proof. From (10), we have

$$\begin{aligned}
&\left(\frac{1}{2}\right)\|u_n\|_\mu^2 - \left(\frac{1}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma (1 + \lambda g(x))\,dx\\
&\quad = \delta + o_n(1),\\
&\|u_n\|_\mu^2 - \int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma (1 + \lambda g(x))\,dx = o_n(1),\\
&\qquad\qquad \text{for } n \text{ large},
\end{aligned} \tag{11}$$

where $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. Then,

$$\begin{aligned}
\delta + o_n(1) &= I_{\lambda,\mu}(u_n) - \left(\frac{1}{\gamma}\right)\left\langle I'_{\beta,\lambda,\mu}(u_n), u_n\right\rangle\\
&= \left(\frac{(\gamma-2)}{2\gamma}\right)\|u_n\|_\mu^2,
\end{aligned} \tag{12}$$

and (u_n) is bounded in $\mathscr{D}$. Going if necessary to a subsequence, we can assume that there exists $u \in \mathscr{D}$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } \mathscr{D}, \\ &u_n \longrightarrow u \quad \text{in } L^\gamma\left(\mathbb{R}^N, |y|^{-b}\, dx\right), \\ &u_n \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{13}$$

Consequently, we get, for all $v \in C_0^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$,

$$\begin{aligned} \int_{\mathbb{R}^N} \Big(&\nabla u \nabla v - \mu |y|^{-2} uv \\ &- |y|^{-b} |u|^{\gamma-2} uv (1 + \lambda g(x)) \Big) = 0, \end{aligned} \tag{14}$$

which means that

$$I'_{\beta,\lambda,\mu}(u) = 0. \tag{15}$$

□

3. Existence Result

Firstly, we require the following lemmas.

Lemma 4. *Let $(u_n) \subset \mathscr{D}$ be a $(PS)_\delta$ sequence of $I_{\lambda,\mu}$ for some $\delta \in \mathbb{R}$. Then,*

$$u_n \rightharpoonup u \quad \text{in } \mathscr{D} \tag{16}$$

and either

$$\begin{aligned} &u_n \longrightarrow u \\ &\text{or } \delta \geq I_{\lambda,\mu}(u) + \left(\frac{(\gamma-2)}{2\gamma}\right)\left(S_{\mu,\gamma}\right)^{\gamma/(\gamma-2)}. \end{aligned} \tag{17}$$

Proof. We know that (u_n) is bounded in $\mathscr{D}$. Up to a subsequence if necessary, we have that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } \mathscr{D} \\ &u_n \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{18}$$

Denote $v_n = u_n - u$, and then $v_n \rightharpoonup 0$. As in Brézis and Lieb [16], we have

$$\begin{aligned} \int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma &= \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + \int_{\mathbb{R}^N} |y|^{-b} |u|^\gamma, \\ \|u_n\|_\mu^2 &= \|v_n\|_\mu^2 + \|u\|_\mu^2. \end{aligned} \tag{19}$$

From Lebesgue theorem and by using the assumption (G), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x) |y|^{-b} |u_n|^\gamma dx \\ &\quad = \lim_{n\to\infty} \int_{\mathbb{R}^N} g(x) |y|^{-b} |u|^\gamma dx. \end{aligned} \tag{20}$$

Then, we deduce that

$$\begin{aligned} I_{\lambda,\mu}(u_n) &= I_{\lambda,\mu}(u) + \left(\frac{1}{2}\right)\|v_n\|_\mu^2 \\ &\quad - \left(\frac{1}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + o_n(1), \\ \left\langle I'_{\lambda,\mu}(u_n), u_n \right\rangle &= \|v_n\|_\mu^2 - \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + o_n(1). \end{aligned} \tag{21}$$

From the fact that $v_n \rightharpoonup 0$ in $\mathscr{D}$, we can assume that

$$\lim_{n\to\infty} \|v_n\|_\mu^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma = \alpha \geq 0. \tag{22}$$

Assuming that $\alpha > 0$, we have by definition of $S_{\mu,\gamma}$

$$\alpha \geq S_{\mu,\gamma} l^{(2/\gamma)}, \tag{23}$$

and so

$$\alpha \geq \left(S_{\mu,\gamma}\right)^{\gamma/(\gamma-2)}. \tag{24}$$

Then, we get

$$\delta \geq I_{\lambda,\mu}(u) + \left(\frac{(\gamma-2)}{2\gamma}\right)\left(S_{\mu,\gamma}\right)^{\gamma/(\gamma-2)}. \tag{25}$$

Therefore, if not, we obtain $\alpha = 0$. That is, $u_n \to u$ in $\mathscr{D}$. □

Lemma 5. *Suppose that $2 < k \leq N$, $\mu < \overline{\mu}_k$, and (G) hold. If $(2,\gamma) \in \mathscr{R}_1 \cup \mathscr{R}_2$, then there exist $\Lambda^* > 0$ and ϱ and ν positive constants such that, for all $\lambda > \Lambda^*$,*

(i) *there exist $\omega \in \mathbb{R}^N$ such that $I_{\lambda,\mu}(\omega) < 0$,*

(ii) *we have*

$$I_{\lambda,\mu}(u) \geq \nu > 0 \quad \text{for } \|u\|_\mu = \varrho_0. \tag{26}$$

Proof. (i) Let $t_0 > 0$ where t_0 is small, and $\phi \in C_0^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$ such that $\phi \not\equiv 0$. Choosing $\Lambda^* = |t_0\phi|^{1-\gamma}$, then, if $\lambda > \Lambda^*$ large enough,

$$\begin{aligned} I_{\lambda,\mu}(t_0\phi) &:= \left(\frac{t_0^2}{2}\right)\|\phi\|_\mu^2 - \left(\frac{t_0^\gamma}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |\phi|^\gamma 1 \\ &\quad - \left(\frac{t_0^\gamma}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |\phi|^\gamma \lambda g(x) \\ &< \left(\frac{t_0^2}{2}\right)\|\phi\|_\mu^2 - \left(\frac{t_0^\gamma}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |\phi|^\gamma 1 \\ &\quad - \left(\frac{t_0}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b} |\phi| g(x) < 0. \end{aligned} \tag{27}$$

Thus, if $\omega = t_0\phi$, we obtain that $I_{\lambda,\mu}(\omega) < 0$.

(ii) By the Holder inequality and the definition of $S_{\mu,\gamma}$ and since $\gamma > 2$, we get for all $u \in \mathscr{D} \setminus \{0\}$

$$\begin{aligned} I_{\lambda,\mu}(u) &:= \left(\frac{1}{2}\right)\|u\|_\mu^2 \\ &\quad - \left(\frac{1}{\gamma}\right)\int_{\mathbb{R}^N} |y|^{-b}|u|^\gamma (1+\lambda g(x))\,dx \\ &\geq \left(\frac{1}{2}\right)\|u\|_\mu^2 - \left(\frac{1}{\gamma}\right) S_{\mu,\gamma}\|u\|_\mu^\gamma \left(1+\lambda |g|_\infty\right). \end{aligned} \tag{28}$$

If $\lambda > \Lambda^*$, then there exist $\nu > 0$ and $\varrho_0 > 0$ small enough such that

$$I_{\lambda,\mu}(u) \geq \nu > 0 \quad \text{for } \|u\|_\mu = \varrho_0. \tag{29}$$

We also assume that t_0 is small enough such that $\|t_0\phi\|_\mu < \varrho_0$. Thus, we have

$$\begin{aligned} c_1 = \inf\left\{I_{\lambda,\mu}(u) : u \in B_{\varrho_0}\right\} < 0, \\ \text{where } B_{\varrho_0} = \left\{u \in \mathscr{D},\ \mathscr{N}(u) \leq \varrho_0\right\}. \end{aligned} \tag{30}$$

Using Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_0}$ with respect to the norm of $\mathscr{D}$, we can prove that there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \overline{B}_{\rho_0}$ such that $u_n \rightharpoonup u_1$ for some u_1 with $\mathscr{N}(u_1) \leq \rho_0$.

Now, we claim that $u_n \to u_1$. If not, by Lemma 4, we have

$$\begin{aligned} c_1 &\geq I_{\lambda,\mu}(u_1) + \left(\frac{(\gamma-2)}{2\gamma}\right)\left(S_{\mu,\gamma}\right)^{\gamma/(\gamma-2)} \\ &\geq c_1 + \left(\frac{(\gamma-2)}{2\gamma}\right)\left(S_{\mu,\gamma}\right)^{\gamma/(\gamma-2)} > c_1, \end{aligned} \tag{31}$$

which is a contradiction.

Then, we obtain a critical point u_1 of $I_{\lambda,\mu}$ for all $\lambda > \Lambda^*$ large enough satisfying

$$c_1 = \left(\frac{(\gamma-2)}{2\gamma}\right)\|u_1\|_\mu^2 > 0. \tag{32}$$

□

Proof of Theorem 1. From Lemmas 4 and 5, we can deduce that there exists at least a nontrivial solution u_1 for our problem $(\mathscr{P}_{\lambda,\mu})$ with positive energy. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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