Designing bound states in a band as a model for a quantum network

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We provide a model of a one dimensional quantum network, in the framework of a lattice using Von Neumann and Wigner's idea of bound states in a continuum. The localized states acting as qubits are created by a controlled deformation of a periodic potential. These wave functions lie at the band edges and are defects in a lattice. We propose that these defect states, with atoms trapped in them, can be realized in an optical lattice and can act as a model for a quantum network.

I. INTRODUCTION

In classical computation, bits 0 and 1 are used to represent data. Logic gates are employed for computation and data manipulation. At present, considerable work is being done to develop quantum computers. It is expected that they will increase the speed and efficiency of computation [1] compared to a classical computer. The basic building blocks in quantum computing are known as quantum bits or qubits and are represented by the eigenstates of a system. For example, qubits are described by the up and down states of spin-half particles, represented by $|0\rangle$ and $|1\rangle$. The unitary operators play the role of logic gates and we can perform various operations similar to classical computing. The major areas of interest in the field of quantum computation are the development of models, having states which can be used as qubits and the construction of unitary operators to manipulate these qubits. It is important that these states should be well isolated from the outside environment to avoid environmental decoherence.

In this paper, we propose a theoretical model for preparing qubits *i.e.*, $|0 >$ and $|1 >$, using periodic potentials. These potentials are characterized by a band spectrum, which contains bands of allowed energies interspersed with forbidden energy gaps. The wave functions are not integrable and they extend to spatial infinity. Hence, they cannot be envisaged as qubits. We will explicitly demonstrate that a periodic potential can be appropriately deformed to accommodate bound states in its band spectrum. This deformation can be treated as the perturbation of the original potential. The effect of which is the creation of the localized states which can be perceived as defects in the lattice. Atoms trapped in these defects can be used to describe qubits. This treatment of the trapped atoms in optical lattices, as qubits in a quantum network, is similar to the quantum computation model proposed by Angelakis *et. al.,*[2]. Here, they have used photonic crystals to confine photons in the defect states created inside the band gap, to represent qubits. In quantum networking atoms are trapped in the nodes and are used to store information. We propose that an array of such localized states in the deformed optical lattice, with atoms trapped in them, can be used as a quantum network [3, 4].

At present, various optical lattices are routinely realized in the laboratories and atoms can be trapped in a potential well with relative ease. For our model, we consider one of the well - known family of periodic potentials namely, the Lamé potentials [5, 6],

$$V(x) = j(j+1)m \operatorname{sn}^2(x, m). \tag{1}$$

The function $\operatorname{sn}(x, m)$ is the Jacobi elliptic function [7, 8] with elliptic modulus $0 < m < 1$. These potentials are exactly solvable for integer values of j and for a given j there are $(2j + 1)$ band-edges. Both the form of these band-edge solutions and explicit solutions for smaller values of j are given in [6, 9, 10]. The Lamé potential has been proposed as a model for quasi 1 - d confinement of Bose - Einstein condensates (BEC) in a standing light wave [12]. Possible application of BECs for quantum computation are currently being explored and the bound states created by deforming a potential like the Lamé potential may have useful implications.

The method used to construct these bound states is the same as that used by Pappademos *et. al.,* to construct quantum mechanical bound states in a classical continuous energy spectrum. Such states were first discovered by

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Von Neumann and Wigner [13] and many such examples were found later [14 - 16]. It became clear that such states arose due to the delicate interplay between the rate at which the oscillatory potential falls off and the time taken by the various maxima of the potential to interact and create a bound state. Existence of such states was reported by Capasso *et. al.,* in semiconductor hetrostructures [17].

SUSYQM [18, 19] was applied to generate bound states in the continuum of the energy spectrum for a spherically symmetric potential [20, 21]. We apply the same technique to periodic potentials and construct bound states in their band spectrum, with the same energies as the band-edge energies. The solutions thus obtained are located at the band-edges and not inside the forbidden energy gap as discussed in [2, 22, 23].

In the next section, we give an overview of SUSYQM and briefly describe the steps involved in constructing a bound state in the continuum (BIC). In section III, we use the same procedure to construct bound states for the Lamé potential. In the last section, we present our conclusions.

II.   SUPERSYMMETRIC QUANTUM MECHANICS

For a given 1 - d potential $V(x)$ with eigenfunctions $u_n(x)$ and eigenvalues E_n $(n = 0, 1, 2...)$, we can generate a new family of potentials $\tilde{V}(x; \lambda)$, isospectral to $V(x)$. The parameter λ is used to label the potentials in the isospectral family and takes values lying in the range $\lambda > 0$ and $\lambda < -1$. Setting $\hbar = 2m = 1$ we can write the superpotential as,

$$W(x) = -\frac{u_0'}{u_0}. \tag{2}$$

The original potential $V(x)$ can be expressed in terms of $W(x)$ as,

$$V(x) = W^2(x) - W'(x). \tag{3}$$

Its isospectral partner $\tilde{V}(x; \lambda)$ is given by

$$\tilde{V}(x; \lambda) = W^2(x) + W'(x). \tag{4}$$

Let $\tilde{u}_n(x)$ be the eigenfunctions of $\tilde{V}(x; \lambda)$ $(n = 1, 2...)$. We have $\tilde{u}_n(x) = Au_n(x)$ where $A = d/dx + W(x)$. (Hence, $A^\dagger = -d/dx + W(x)$). Note that $\tilde{u}_0(x)$, the ground state cannot be obtained in this manner since $Au_0(x) = 0$. Thus, $\tilde{V}(x; \lambda)$ is isospectral to $V(x)$, except that its spectrum does not contain $\tilde{u}_0(x)$. Hence, to introduce the ground state into its spectrum and form a complete set of eigenstates, we need to find the most general superpotential $\tilde{W}(x)$, so that

$$\tilde{V}(x; \lambda) = \tilde{W}^2(x) + \tilde{W}'(x). \tag{5}$$

We can now show that

$$\tilde{W}(x) = W(x) + \frac{d}{dx} \ln(I_0(x) + \lambda) \tag{6}$$

where

$$I_0 = \int_0^x u_0^2(y) dy. \tag{7}$$

Thus, in this process of reinstating the ground state, we obtain the expression for the potential $\tilde{V}(x; \lambda)$ in terms of I_0 as,

$$\tilde{V}(x; \lambda) = V(x) - 2[\ln(I_0 + \lambda)]'' = V(x) - \frac{4u_0 u_0'}{I_0 + \lambda} + \frac{2u_0^4}{(I_0 + \lambda)^2}. \tag{8}$$

Note that $\tilde{V}(x; \lambda)$ is $V(x)$ plus a perturbative term and λ can be take as the perturbative parameter. The perturbed potential is isospectral to the old potential and its eigenstates are given below in Eqs. (9) and (10).

In standard SUSYQM, isospectral families of potentials which allow only bound states have been constructed, and $u_0(x)$ was taken to be the ground state. In [21], the above method was generalized to the case where the potentials have a continuous energy spectrum and $u_0(x)$ has been taken to be any non-singular eigenstate of $V(x)$.

This procedure was used to construct BIC for the spherically symmetric potential and the results were stated as a theorem. We refer the reader to [20] and the references therein for further details and merely reproduce the result here.

Let $u_0(x)$ and $u_1(x)$ be any two nonsingular solutions of the Schrödinger equation for the potential $V(x)$ corresponding to arbitrarily selected energies E_0 and E_1 respectively. Construct a new potential $\tilde{V}(x; \lambda)$ as prescribed by Eq. (8). Then, the two functions

$$\tilde{u}_0(x) = \frac{u_0}{I_0 + \lambda} \tag{9}$$

and

$$\tilde{u}_1(x) = (E_1 - E_0)u_1(x) + \tilde{u}_0(x)W_r(u_0(x), u_1(x)) \tag{10}$$

are solutions of the Schrödinger equation for the new potential $\tilde{V}(x; \lambda)$, corresponding to the same energies E_0 and E_1. Here $W_r(u_0(x), u_1(x))$ is the Wronskian.

Note that the original potential $V(x)$ had no integrable solutions but the new potential has one square-integrable solution $\tilde{u}_0(x)$, with the rest being non - integrable. The creation of the bound state can be elucidated by the fact that I_0 in Eq. (7) diverges owing to the non-integrability of u_0. Hence, as $I_0 \to \infty$, $\tilde{u}_0(x) \to 0$, resulting in a square-integrable wave function in the continuum.

We can create another bound state by using the non-normalizable state $\tilde{u}_1(x)$ in place of $u_0(x)$ and deforming $\tilde{V}(x; \lambda)$, using the same procedure described above. We then obtain a potential $\tilde{\tilde{V}}(x; \lambda, \lambda_1)$, isospectral to $V(x)$ which has two bound states in the continuous spectrum with energies E_0 and E_1. The parameter λ_1 is a real number lying in the range $\lambda_1 > 0$ and $\lambda_1 < -1$.

The expressions for the new potential $\tilde{\tilde{V}}(x)$ and the two square-integrable states $\tilde{\tilde{u}}_0(x)$ and $\tilde{\tilde{u}}_1(x)$, with energies E_0 and E_1 respectively are given by

$$\tilde{\tilde{V}}(x) = \tilde{V}(x) - 2[\ln(I_1 + \lambda_1)]'' = \tilde{V}(x) - \frac{4\tilde{u}_1 \tilde{u}_1'}{I_1 + \lambda_1} + \frac{2\tilde{u}_1^4}{(I_1 + \lambda_1)^2}, \tag{11}$$

$$\tilde{\tilde{u}}_0(x) = (E_0 - E_1)\tilde{u}_0(x) + \tilde{\tilde{u}}_1(x)W_r(\tilde{u}_1(x), \tilde{u}_0(x)) \tag{12}$$

and

$$\tilde{\tilde{u}}_1(x) = \frac{\tilde{u}_1}{I_1 + \lambda_1} \tag{13}$$

where

$$I_1 = \int_0^x \tilde{u}_1^2(y)dy \tag{14}$$

and $W_r(\tilde{u}_1(x), \tilde{u}_0(x))$ is the Wronskian. In the next section, we apply the above technique to periodic potentials on the half-line and construct bound states in the band spectrum. For this purpose, we use the band-edge wave functions to deform the original periodic potential. The bound states thus created have the same band-edge energies.

III.   THE LAMÉ POTENTIAL

The Lamé potential (Eq. (1)) with $j = 2$ is,

$$V(x) = 6m\,\text{sn}^2(x, m). \tag{15}$$

It has two bands and a continuum. The expressions for the five band - edge wave functions and energies are given below [6] [9, 10] [27] with $\psi_0(x)$ and $\psi_1(x)$ representing the lower and upper band - edge wave functions of the first band and so on. We have

$$\psi_0(x) = 3m + 3 - \delta - 3m\,\text{sn}^2(x, m), \ E_0 = 2\delta - 2m - 2, \tag{16}$$

$$\psi_1(x) = \text{cn}(x, m)\,\text{dn}(x, m), \ E_1 = m + 1, \tag{17}$$

$$\psi_2(x) = \text{dn}(x, m)\,\text{sn}(x, m), \ E_2 = 4m + 1, \tag{18}$$

$$\psi_3(x) = \text{cn}(x, m)\,\text{sn}(x, m), \ E_3 = m + 4, \tag{19}$$

$$\psi_4(x) = 3m + 3 - 3\delta - 3m\,\text{sn}^2(x, m), \ E_4 = 2\delta + 2m + 2, \tag{20}$$

where, the functions $\mathrm{cn}\,(x,m)$ and $\mathrm{dn}\,(x,m)$ are the Jacobi elliptic functions with modulus parameter m. For constructing the bound states, we examine only the half-line problem and hence consider the band-edge wave functions which vanish at the origin. In the above given expressions, only Eqs. (18) and (19) which represent the lower and upper band edge wave functions of the second band, satisfy this condition, since $\mathrm{sn}\,(0,m) = 0$. We follow the steps described in the previous section and construct two bound states with energies E_2 and E_3. The entire procedure is done numerically and we give the plots of the deformed potential and bound states thus obtained, in the sequel.

We deform the potential given in Eq. (15), using the expression for $\psi_2(x)$ in Eq. (18), to obtain a one-parameter bound state solution which depends on the parameter λ. For this purpose, we first plot I_0 versus x in Fig. 1. As expected I_0 turns out to be a diverging integral. Using I_0 and Eqs. (8), (9) and (10), we plot the deformed potential $\tilde{V}(x)$ and the deformed wave functions $\tilde{\psi}_2(x)$ and $\tilde{\psi}_3(x)$. These are give in Figs. 2, 3 and 4 respectively. For comparison, the original potential and wave functions are plotted in dotted line.

It is clear from Figs. 3 and 4 that $\tilde{\psi}_2(x)$ is a normalizable state and $\tilde{\psi}_3(x)$ is not normalizable. Thus, with this deformation we have obtained only one bound state. In order to construct two bound states, we deform $\tilde{V}(x)$ with $\tilde{\psi}_3(x)$ using Eqs (11) - (14). Plots of I_1, $\tilde{\tilde{V}}(x)$, $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$ versus x are given in Figs. 5, 6, 7 and 8 respectively.

From Figs.7 and 8, it is clear that both $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$ are integrable. These states have energies E_2 and E_3 and the potential $\tilde{\tilde{V}}(x)$ is isospectral to the original potential $V(x)$. The deformed potential and the bound states depend on the parameters λ and λ_1 and as we increases their values, the deformed states and the potential tend towards the corresponding original states and the potential. In Figs. 9 and 10, we give the plots of the deformed wave functions $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$, for two different values of λ and λ_1. (Without loss of generality we have set $\lambda = \lambda_1 = 1$ and $\lambda = \lambda_1 = 10$ in these figures).

IV. CONCLUSIONS

We have shown that we can construct bound states in the band spectrum of a periodic potential using SUSYQM. It is clear from the above procedure that we can create a class of bound states by successively deforming the potential $V(x)$, provided there exist band-edge wave functions, of the original potential, which satisfy the boundary condition $\psi_n(0) = 0$.

We can use these square-integrable states as qubits in quantum computation. The localized states with atoms trapped in them can be treated as the qubits $|0\rangle$ and $|1\rangle$. If it is possible to get n such defect states in the optical lattice we have an array of qubits which can be used as an optical network [4]. The perturbative parameters can be used to adjust the overlap of these trapped atoms and also to control the deformation of the periodic potential. Moreover, these localized states are in the band and hence, protected from the external influences. Since optical lattices are easy to create and manipulate in the laboratory, this can be a useful model for quantum networking [24, 25, 26].

In conclusion, we have shown that it is possible to deform a periodic potential to accommodate localized states at the band edges. It is proposed that by trapping atoms in these states we can construct qubits and an array of n such qubits can be used as a quantum network.

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27. The solutions given in these references [6, 9, 10]are for supersymmetric potential, where a constant is added to the potential to make the ground state energy zero.
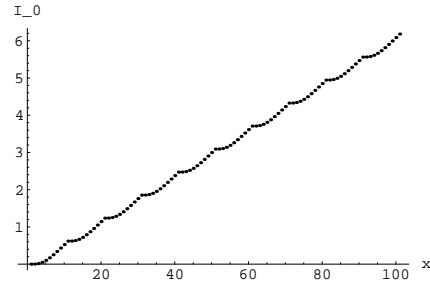
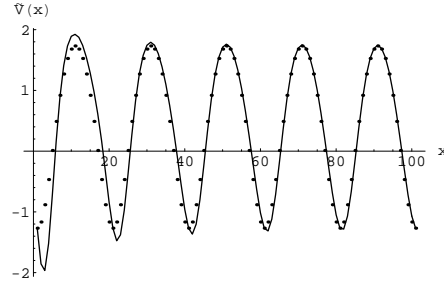FIG. 1: Plot of the diverging integral I_0 versus x.

FIG. 2: Deformed potential $\tilde{V}(x)$ versus x, with $\lambda = 1$. The dotted line represents the original potential $V(x)$.
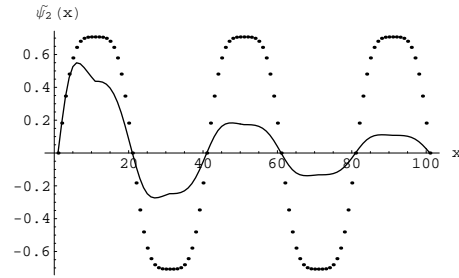

FIG. 3: Deformed wave function $\tilde{\psi}_2(x)$ versus x, with $\lambda = 1$. The dotted line represents the original band-edge wave function $\psi_2(x)$.
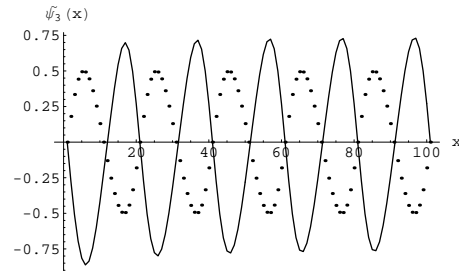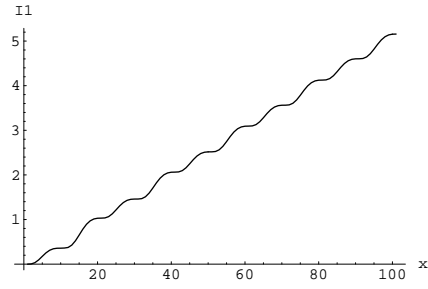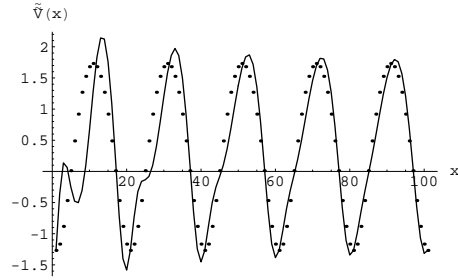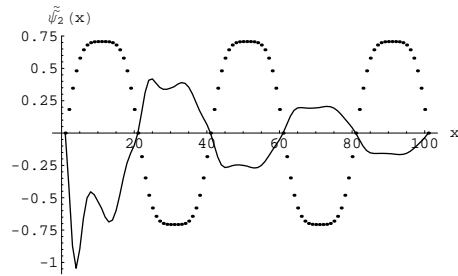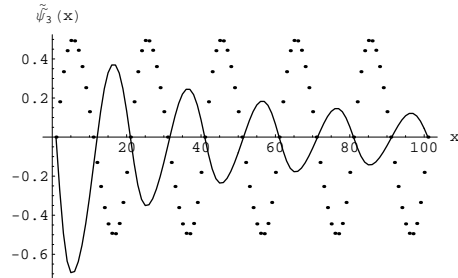

FIG. 4: Deformed wave function $\tilde{\psi}_3(x)$ versus x, with $\lambda = 1$. The dotted line represents the original band-edge wave function $\psi_3(x)$.

FIG. 5: Plot of the diverging integral I_1 versus x.

FIG. 6: Deformed potential $\tilde{\tilde{V}}(x)$ versus x, with $\lambda_1 = 1$. The dotted line represents the original potential $V(x)$.

FIG. 7: Deformed wave function $\tilde{\tilde{\psi}}_2(x)$ versus x, with $\lambda_1 = 1$. The dotted line represents the original band-edge wave function $\psi_2(x)$.

FIG. 8: Deformed wave function $\tilde{\tilde{\psi}}_3(x)$ versus x, with $\lambda_1 = 1$. The dotted line represents the original band-edge wave function $\psi_3(x)$.
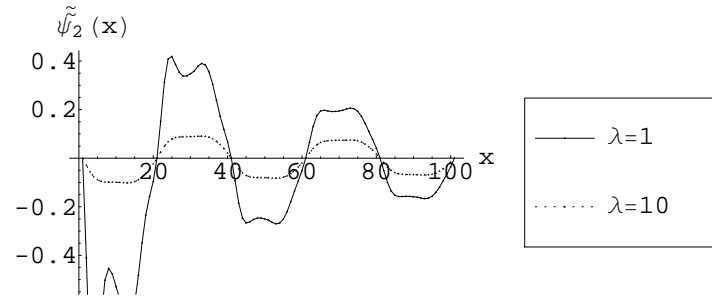
FIG. 9: Deformed wave function $\tilde{\tilde{\psi}}_2(x)$ for $\lambda_1 = \lambda = 1$ (dotted line) and for $\lambda_1 = \lambda = 10$ (thick line).
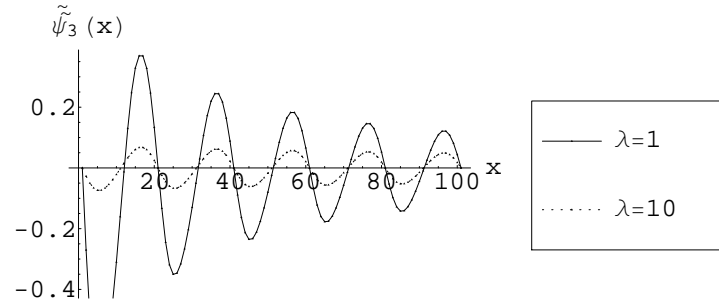

FIG. 10: Deformed wave function $\tilde{\tilde{\psi}}_3(x)$ for $\lambda_1 = \lambda = 1$ (dotted line) and for $\lambda_1 = \lambda = 10$ (thick line).