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Self-gravitating ring of matter in orbit around a black hole: the innermost stable circular orbit

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Abstract We study analytically a black-hole-ring system which is composed of a stationary axisymmetric ring of particles in orbit around a perturbed Kerr black hole of mass M. In particular, we calculate the shift in the orbital frequency of the innermost stable circular orbit (ISCO) due to the finite mass m of the orbiting ring. It is shown that for thin rings of half-thickness $r \ll M$, the dominant finite-mass correction to the characteristic ISCO frequency stems from the self-gravitational potential energy of the ring (a term in the energy budget of the system which is *quadratic* in the mass m of the ring). This dominant correction to the ISCO frequency is of order $O(\mu \ln(M/r))$, where $\mu \equiv m/M$ is the dimensionless mass of the ring. We show that the ISCO frequency *increases* (as compared to the ISCO frequency of an orbiting test-ring) due to the finite-mass effects of the self-gravitating ring.

1 Introduction

The geodesic motions of test particles in black-hole spacetimes are an important source of information on the structure of the spacetime geometry [1–11]. Of particular importance is the innermost stable circular orbit (ISCO). This orbit is defined by the onset of a dynamical instability for circular geodesics. In particular, the ISCO separates stable circular orbits from orbits that plunge into the central black hole [2]. This special geodesic therefore plays a central role in the two-body dynamics of in-spiralling compact binaries since it marks the critical point where the character of the motion sharply changes [5]. In addition, this marginally stable orbit is usually regarded as the inner edge of accretion disks in composed black-hole-disk systems [2].

An important physical quantity which characterizes the ISCO is the orbital angular frequency Ω_{isco} as measured by asymptotic observers. This characteristic frequency is often regarded as the end-point of the in-spiral gravitational templates [5]. For a test particle in the Schwarzschild-black-hole spacetime, this frequency is given by the well-known relation [1–11]

$$M\Omega_{\text{isco}} = 6^{-3/2}, \tag{1}$$

where M is the mass of the central black hole.

Realistic astrophysical scenarios often involve a composed two-body system in which the mass m of the orbiting object is smaller but *non*-negligible as compared to the mass M of the central black hole [5]. In these situations the zeroth-order (test-particle) approximation is no longer valid and one should take into account the gravitational self-force (GSF) corrections to the orbit [12–26]. These first-order corrections take into account the *finite* mass m of the orbiting object. The gravitational self-force has two distinct contributions: (1) It is responsible for dissipative (radiation-reaction) effects that cause the orbiting particle to emit gravitational waves. The location of the ISCO may become blurred due to these non-conservative effects [5,12,13]. (2) The self-force due to the finite mass of the particle is also responsible for conservative effects which preserve the characteristic constants of the orbital motion. These conservative effects produce a non-trivial shift in the ISCO frequency, which characterizes the two-body dynamics.

It should be emphasized that the computation of the conservative shift in the characteristic ISCO frequency (due to the *finite* mass of the orbiting object) is a highly non-trivial task. A notable event in the history of the two-body problem in general relativity took place three years ago: after two decades of intensive efforts by many groups of researches to evaluate the conservative self-force corrections to the orbital parameters, Barack and Sago [23,24] have succeeded in computing the shift in the ISCO frequency due to the finite mass of the orbiting object. Their numerical result for the corrected ISCO frequency can be expressed in the form [22–24]

$$M\Omega_{\text{isco}} = 6^{-3/2}(1 + c \cdot \mu) \quad \text{with} \quad c \simeq 0.251, \tag{2}$$

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where

$$\mu \equiv m/M \ll 1 \tag{3}$$

is the dimensionless ratio between the mass of the orbiting object (the 'particle') and the mass of the black hole. The result (2) provides valuable information about the conservative dynamics of the composed two-body system in the strong-gravity regime.

It is worth emphasizing that the $O(\mu)$ correction term to the ISCO frequency [see Eq. (2)] stems from similar correction terms that appear in the metric components of the perturbed black-hole spacetime [23,24]. These correction terms to the metric components of the "bare" Schwarzschild spacetime are also linear (in the extreme mass-ratio regime) in the dimensionless mass μ of the orbiting particle [23].

In the present study we shall analyze a closely related (but mathematically much simpler) problem: that of a *self-gravitating* thin ring of matter in equatorial orbit around a central black hole [27,28][1]. This composed system, like the original black-hole-particle system, is characterized by a perturbative (finite-mass) correction to the ISCO frequency [see Eq. (15) below]. In fact, as we shall show below, the leading-order shift in the ISCO frequency can be computed *analytically* for this axisymmetric black-hole-ring system.

Before proceeding into details, it is important to emphasize that there is one important difference between the black-hole-particle system studied in [12–26] and the black-hole-ring system that we shall study here: As emphasized in Ref. [23,24], the work [23,24] is complementary to the analysis of [12,13] in that [23,24] considered only conservative GSF corrections and ignored dissipative (radiation-reaction) GSF effects. It is only then that the ISCO has a sharp location. It should be emphasized that, for the black-hole-particle system, the dissipative effects actually dominate over the conservative ones [12,13,23,24]. Thus, the radius and orbital frequency of the ISCO in the black-hole-particle system are not sharply defined. On the other hand, due to the axial symmetry of the black-hole-ring system, there is no emission of gravitational waves, and thus there are no dissipative effects in the black-hole-ring system. The black-hole-ring system is therefore characterized by purely conservative finite-mass corrections to the dynamics. Consequently, the radius and orbital frequency of the ISCO in the black-hole-ring system are sharply defined [see Eqs. (9) and (15) below].

It is worth emphasizing that recently [9] we analyzed a simplified black-hole-ring toy model. Following the original analysis of [27,28], in [9] we ignored the *self*-gravitational potential energy of the ring. This self-energy term represents the inner interactions between the *many* particles that compose the orbiting ring. Since our goal in [9] was to model the conservative dynamics of the two-body (black-hole-particle) system (with a *single* orbiting particle), we did not consider in [9] this *many*-particle self-gravitational term. The omission of this self-gravitational potential energy term has allowed us to focus in [9] on the frame-dragging effect caused by the orbiting object. In this respect, the ring considered in [9,27,28] should be regarded as a quasi-test ring.

Our main goal in the present study is to analyze the influence of this *self*-gravitational potential energy term on the dynamics of the ring (as emphasized above, following the original analysis of [27,28], this self-energy term was ignored in the toy model studied in [9]). As we shall show below, this self-gravitational potential energy of the ring determines the leading-order correction to the ISCO frequency in the thin-ring regime.

2 The black-hole-ring system

We consider a black-hole-ring system which is composed of a stationary axisymmetric ring of particles in orbit around a black hole. This system is characterized by five physical parameters [27,28]: The mass M of the black hole, the angular momentum per unit mass a of the black hole, the rest mass m of the ring, the proper circumferential radius R of the ring, and the half-thickness r of the ring. We shall assume that the ring is thin and weakly self-gravitating in the sense that

$$\max(r/m, 1) \ll \ln(M/r) \ll M/m. \tag{4}$$

The total energy (energy-at-infinity) of the ring in the black-hole spacetime is given by [1,27–29][2]

[1] Recently [9,10] we proposed to model the *conservative* behavior of the two-body system using a simple toy model which is composed of an axisymmetric ring of particles in orbit around a perturbed Schwarzschild black hole. This composed system was first analyzed by Will [27]. Following the original analysis of [27], in [9,10] we have taken into account the physical effect of frame-dragging caused by the orbiting ring (this physical effect introduces a term quadratic in the mass m of the orbiting ring into the energy budget of the system) but ignored the self-gravitational potential energy of the ring (an energy term which is also quadratic in the mass m of the ring). We have therefore regarded the ring in [27] as a *quasi-test ring*. We have shown [9,10] that this toy model (with *neglecting* of the self-gravitational potential energy of the ring) may capture some important features of the conservative two-body (black-hole-particle) dynamics. (It should be emphasized that this toy model, being axially symmetric, cannot describe the most important feature of the two-body dynamics: the emission of gravitational waves.) In particular, the shift in the ISCO frequency found for the quasi-test ring [9] (a shift which is caused by the physical effect of frame dragging) is remarkably close to the shift in the ISCO frequency found for the original black-hole-particle system [23] [compare (2) with Eq. (10) of [9]]. In addition, it was shown [10,26] that the black-hole-quasi-test-ring system and the black-hole-particle system share the *same* quadratic divergent behavior of the physical quantities near the light ring, where the circular orbits become null.

[2] The first term on the r.h.s. of Eq. (5) should be multiplied by a correction factor $1 + O(r/R)$ which stems from the finite thickness of the ring. Note that in the thin-ring regime, $r/m \ll \ln(R/r)$ [see Eq. (4)], this correction term is dominated over by the self-gravitational

$$E(R;M,a,m,r) = m\cdot\frac{R^{3/2}-2MR^{1/2}\pm aM^{1/2}}{R^{3/4}(R^{3/2}-3MR^{1/2}\pm 2aM^{1/2})^{1/2}} -\frac{m^2}{2\pi R}\ln(R/r)+O(m^2/M), \tag{5}$$

where the upper/lower signs correspond to co-rotating/counter-rotating orbits, respectively. The first term on the r.h.s. of (5) represents the first-order contribution of the ring to the total mass of the system [1]. In the Newtonian (large-R) limit it becomes $m-M_{\rm ir}m/2R$, which can be identified as the rest mass of the ring plus the (negative) potential energy of the black-hole-ring system plus the rotational energy of the ring. The second term on the r.h.s. of Eq. (5) represents the (second-order) self-gravitational potential energy of the ring [29].

The sub-leading correction term $O(m^2/M)$ in Eq. (5) represents a non-linear contribution to the energy of the ring which stems from $O(m/M)$ corrections to the metric components of the "bare" Kerr spacetime [27,28]. In the $\ln(M/r)\gg 1$ regime [the thin-ring regime; see Eq. (4)] this $O(m^2/M)$ term is much smaller than the self-gravitational potential energy of the ring which is of order $O(\frac{m^2}{M}\ln(R/r))$; see Eq. (5). Thus, the dominant contribution to the ISCO frequency shift [a term of order $O(\frac{m}{M^2}\ln(M/r))$; see Eq. (15) below] would stem from the self-gravitational potential energy of the thin ring. This correction term would dominate over a sub-leading correction term [of order $O(\frac{m}{M^2})$] which stems from finite-mass corrections to the metric components of the "bare" Kerr metric [27,28].

3 The innermost stable circular orbit

A standard way to identify the location of the ISCO is by finding the minimum of the orbital energy [5,25,30][3]. A simple differentiation of (5) with respect to R yields the characteristic equation[4, 5]

$$\frac{R^{1/4}(R^2-6MR\pm 8aM^{1/2}R^{1/2}-3a^2)}{(R^{3/2}-3MR^{1/2}\pm 2aM^{1/2})^{3/2}} +\mu\cdot[\pi^{-1}\ln(R/r)+O(1)]=0 \tag{6}$$

for the location of the ISCO. The zeroth-order solution (with $\mu\equiv 0$) of the characteristic equation (6) is given by [1]

$$R_0=M\{3+Z_2\mp[(3-Z_1)(3+Z_1+2Z_2)]^{1/2}\}, \tag{7}$$

where

$$Z_1\equiv 1+(1-a^2/M^2)^{1/3}[(1+a/M)^{1/3}+(1-a/M)^{1/3}] \quad \text{and} \quad Z_2\equiv(3a^2/M^2+Z_1^2)^{1/2}. \tag{8}$$

The corrected (first-order) solution of the characteristic equation (6) is then given by

$$R_{\rm isco}=R_0\{1-\mu\cdot f(a)\cdot[\ln(M/r)+O(1)]\}, \tag{9}$$

where

$$f(a)\equiv\frac{1}{2\pi}\left(1-\frac{3M}{R_0}\pm\frac{2aM^{1/2}}{R_0^{3/2}}\right)^{1/2}. \tag{10}$$

The expression (9) provides the location of the ISCO for the composed black-hole-ring system in the finite mass-ratio (finite-μ) regime. In the Schwarzschild ($a\to 0$) limit one finds

$$R_{\rm isco}(a=0)=6M\left[1-\mu\cdot\frac{1}{2\sqrt{2}\pi}\ln(M/r)\right]. \tag{11}$$

In the extremal ($a\to M$) limit one finds

$$R_{\rm isco}(a=M)=9M\left[1-\mu\cdot\frac{2}{3\sqrt{3}\pi}\ln(M/r)\right]. \tag{12}$$

for counter-rotating orbits, and

$$R_{\rm isco}(a=M(1-\epsilon)) = M\left\{1+(4\epsilon)^{1/3}\left[1-\mu\cdot\frac{\sqrt{3}}{4\pi}\ln(M/r)\right]\right\} \tag{13}$$

for co-rotating orbits, where $\epsilon\ll 1$.

The angular velocity of the ring is given by [2,27,28]

$$\Omega=\frac{\sqrt{M/R^3}}{\pm 1+a\sqrt{M/R^3}}[1+O(\mu)]. \tag{14}$$

The correction term $O(\mu)$ in Eq. (14) represents a non-linear contribution to the angular velocity of the ring [27,28] which stems from $O(\mu)$ corrections to the metric components of the "bare" Kerr spacetime. In the $\ln(M/r)\gg 1$ regime [the thin-ring regime; see Eq. (4)] this $O(\mu)$ term is much smaller than the leading-order $O(\mu\ln(M/r))$ correction term to $\Omega_{\rm isco}$ which stems from the self-gravitational potential energy correction to $R_{\rm isco}$ [a term of order $O(\mu\ln(M/r))$; see Eq. (9)].

Substituting (9) into (14), one finds

$$\Omega_{\rm isco}=\Omega_0\{1+\mu\cdot g(a)\cdot[\ln(M/r)+O(1)]\} \tag{15}$$

Footnote 2 continued
potential energy of the thin ring which is of order $O(m^2\ln(R/r)/R)$ [see the second term on the r.h.s. of Eq. (5)].

[3] The critical orbit so defined is sometimes referred to as the MECO (maximum binding-energy circular orbit) [5,25,30].

[4] We consider here a family of orbiting rings with a fixed (constant) value of the parameter r, the half-thickness of the rings. Alternatively, one may consider a family of constant proper-density rings. In this case, the parameter r would have the dependence $r=const\times R^{-1/2}$, where R is the circumferential radius of the ring. For such a family of constant proper-density rings, the square brackets in Eq. (6) would simply acquire a sub-leading correction term of order $O(1)$, which, in the thin-ring regime $r\ll R$ [see Eq. (4)], is dominated over by the leading-order correction term $\ln(R/r)\gg 1$ [see Eq. (6)].

[5] The first term on the l.h.s. of Eq. (6) should be multiplied by a correction factor $1+O(r/R)$ with $r/R\ll 1$ [see Eq. (4)].

for the perturbed ISCO frequency of the ring, where

$$\Omega_0 \equiv \frac{\sqrt{M/R_0^3}}{\pm 1 + a\sqrt{M/R_0^3}} \tag{16}$$

is the zeroth-order frequency of an orbiting test-ring, and

$$g(a) \equiv \frac{3}{4\pi}(1 - a\Omega_0)\left(1 - \frac{3M}{R_0} \pm \frac{2aM^{1/2}}{R_0^{3/2}}\right)^{1/2}. \tag{17}$$

The expression (15) provides the characteristic frequency of the ISCO for the composed black-hole-ring system in the finite mass-ratio (finite-μ) regime. In the Schwarzschild ($a \to 0$) limit one finds

$$M\Omega_{\text{isco}}(a = 0) = 6^{-3/2}\left[1 + \mu \cdot \frac{3}{4\sqrt{2\pi}} \ln(M/r)\right]. \tag{18}$$

[Compare (18) with the corresponding result (2) for the (Schwarzschild-) black-hole-particle system. In both cases the ISCO frequency increases due to the *finite* mass of the orbiting object. Note, however, that the $O(\mu)$ correction in Eq. (2) stems from $O(\mu)$ corrections to the metric components of the bare Schwarzschild metric while the dominant $O(\mu \ln(M/r))$ correction in Eq. (18) stems from the self-gravitational potential energy of the orbiting ring]. In the extremal ($a \to M$) limit one finds

$$M\Omega_{\text{isco}}(a = M) = -\frac{1}{26}\left[1 + \mu \cdot \frac{9\sqrt{3}}{26\pi} \ln(M/r)\right] \tag{19}$$

for counter-rotating orbits, and

$$M\Omega_{\text{isco}}(a = M(1 - \epsilon)) = \frac{1}{2}\left\{1 - \frac{3}{4}(4\epsilon)^{1/3}\left[1 - \mu \cdot \frac{\sqrt{3}}{4\pi} \ln(M/r)\right]\right\} \tag{20}$$

for co-rotating orbits.

It is worth emphasizing that the shift-function $g(a)$ is a non-negative function for all a-values. Thus, taking cognizance of Eq. (15) one concludes that the ISCO frequency *increases* (in its absolute value) due to the finite-mass effects of the orbiting ring:

$$|\Omega_{\text{isco}}| > |\Omega_0|. \tag{21}$$

4 Summary and discussion

We have analyzed a stationary and axisymmetric black-hole-ring system which is composed of a *self-gravitating* ring of matter in orbit around a central black hole. In particular, we have calculated the shift in the fundamental frequency of the innermost stable circular orbit (ISCO) due to the finite mass of the ring. For thin rings with $\ln(M/r) \gg \max(r/m, 1)$ [see Eq. (4)], the dominant finite-mass correction to the ISCO frequency stems from the self-gravitational potential energy of the ring. This correction term is of order $O[\mu \ln(M/r)]$ and, in the thin-ring regime (4), it dominates over a sub-leading correction term of order $O(\mu)$ which stems from $O(\mu)$ corrections to the metric components of the "bare" black hole. We have shown [see Eqs. (15) and (17)] that the characteristic ISCO frequency *increases* due to the finite-mass effects of the self-gravitating orbiting ring.

It is worth emphasizing that the composed black-hole-ring system, being axially symmetric, is characterized by purely conservative gravitational effects. That is, there is no emission of gravitational waves in this axially symmetric system. Thus, this composed two-body system probably has a limited observational relevance. The black-hole-ring system should instead be regarded as a simple toy model for the astrophysically more relevant two-body (black-hole-particle) system.

In this respect, the black-hole-self-gravitating-ring model has two important advantages over the astrophysically more realistic black-hole-particle system:

1. The original black-hole-particle system is a highly non-symmetrical system. This lack of symmetry makes the computation of the ISCO frequency shift a highly non-trivial task. In fact, one is forced to use numerical techniques [23,24] in order to compute the ISCO shift in this system [see Eq. (2)]. On the other hand, the black-hole-ring toy model is an axially symmetric system. As we have shown above, this axial symmetry of the black-hole-ring model simplifies the calculation of the ISCO frequency shift. In fact, we have seen that, due to the axial symmetry of the black-hole-ring system, one can obtain an *analytic* formula [see Eq. (15)] for the ISCO frequency shift in this composed system. We believe that any new analytical solution, even one for a simplified (more symmetrical) problem, is certainly a useful contribution to this field.
2. To the best of our knowledge, the highly important result (2) for the ISCO frequency shift in the Schwarzschild-black-hole-particle system has so far not been extended to the case of rotating Kerr black holes. This lack of results for generic Kerr black holes is probably due to the numerical complexity of the problem. On the other hand, as we have seen above [see Eq. (15)], the calculation of the ISCO frequency shift in the composed black-hole-ring model can be extended to the regime of *generic* (that is, rotating) Kerr black holes. In this respect, it is worth noting that our conclusion (18) that the ISCO frequency of the Schwarzschild-black-hole-ring system increases due to the finite mass of the orbiting object is in agreement with the corresponding result (2) for the original Schwarzschild-black-hole-particle system [23,24]. This

qualitative agreement may indicate that the increase in the ISCO frequency (due to the finite mass of the orbiting object) may be a generic feature of the conservative two-body dynamics [see Eq. (21)].

Finally, it is worth mentioning the well known fact that many astrophysical black holes have accretion disks around them [31,32]. The radial location of the test-particle ISCO [see Eq. (7)] is usually regarded as the inner edge of the accretion disk in these composed black-hole-disk systems. It is expected, however, that self-gravitational effects (due to the finite mass of the accretion disk) would modify the location (the radius) of the disk's inner edge in these astrophysical black-hole-disk systems. In this respect, our result (9) for the location of the ISCO in the composed black-hole-self-gravitating-ring system should be regarded as a first approximation for the location of the ISCO (the location of the disk's inner edge) in realistic black-hole-disk systems. We believe that our analytic treatment of the black-hole-ring system will be useful and stimulating for further studies of the physical properties (and, in particular, the location of the ISCO) of astrophysical black-hole-disk systems.

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