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Research Article

Uniqueness of Entire Functions Sharing Polynomials with Their Derivatives

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We use the theory of normal families to prove the following. Let $Q_1(z) = a_1z^p + a_{1,p-1}z^{p-1} + \cdots + a_{1,0}$ and $Q_2(z) = a_2z^p + a_{2,p-1}z^{p-1} + \cdots + a_{2,0}$ be two polynomials such that $\deg Q_1 = \deg Q_2 = p$ (where p is a nonnegative integer) and $a_1, a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let $f(z)$ be a transcendental entire function. If $f(z)$ and $f'(z)$ share the polynomial $Q_1(z)$ CM and if $f(z) = Q_2(z)$ whenever $f'(z) = Q_2(z)$, then $f \equiv f'$. This result improves a result due to Li and Yi.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$,and let $P(z)$ be a polynomial or a finite complex number. $\deg P(z)$ denotes the degree of the polynomial $P(z)$. To simplify the statement of our results in this paper, deviating from the common definition, we consider the zero polynomial as a polynomial of degree 0. If $g(z) - P(z) = 0$ whenever $f(z) - P(z) = 0$, we write $f(z) = P(z) \Rightarrow g(z) = P(z)$. If $f(z) = P(z) \Rightarrow g(z) = P(z)$ and $g(z) = P(z) \Rightarrow f(z) = P(z)$, we write $f(z) = P(z) \Leftrightarrow g(z) = P(z)$ and say that $f(z)$ and $g(z)$ share $P(z)$ (IM ignoring multiplicity). If $f(z)-P(z)$ and $g(z)-P(z)$ have the same zeros with the same multiplicities, we write $f(z) = P(z) \rightleftharpoons g(z) = P(z)$ and say that $f(z)$ and $g(z)$ share $P(z)$ (CM counting multiplicity) (see, [1]). In addition, we use notations $\sigma(f)$, $\sigma_2(f)$ to denote the order and the hyperorder of $f(z)$, respectively, where

$$\sigma(f) = \limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r}, \qquad \sigma_2(f) = \limsup_{r\to\infty}\frac{\log^+\log^+T(r,f)}{\log r}. \tag{1.1}$$

It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [1, 2].

In 1977, Rubel and Yang [3] proved the well-known theorem.

Theorem A. *Let a and b be two complex numbers such that $b \neq a$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \rightleftharpoons f'(z) = a$ and $f(z) = b \rightleftharpoons f'(z) = b$, then $f(z) \equiv f'(z)$.*

This result has undergone various extensions and improvements (see, [1]).

In 1979, Mues and Steinmetz [4] proved the following result.

Theorem B. *Let a and b be two complex numbers such that $b \neq a$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \Leftrightarrow f'(z) = a$ and $f(z) = b \Leftrightarrow f'(z) = b$, then $f(z) \equiv f'(z)$.*

In 2006, Li and Yi [5] proved the following related result.

Theorem C. *Let a and b be two complex numbers such that $b \neq a, 0$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \rightleftharpoons f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $f(z) \equiv f'(z)$.*

Remark 1.1. Meanwhile, Li and Yi [5] give an example to show that $b \neq 0$ cannot be omitted in Theorem C.

In recent years, there have been several papers dealing with entire functions that share a polynomial with their derivatives.

In 2006, Wang [6] proved the following result.

Theorem D. *Let $f(z)$ be a nonconstant entire function, and let $Q(z)$ be a polynomial of degree $q \geq 1$. Let $k \geq q+1$ be an integer. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$ and if $f^{(k)}(z) = Q(z)$ for every $z \in \mathbb{C}$ with $f(z) = Q(z)$, then $f(z) \equiv f'(z)$.*

In 2007, Li and Yi [7] proved the following result.

Theorem E. *Let $f(z)$ be a nonconstant entire function of hyperorder $\sigma_2(f) < 1/2$, and let $Q(z)$ be a nonconstant polynomial. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$, then*

$$\frac{f'(z) - Q(z)}{f(z) - Q(z)} \equiv c \tag{1.2}$$

for some constant $c \neq 0$.

In 2008, Grahl and Meng [8] proved the following result.

Theorem F. *Let $f(z)$ be a nonconstant entire function, and let $Q(z)$ be a nonconstant polynomial. Let $k \geq 2$ be an integer. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$ and if for some positive M we have $|f^{(k)}(z)| \leq M(1 + |Q(z)|)$ for every $z \in \mathbb{C}$ with $f(z) = Q(z)$, then*

$$\frac{f'(z) - Q(z)}{f(z) - Q(z)} \tag{1.3}$$

is constant.

From the ideas of Theorem D to Theorem F, it is natural to ask whether the values a, b in Theorem C can be replaced by two polynomials Q_1, Q_2. The main purpose of this paper is to investigate this problem. We prove the following result.

Theorem 1.2. *Let* $Q_1(z) = a_1z^p + a_{1,p-1}z^{p-1} + \cdots + a_{1,0}$ *and* $Q_2(z) = a_2z^p + a_{2,p-1}z^{p-1} + \cdots + a_{2,0}$ *be two polynomials such that* $\deg Q_1(z) = \deg Q_2(z) = p$ *(where* p *is a nonnegative integer) and* a_1, $a_2(a_2 \neq 0)$ *are two distinct complex numbers. Let* $f(z)$ *be a transcendental entire function. If* $f(z) = Q_1(z) \rightleftharpoons f'(z) = Q_1(z)$ *and* $f'(z) = Q_2 \Rightarrow f(z) = Q_2(z)$*, then* $f(z) \equiv f'(z)$.

Remark 1.3. The following example shows the hypothesis that f is transcendental cannot be omitted in Theorem 1.2.

Example 1.4. Let $f(z) = z^3$, $Q_1(z) = 2z^3 - 3z^2$ and $Q_2(z) = z^3$. Then

$$\frac{f'(z) - Q_1(z)}{f(z) - Q_1(z)} = 2, \quad f'(z) = Q_2(z) \Longrightarrow f(z) = Q_2(z). \tag{1.4}$$

While $f(z)$ does not satisfy the result of Theorem 1.2.

Remark 1.5. The case $p = 0$ of Theorem 1.2 yields Theorem C.

It seems that we cannot get the result by the methods used in [4, 5]. In order to prove our theorem, we need the following result which is interesting in its own right.

Theorem 1.6. *Let* $Q_1(z) = a_1z^p + a_{1,p-1}z^{p-1} + \cdots + a_{1,0}$ *and* $Q_2(z) = a_2z^p + a_{2,p-1}z^{p-1} + \cdots + a_{2,0}$ *be two polynomials such that* $\deg Q_1(z) = \deg Q_2(z) = p$ *(where* p *is a nonnegative integer) and* a_1, $a_2(a_2 \neq 0)$ *are two distinct complex numbers. Let* $f(z)$ *be a nonconstant entire function, and* $f(z) = Q_1(z) \Rightarrow f'(z) = Q_1(z)$ *and* $f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z)$*, then* $f(z)$ *is of finite order.*

2. Some Lemmas

In order to prove our theorems, we need the following lemmas.

Let h be a meromorphic function in $\mathbb{C}$. h is called a normal function if there exists a positive M such that $h^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2} \tag{2.1}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on compact subsets of D; see [9].

Normal families, in particular, of holomorphic functions often appear in operator theory on spaces of analytic functions; for example, see in [10, Lemma 3] and in [11, Lemma 4].

Lemma 2.1 (see [12]). *Let $\mathcal{F}$ be a family of analytic functions in the unit disc Δ with the property that for each $f(z) \in \mathcal{F}$, all zeros of $f(z)$ have multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) \in \mathcal{F}$ and $f(z) = 0$. If $\mathcal{F}$ is not normal in Δ, then for $0 \leq \alpha \leq k$, there exist*

(1) *a number $r \in (0,1)$,*

(2) *a sequence of complex numbers z_n, $|z_n| < r$,*

(3) *a sequence of functions $f_n \in \mathcal{F}$, and*

(4) *a sequence of positive numbers $\rho_n \to 0$*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ converges locally and uniformly (with respect to the spherical metric) to a nonconstant analytic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

Lemma 2.2 (see [13]). *A normal meromorphic function has order at most two. A normal entire function is of exponential type and thus has order at most one.*

Lemma 2.3 (see [9, Marty's criterion]). *A family $\mathcal{F}$ of meromorphic functions on a domain D is normal if and only if, for each compact subset $K \subseteq D$, there exists a constant M such that $f^{\#}(z) \leq M$ for each $f \in \mathcal{F}$ and $z \in K$.*

Lemma 2.4 (see [2]). *Let $f(z)$ be a meromorphic function, and let $a_1(z)$, $a_2(z)$, $a_3(z)$ be three distinct meromorphic functions satisfying $T(r, a_i) = S(r, f)$, $i = 1, 2, 3$. Then*

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_3}\right) + S(r,f). \tag{2.2}$$

Lemma 2.5 (see [5]). *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D, and let a and b be two finite complex numbers such that $b \neq a, 0$. If for each $f \in \mathcal{F}$, $f(z) = a \Rightarrow f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $\mathcal{F}$ is normal in D.*

3. Proof of Theorem 1.6

If $Q_1 \equiv 0$, by $\deg Q_1 = \deg Q_2$, we obtain $p = 0$, $a_1 = 0$, $Q_2 \equiv a_2 (a_2 \neq 0)$. From the conditions of Theorem 1.6, we obtain $f(z) = 0 \Rightarrow f'(z) = 0$ and $f'(z) = a_2 \Rightarrow f(z) = a_2$. By Lemmas 2.5 and 2.3 we obtain that f is a normal function in D. By Lemma 2.2 we obtain that f is a finite order function.

If $Q_1 \not\equiv 0$, by $\deg Q_1 = \deg Q_2$ and $a_2 \neq 0$, we obtain $a_1 \neq 0$. Now we consider the function $F = f/Q_1 - 1$, and we distinguish two cases.

Case 1. If there exists a constant M such that $F^{\#}(z) \leq M$, by Lemmas 2.3 and 2.2, then F is of finite order. Hence $f = (F+1)Q_1$ is of finite order as well.

Case 2. If there does not exist a constant M such that $F^{\#}(z) \leq M$, then there exists a sequence $(w_n)_n$ such that $w_n \to \infty$ and $F^{\#}(w_n) \to \infty$ for $n \to \infty$.

Since Q_1 is a polynomial, there exists an r_1 such that

$$\left|\frac{Q_1'(z)}{Q_1(z)}\right| \leq \frac{2p}{|z|} \quad \forall z \in \mathbb{C} \text{ satisfying } |z| \geq r_1. \tag{3.1}$$

Obviously, if $z \to \infty$, then $2p/|z| \to 0$. Let $r > r_1$, and $D = \{z : |z| \geq r\}$, then F is analytic in D. Without loss of generality, we may assume $|w_n| \geq r+1$ for all n. We define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f(w_n + z)}{Q_1(w_n + z)} - 1. \tag{3.2}$$

Let $z \in D_1$ be fixed; from the above equality, if $F(w_n + z) = 0$, then $f(w_n + z) = Q_1(w_n + z)$. Noting that $f = Q_1 \Rightarrow f' = Q_1$, then we obtain the following: if $n \to \infty$, then

$$\begin{aligned} \left|F_n'(z)\right| &= \left|\left(\frac{f(w_n + z)}{Q_1(w_n + z)}\right)'\right| = \left|\frac{f'(w_n + z)}{Q_1(w_n + z)} - \frac{f(w_n + z)}{Q_1(w_n + z)}\frac{Q_1'(w_n + z)}{Q_1(w_n + z)}\right| \\ &\leq \left|\frac{f'(w_n + z)}{Q_1(w_n + z)}\right| + \left|\frac{f(w_n + z)}{Q_1(w_n + z)}\right|\left|\frac{Q_1'(w_n + z)}{Q_1(w_n + z)}\right| < 2. \end{aligned} \tag{3.3}$$

Obviously, $F_n(z)$ are analytic in D_1 and $F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. It follows from Lemma 2.3 that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1, with ($\alpha = k = 1$ and $A = 2$). Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$, such that $z_n \to 0$ and $\rho_n \to 0$ and such that the sequence $(g_n)_n$ defined by

$$g_n(\xi) = \rho_n^{-1} F_n(z_n + \rho_n\xi) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)} - 1\right\} \longrightarrow g(\xi) \tag{3.4}$$

converges locally and uniformly in $\mathbb{C}$ where $g(\xi)$ is a nonconstant analytic function and $g^{\#}(\xi) \leq g^{\#}(0) = A + 1 = 3$. By lemma 2.2, the order of $g(\xi)$ is at most 1.

First, we will prove that $g = 0 \Rightarrow g' = 1$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. Then by Hurwitz's theorem, there exist ξ_n, $\xi_n \to \xi_0$ as $n \to \infty$ such that for n sufficiently large

$$g_n(\xi_n) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\} = 0. \tag{3.5}$$

This implies $f(w_n + z_n + \rho_n\xi_n) = Q_1(w_n + z_n + \rho_n\xi_n)$. From the above, we obtain

$$g_n'(\xi) = \frac{f'(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)} - \frac{f(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)}\frac{Q_1'(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)}. \tag{3.6}$$

Let $G_n(\xi) = f'(w_n + z_n + \rho_n\xi)/Q_1(w_n + z_n + \rho_n\xi)$, by (3.1), (3.3) and (3.4), it is easy to obtain $\lim_{n\to\infty}G_n(\xi) = \lim_{n\to\infty}g_n'(\xi) = g'(\xi)$. Noting that $f = Q_1 \Rightarrow f' = Q_1$, we have

$$G_n(\xi_n) = \frac{f'(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} = 1 \quad (n \to \infty) \tag{3.7}$$

Thus

$$g'(\xi_0) = \lim_{n\to\infty} G_n(\xi_n) = 1. \tag{3.8}$$

This shows that $g = 0 \Rightarrow g' = 1$.

Next we will prove that $g'(\xi) \neq a_2/a_1$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g'(\xi_0) = a_2/a_1$. If $g'(\xi) \equiv a_2/a_1$, then $g(\xi) = a_2/a_1\xi + c$, where c is a constant, together with the fact that $g = 0 \Rightarrow g' = 1$ gives $a_2/a_1 = 1$, which contradicts to the assumptions. Thus $g'(\xi) \not\equiv a_2/a_1$. Since $G_n(\xi) - Q_2(w_n + z_n + \rho_n\xi)/Q_1(w_n + z_n + \rho_n\xi) \to g'(\xi) - a_2/a_1$ as $n \to \infty$ and $g'(\xi_0) = a_2/a_1$, by Hurwitz's theorem, there exist $\xi_n \to \xi_0$ as $n \to \infty$ such that for n sufficiently large

$$\begin{aligned} G_n(\xi_n) - \frac{Q_2(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} &= 0 \\ \Longrightarrow f'(w_n + z_n + \rho_n\xi_n) = Q_2(w_n + z_n + \rho_n\xi_n). \end{aligned} \tag{3.9}$$

Noting that $f' = Q_2 \Rightarrow f = Q_2$, from (3.4) and (3.9) (for n sufficiently large), we have

$$g_n(\xi_n) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\} = \rho_n^{-1}\left\{\frac{Q_2(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\}. \tag{3.10}$$

Since $a_2 \neq a_1 (a_1 \neq 0)$, $\deg Q_1 = \deg Q_2 = p$ and $\rho_n \to 0$, by (3.10), we get

$$g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n) = \infty, \tag{3.11}$$

which contradicts $g'(\xi_0) = a_2/a_1$. This shows that $g'(\xi) \neq a_2/a_1$ on $\mathbb{C}$.

Since g is of order at most one, so is g', it follows that

$$g'(\xi) = \frac{a_2}{a_1} + e^{b_0 + b_1\xi}, \tag{3.12}$$

where b_0, b_1 are two finite constants. We divide this case into two subcases.

Subcase 1. If $b_1 = 0$, from (3.12), we have

$$g(\xi) = \left(\frac{a_2}{a_1} + e^{b_0}\right)\xi + c_0, \tag{3.13}$$

where c_0 is a constant. Since $g = 0 \Rightarrow g' = 1$, from (3.13) we have $a_2/a_1 + e^{b_0} = 1$. By a simple calculation, we have $g^{\#}(0) = 1/(1 + |c_0|^2)$, which contradicts $g^{\#}(0) = 3$.

Subcase 2. If $b_1 \neq 0$, by

$$g'(\xi) = \frac{a_2}{a_1} + e^{b_0 + b_1\xi}, \tag{3.14}$$

we obtain

$$g(\xi) = \frac{a_2}{a_1}\xi + \frac{1}{b_1}e^{b_0 + b_1\xi} + B, \tag{3.15}$$

where B is a constant. Obviously, $g(\xi) = 0$ has infinitely many solutions. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. By (3.14), (3.15), and $g = 0 \Rightarrow g' = 1$, we get a unique $\xi_0 = (a_2 - a_1 - b_1 B a_1)/b_1 a_2$. Which is a contradiction.

Thus f is of finite order. This completes the proof of the theorem.

4. Proof of Theorem 1.2

Now we distinguish two cases.

Case 1. If $p = 0$, by $\deg Q_1 = \deg Q_2 = 0$, we deduce $Q_1 \equiv a_1$ and $Q_2 \equiv a_2 (a_2 \neq a_1, 0)$. By Theorem C, we obtain $f \equiv f'$.

Case 2. If $p \geq 1$, by $\deg Q_1 = \deg Q_2 = p$ and $a_2 \neq 0$, we deduce $a_1 \neq 0$. So Q_1 is a nonconstant polynomial. By Theorem 1.6, we know that f is of finite order. Thus, the hyperorder $\sigma_2(f) = 0$. Then, by Theorem E, we have

$$\lambda = \frac{f' - Q_1}{f - Q_1}, \tag{4.1}$$

where λ is a nonzero constant. We rewrite it as

$$f' = \lambda f + (1 - \lambda) Q_1. \tag{4.2}$$

If $\lambda = 1$, we obtain $f \equiv f'$.

Now, we assume that $\lambda \neq 1$. Solving (4.2), we obtain

$$f(z) = Ae^{\lambda z} + P(z), \tag{4.3}$$

where A is a nonzero constant, and $P(z)$ is a polynomial. Thus, we have

$$f'(z) = A\lambda e^{\lambda z} + P'(z). \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we get

$$(\lambda - 1)Q_1 - \left(\lambda P - P'\right) \equiv 0. \tag{4.5}$$

Next, we will prove that $P'(z) \equiv Q_2(z)$. Suppose that $P'(z) \not\equiv Q_2(z)$, by (4.4) we obtain

$$\overline{N}\left(r, \frac{1}{f'(z) - Q_2(z)}\right) = \overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right). \tag{4.6}$$

Since $f(z)$ is a transcendental entire function and $P'(z) - Q_2(z)$ is a polynomial, we deduce $T(r, P'(z) - Q_2(z)) = S(r, f)$. It is well known that 0 and ∞ are the Picard values of $e^{\lambda z}$. By Lemma 2.4, we obtain

$$T\left(r, \lambda A e^{\lambda z}\right) \le \overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) + S(r, f). \tag{4.7}$$

By the Nevanlinna First Fundamental Theorem, we immediately obtain

$$\overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) \le T\left(r, \lambda A e^{\lambda z}\right) + S(r, f). \tag{4.8}$$

If we combine (4.7) and (4.8), we obtain

$$\overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) = T\left(r, \lambda A e^{\lambda z}\right) + S(r, f) \ne S(r, f). \tag{4.9}$$

Since $P'(z) \not\equiv Q_2(z)$, we suppose z_0 is a zero of $f' - Q_2$. By the assumption $f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z)$, we have $f(z_0) = Q_2(z_0)$. Substituting z_0 into (4.3) and (4.4), we have

$$(\lambda - 1)Q_2(z_0) = \lambda P(z_0) - P'(z_0). \tag{4.10}$$

If $(\lambda - 1)Q_2 - (\lambda P - P') \not\equiv 0$, noting that $(\lambda - 1)Q_2 - (\lambda P - P')$ is a polynomial, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f' - Q_2}\right) &\le \overline{N}\left(r, \frac{1}{(\lambda - 1)Q_2 - (\lambda P - P')}\right) \\ &\le T\left(r, (\lambda - 1)Q_2 - (\lambda P - P')\right) = S(r, f), \end{aligned} \tag{4.11}$$

which contradicts with (4.9). Hence,

$$(\lambda - 1)Q_2 - \left(\lambda P - P'\right) \equiv 0. \tag{4.12}$$

Comparing the above equality to (4.5), we have $Q_1 \equiv Q_2$, a contradiction. Thus, we prove $P'(z) \equiv Q_2(z)$. It is easy to see $\deg Q_2 = \deg P'$. By (4.5) we obtain $\deg Q_1 = \deg P$. Finally we deduce $\deg Q_1 \ne \deg Q_2$. This is a contradiction. So $\lambda \ne 1$ is impossible. This completes the proof of Theorem 1.2.

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