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Optimal control for evolutionary imperfect transmission problems

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Abstract

We study the optimal control problem of a second order linear evolution equation defined in two-component composites with ε-periodic disconnected inclusions of size ε in presence of a jump of the solution on the interface that varies according to a parameter γ. In particular here the case $\gamma < 1$ is analyzed. The optimal control theory, introduced by Lions (Optimal Control of System Governed by Partial Differential Equations, 1971), leads us to characterize the control as the solution of a set of equations, called optimality conditions. The main result of this paper proves that the optimal control of the ε-problem, which is the unique minimum point of a quadratic cost functional J_ε, converges to the optimal control of the homogenized problem with respect to a suitable limit cost functional J_∞. The main difficulties are to find the appropriate limit functional for the control of the homogenized system and to identify the limit of the controls.

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1 Introduction

In this paper we study the optimal control of a linear hyperbolic problem with oscillating coefficients on a domain Ω of $\mathbb{R}^n$ made up of two components, a connected one $\Omega_{1\varepsilon}$ and a second one $\Omega_{2\varepsilon}$, which is the union of ε-periodic disconnected inclusions of size ε. On the interface $\Gamma^\varepsilon = \partial\Omega_{2\varepsilon}$ separating the two components, we prescribe the continuity of the conormal derivatives and a jump of the solution proportional to the conormal derivatives through a function of order ε^γ, meanwhile, a Dirichlet condition is imposed on the exterior boundary $\partial\Omega$ (see Figure 1). The order of magnitude of this parameter, with respect to the period ε, determines the influence of the contact barrier in the propagation properties of the medium. Indeed this problem models the wave propagation in a medium made up of two components with very different coefficients of propagation, which gives rise to a jump in the boundary condition on the interface. This interface condition is the mathematical interpretation of imperfect interface characterized by the discontinuity of the displacement (see [1–10] and references therein).

This work connects the corresponding homogenization and correctors results proved respectively in [5] and [6]. The first question of this paper deals with the existence of an optimal control of the ε-problem with respect to a quadratic functional. If such a control exists, the second and more interesting question is: does the optimal control of the ε-problem converge as $\varepsilon \to 0$ to the optimal control of the homogenized problem with

Figure 1 Ω_ε.
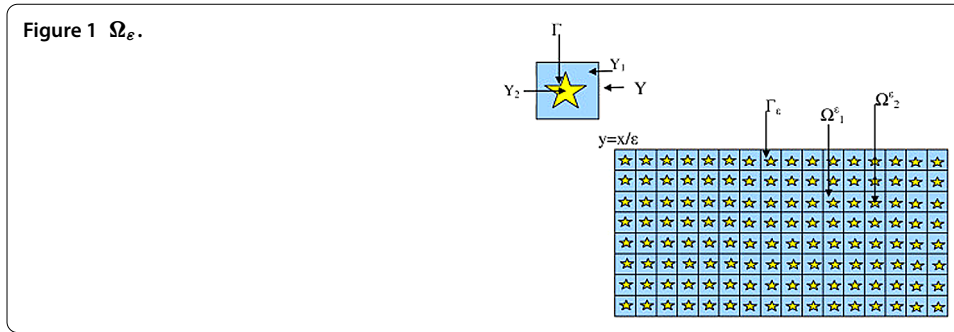
respect to a suitable cost functional? The optimal control of one or more aspects of a problem entails the minimization of a cost functional which describes physical quantities involved in the specific problem. To give a positive answer to both questions, we refer to the techniques used by Lions in [11]. These ones consist of the construction of the adjoint problem and the research of a set of equations, called optimality conditions, characterizing the optimal control and the related cost functional. As already shown by Hummel in [12], for the homogenization results in the elliptic case, one cannot expect to have boundedness of the solutions when $\gamma > 1$. Hence it would be natural here to suppose $\gamma \leq 1$. Nevertheless in this paper we analyze the case $\gamma < 1$ being the case $\gamma = 1$ more delicate. Indeed it is already known from previous studies (see [5]) that the asymptotic behavior of the ε-problem differs in terms of the homogenized problems in the two cases $\gamma < 1$ and $\gamma = 1$. The second one is the most complicated one, since the limit problem is a coupled system of a P.D.E. and a O.D.E. and gives rise to what is called a memory effect. When searching an optimal control result for the case $\gamma = 1$ we cannot adapt the same arguments used for the more general case $\gamma < 1$. In fact the homogenized problem is no more symmetric, hence the adjoint of the homogenized problem does not coincide with the limit of the adjoint problem at ε-level. We will use other techniques to study the case $\gamma = 1$.

The plan of the paper is as follows.

In Section 2 we recall some useful properties of a specific functional space, introduced in [9, 13] by Donato and Monsurrò in the elliptic framework, suitable for the solutions of this kind of interface problems. Successively, we recall some further properties, involving evolution triples, needed in the time-dependent framework. They have been proved by Donato *et al.* in [5].

In Section 3, we state the main result, Theorem 2.1, whose proof is performed into several steps. At first we describe the homogenization result for the hyperbolic problem; we refer the reader also to [5] where all the proofs can be found. Then we study the control problem. Our approach to the optimal control problem for a hyperbolic equation consists in applying Pontryagin maximum principle to obtain the expression for the optimal control w_ε at level ε in terms of the adjoint state p_ε, solution of the dual problem. We identify the problem satisfied by the limit (u, w) of the sequence of optimal pairs $\{(u_\varepsilon, w_\varepsilon)\}_\varepsilon$, where u_ε denotes the state of the system to be controlled, and also the problem satisfied by the limit p of $\{p_\varepsilon\}_\varepsilon$. We observe that p is the adjoint state corresponding to an optimal control problem for the limit equation and, by the maximum principle, we infer that the pair (u, w) is optimal. The indirect approach, using the adjoint problems and the maximum principle, is necessary since the convergences occur only in L^2-weak topology and not, as usual in homogenization theory, in L^2-strong topology. The crucial point is to obtain compactness

properties for the sequence of the optimal controls w_ε. Finally, we prove that the limit of the minimum points of the cost functional J_ε at level ε is the minimum point of an appropriate limit cost functional J_∞. Let us point out that the cost functional J_∞ describe the physical properties of the wave equation for a composite occupying the whole Ω, without any interface. Moreover, we observe that, without any technical difficulties, we can obtain the same optimal control result with a more general quadratic cost functional.

Optimal control problems and the exact controllability in domains with highly oscillating boundary are considered in [14–20]. Moreover, we refer to [21] for control of hyperbolic problems with oscillating coefficients in a fixed domain and to [22, 23] for control of hyperbolic problems in perforated domains.

In [7] and [8] the authors study, respectively, the approximate control and the correctors for a class of parabolic equations with interfacial contact resistance. For the sake of completeness, we recall also [4] where all the results for transmission problems in the elliptic case are collected and [24, 25] where the authors treated the homogenization in other types of perforated domains. For the study of similar problems, where the same jump condition is taken into account, we quote here also [1, 2, 12, 26–30] and the references therein.

2 Statement of the problem and main result

Let Ω be an open bounded subset of $\mathbb{R}^n$ ($n \geq 2$) and $Y =]0, l_1[\times \cdots \times]0, l_n[$ the reference cell.

We denote by Y_1 and Y_2 two nonempty open and disjoint subsets of Y such that

$$Y = Y_1 \cup \overline{Y_2},$$

with Y_1 connected and $\Gamma := \partial Y_2$ of class C^2. For any $k \in Z^n$ we define the translated sets Y_i^k and Γ_k as follows:

$$Y_i^k := k_l + Y_i, \qquad \Gamma_k := k_l + \Gamma,$$

where $k_l = (k_1 l_1, \ldots, k_n l_n)$ and $i = 1, 2$. Let $\{\varepsilon\}$ be a sequence of positive real numbers converging to zero and for any given ε let us set

$$K_\varepsilon := \left\{ k \in Z^n \mid \varepsilon \Gamma_k \cap \Omega \neq \emptyset \right\}.$$

Then we define the two components of Ω and the interface respectively as follows:

$$\Omega_{i\varepsilon} := \Omega \cap \left\{ \bigcup_{k \in K_\varepsilon} \varepsilon Y_i^k \right\}, \quad i = 1, 2, \quad \text{and} \quad \Gamma^\varepsilon := \partial \Omega_{2\varepsilon}.$$

We assume that

$$\partial \Omega \cap \left(\bigcup_{k \in Z^n} (\varepsilon \Gamma_k) \right) := \emptyset. \tag{2.1}$$

We explicitly observe that by construction, the set Ω is decomposed into two components $\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}}$ where $\Omega_{1\varepsilon}$ is a connected set while $\Omega_{2\varepsilon}$ is a disconnected union of ε-periodic

disjoint translated sets of εY_2. Moreover, Γ_ε is the interface separating the two components with $\partial \Omega \cup \Gamma_\varepsilon = \emptyset$ (see Figure 1).

In the sequel we denote by

- $\tilde{}$ the zero extension to the whole of Ω of functions defined on $\Omega_{1\varepsilon}$ or $\Omega_{2\varepsilon}$;
- χ_E the characteristic function of any measurable set $E \in \mathbb{R}^n$;
- $m_\omega(v) = \frac{1}{\omega} \int_\omega v \, dx$ the average on Y of any function $v \in L^1(\omega)$.

Let us recall (see for instance [31]) that as $\varepsilon \to 0$, for $i = 1, 2$,

$$\chi_{\Omega_{i\varepsilon}} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|} \quad \text{weakly in } L^2(\Omega) \tag{2.2}$$

θ_i being the proportion of the material occupying $\Omega_{i\varepsilon}$.

For any $\varepsilon > 0$, let us introduce he functional space V^ε as

$$V^\varepsilon := \left\{ v_1 \in H^1(\Omega_{1\varepsilon}) \mid v_1 = 0 \text{ on } \partial \Omega \right\},$$

which is a Banach space if endowed with the norm

$$\|v_1\|_{V^\varepsilon} := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}.$$

Clearly, since we do not assume any regularity on $\partial \Omega$, the condition on $\partial \Omega$ in the definition of V^ε has to be understood in a density sense. To be more precise, V^ε is the closure, with respect to the $H^1(\Omega_{1\varepsilon})$-norm, of the set of the functions in $C^\infty(\Omega_{1\varepsilon})$ with a compact support contained in Ω. This can be done in view of (2.1).

For any $\varepsilon > 0$, we set

$$W^\varepsilon := \left\{ v = (v_1, v_2) \in L^2\left(0, T; V^\varepsilon\right) \times L^2\left(0, T; H^1(\Omega_{2\varepsilon})\right) \mid \right.$$
$$\left. v' \in L^2\left(0, T; L^2(\Omega_{1\varepsilon})\right) \times L^2\left(0, T; L^2(\Omega_{2\varepsilon})\right) \right\}, \tag{2.3}$$

which is a Hilbert space if equipped with the norm

$$\|v\|_{W^\varepsilon} = \|v_1\|_{L^2(0,T;V^\varepsilon)} + \|v_2\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))} + \|v_1'\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} + \|v_2'\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))}.$$

Let A be a $n \times n$ Y-periodic matrix field with coefficients in $L^\infty(Y)$ such that for any $\lambda \in \mathbb{R}^n$ and a.e. in Y one has

$$\begin{cases} (A(x)\lambda, \lambda) \geq \alpha |\lambda|^2, \\ |A(x)\lambda| \leq \beta |\lambda|, \\ a_{i,j} = a_{j,i} \quad \text{for every } 1 \leq i, j \leq n, \end{cases} \tag{2.4}$$

with $0 < \alpha < \beta$.

Moreover, we suppose that h is a Y-periodic function such that

$$h \in L^\infty(\Gamma) \text{ and } \exists h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(y) \text{ a.e. in } \Gamma. \tag{2.5}$$

For any $\varepsilon > 0$, we set

$$h^\varepsilon(x) := h\left(\frac{x}{\varepsilon}\right) \tag{2.6}$$

and

$$A^\varepsilon(x) := A(x/\varepsilon). \tag{2.7}$$

The aim of this paper is to study the optimal control and its asymptotic behavior as $\varepsilon \to 0$ for a hyperbolic imperfect transmission problem defined in the domain Ω previously described.

More precisely let $z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon})$ be a control to be found in $L^2(0, T; L^2(\Omega_{1\varepsilon})) \times L^2(0, T; L^2(\Omega_{2\varepsilon}))$. For any fixed $T > 0$ let us consider the following problem:

$$\begin{cases} u_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} + z_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times]0, T[, \\ u_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} + z_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times]0, T[, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times]0, T[, \\ u_{1\varepsilon}(0) = U_{1\varepsilon}^0 & \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}(0) = U_{2\varepsilon}^0 & \text{in } \Omega_{2\varepsilon}, \\ u_{1\varepsilon}'(0) = U_{1\varepsilon}^1 & \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}'(0) = U_{2\varepsilon}^1 & \text{in } \Omega_{2\varepsilon}, \end{cases} \tag{2.8}$$

where $\gamma < 1$, $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, and

$$\begin{cases} \text{(i)} & f_\varepsilon = (f_{1\varepsilon}, f_{2\varepsilon}) \in L^2(0, T; L^2(\Omega_{1\varepsilon})) \times L^2(0, T; L^2(\Omega_{2\varepsilon})), \\ \text{(ii)} & U_\varepsilon^0 = (U_{1\varepsilon}^0, U_{2\varepsilon}^0) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \\ \text{(iii)} & U_\varepsilon^1 = (U_{1\varepsilon}^1, U_{2\varepsilon}^1) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}). \end{cases} \tag{2.9}$$

Let us introduce a class of function spaces expressly considered for the solution of this particular kind of interface problems. They were defined for the first time in [3, 32] and [12] in the framework of the study of the analogous elliptic problem. Clearly, the space of the solutions must take into account either the geometry of the domain in which the material is confined or the boundary and interfacial conditions. For every $\gamma \in \mathbb{R}$, we set (see also [9])

$$H_\gamma^\varepsilon := \left\{ v = (v_1, v_2) \mid v_1 \in V^\varepsilon \text{ and } v_2 \in H^1(\Omega_{2\varepsilon}) \right\}.$$

The space H_γ^ε is a Banach space when equipped with the norm

$$\|v\|_{H_\gamma^\varepsilon}^2 := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}^2 + \|\nabla v_2\|_{L^2(\Omega_{2\varepsilon})}^2 + \varepsilon^\gamma \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

It is easy to check that, if $0 < \varepsilon < 1$ and $\gamma_1 \le \gamma_2$, then

$$\|v\|_{H_{\gamma_2}^\varepsilon} \le \|v\|_{H_{\gamma_1}^\varepsilon}.$$

Moreover, for every fixed ε the norms of H_γ^ε and $V^\varepsilon \times H^1(\Omega_{2\varepsilon})$ are equivalent; see [13] for details.

We point out that H_γ^ε is a separable and reflexive Banach space dense in $L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$. Moreover, $H_\gamma^\varepsilon \subseteq L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ with continuous imbedding. On the other hand, one sees that $L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}) \subseteq (H_\gamma^\varepsilon)'$, with $L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ separable Hilbert

space. This means that the triple $(H_\gamma^\varepsilon, L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}), (H_\gamma^\varepsilon)')$ is an evolution triple. We refer the reader to [5] for an in-depth analysis on this aspect. By using an approach to the evolutionary problems based on evolution triples, as far as the weak formulation of problem (2.8) is concerned, we assume as precise formulation of formal problem the following one (see [18]):

$$
\begin{cases}
\text{find } u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\
\langle u_{1\varepsilon}'', v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u_{2\varepsilon}'', v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \\
\quad + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \cdot \nabla v_1 \, dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \cdot \nabla v_2 \, dx \\
\quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2) \, d\sigma_x = \int_{\Omega_{1\varepsilon}} (f_{1\varepsilon} + z_{1\varepsilon}) v_1 \, dx \\
\quad + \int_{\Omega_{2\varepsilon}} (f_{2\varepsilon} + z_{2\varepsilon}) v_2 \, dx \quad \forall (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}) \text{ in } \mathcal{D}'(0, T), \\
u_{1\varepsilon}(0) = U_{1\varepsilon}^0 \quad \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}(0) = U_{2\varepsilon}^0 \quad \text{in } \Omega_{2\varepsilon}, \\
u_{1\varepsilon}'(0) = U_{1\varepsilon}^1 \quad \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}'(0) = U_{2\varepsilon}^1 \quad \text{in } \Omega_{2\varepsilon}.
\end{cases}
\tag{2.10}
$$

An abstract Galerkin method provides the existence and uniqueness result for the solution of problem (2.10) for any $\varepsilon > 0$ and also some *a priori* estimates (see [5]). We point out that the unique solution $u_\varepsilon(z_\varepsilon)$ of problem (2.10) is said the 'state' of the system to be controlled and (2.10) are called 'state equations'.

To any control $z_\varepsilon \in L^2(0, T; L^2(\Omega_{1\varepsilon})) \times L^2(0, T; L^2(\Omega_{2\varepsilon}))$ we associate the cost functional $J_\varepsilon : L^2(0, T; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})) \to \mathbb{R}$ defined in the following way:

$$
J_\varepsilon(z_\varepsilon) := \frac{1}{2} \int_0^T \int_{\Omega_{1\varepsilon}} |u_{1\varepsilon}(z_\varepsilon)|^2 + \frac{1}{2} \int_0^T \int_{\Omega_{2\varepsilon}} |u_{2\varepsilon}(z_\varepsilon)|^2
$$

$$
+ \frac{1}{2} \int_0^T \int_{\Omega_{1\varepsilon}} |z_{1\varepsilon}|^2 + \frac{1}{2} \int_0^T \int_{\Omega_{2\varepsilon}} |z_{2\varepsilon}|^2.
\tag{2.11}
$$

This functional is continuous, strictly convex and coercive. Hence, by applying the direct method in the calculus of variations, the following minimum problem:

$$
\min \{ J_\varepsilon(z) : z \in L^2(0, T; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})) \}
\tag{2.12}
$$

admits a unique solution w_ε which is called the optimal control of problem (2.8), (2.9) with respect to the cost functional (2.11).

The aim of this paper is to study the asymptotic behavior, as $\varepsilon \to 0$ of the sequence of the optimal pairs $(u_\varepsilon, w_\varepsilon)$ under the following assumptions on the data:

$$
\begin{cases}
\text{(i)} \quad \widetilde{U_\varepsilon^0} \rightharpoonup U^0 := (U_1^0, U_2^0) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\
\qquad \text{with } U_2^0 \in H_0^1(\Omega), \\
\text{(ii)} \quad \widetilde{U_\varepsilon^1} \rightharpoonup U^1 := (U_1^1, U_2^1) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\
\text{(iii)} \quad \| U_\varepsilon^0 \|_{H_\gamma^\varepsilon} \leq C,
\end{cases}
\tag{2.13}
$$

with C positive constant independent of ε and

$$
\widetilde{f_\varepsilon} \rightharpoonup f := (f_1, f_2) \quad \text{weakly in } L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)).
\tag{2.14}
$$

Let A_γ^0 be the matrix defined as follows:

(i) for $\gamma < -1$

$$A_\gamma^0 \lambda = \frac{1}{|Y|} \int_{Y_1} A \nabla W_\gamma \, dy \tag{2.15}$$

with $W_\gamma \in H^1(Y_1)$ a solution, for any $\lambda \in \mathbb{R}^n$, of

$$\begin{cases} -\operatorname{div}(A \nabla W_\gamma) = 0 & \text{in } Y_1, \\ W_\gamma - \lambda \cdot y & Y\text{-periodic}, \\ \frac{1}{|Y_1|} \int_{Y_1} (W_\gamma - \lambda \cdot y) \, dy = 0; \end{cases} \tag{2.16}$$

(ii) for $\gamma = -1$

$$A_\gamma^0 \lambda = \frac{1}{|Y|} \int_{Y_1} A \left(\nabla \mathrm{w}^1 + \nabla \mathrm{w}^2 \right) dy \tag{2.17}$$

with $(\mathrm{w}^1, \mathrm{w}^2) \in H^1(Y_1) \times H^1(Y_2)$ a solution, for any $\lambda \in \mathbb{R}^n$, of

$$\begin{cases} -\operatorname{div}(A \nabla \mathrm{w}^1) = 0 & \text{in } Y_1, \\ -\operatorname{div}(A \nabla \mathrm{w}^2) = 0 & \text{in } Y_2, \\ (A \nabla \mathrm{w}^1) \cdot n_1 = -(A \nabla \mathrm{w}^2) \cdot n_2 & \text{on } \Gamma, \\ -A \nabla \mathrm{w}^1 \cdot n_1 = h(\mathrm{w}^1 - \mathrm{w}^2), \\ \mathrm{w}^1 - \lambda \cdot y & Y\text{-periodic}, \\ \frac{1}{|Y_1|} \int_{Y_1} (\mathrm{w}^1 - \lambda \cdot y) \, dy = 0; \end{cases} \tag{2.18}$$

(iii) for $-1 < \gamma < 1$

$$A_\gamma^0 \lambda = \frac{1}{|Y|} \int_{Y_1} A \nabla \mathrm{w} \, dy \tag{2.19}$$

with $\mathrm{w} \in H^1(Y_1)$ a solution, for any $\lambda \in \mathbb{R}^n$, of

$$\begin{cases} -\operatorname{div}(A \nabla \mathrm{w}) = 0 & \text{in } Y_1, \\ (A \nabla \mathrm{w}) \cdot n_1 = 0 & \text{on } \Gamma, \\ \mathrm{w} - \lambda \cdot y & Y\text{-periodic}, \\ \frac{1}{|Y_1|} \int_{Y_1} (\mathrm{w} - \lambda \cdot y) \, dy = 0. \end{cases} \tag{2.20}$$

We establish the following result.

Theorem 2.1 *Let $\gamma < 1$ and let A_ε and h_ε satisfy* (2.4)-(2.7). *Suppose that* (2.9), (2.13), *and* (2.14) *hold and let w_ε the optimal control of problem* (2.8), (2.9), (2.11), *and* (2.12). *Then there exists a function $w \in (L^2(0, T; L^2(\Omega)))^2$ such that*

$$\widetilde{w}_\varepsilon \rightharpoonup w = (w_1, w_2) \quad \text{in } \left(L^2 \left(0, T; L^2(\Omega) \right) \right)^2, \tag{2.21}$$

where

$$w_2 = \frac{\theta_2}{\theta_1} w_1 \tag{2.22}$$

and $\frac{w_1}{\theta_1}$ is the unique solution of the following problem:

$$\min\left\{\frac{1}{2}\int_0^T\int_\Omega |u_1(z)|^2 + \frac{1}{2}\int_0^T\int_\Omega |z|^2 : z \in L^2\big(0, T; L^2(\Omega)\big)\right\}, \tag{2.23}$$

$u_1(z)$ being, for every $z \in L^2(0, T; L^2(\Omega))$, the unique solution of

$$\begin{cases} u_1'' - \operatorname{div}(A_\gamma^0 \nabla u_1) = f_1 + f_2 + z & in \ \Omega \times \,]0, T[, \\ u_1 = 0 & on \ \partial\Omega \times \,]0, T[, \\ u_1(0) = U_1^0 + U_2^0 & in \ \Omega, \\ u_1'(0) = U_1^1 + U_2^1 & in \ \Omega, \end{cases} \tag{2.24}$$

where the homogenized matrix A_γ^0 is defined in (2.15), (2.17) and (2.19).

Remark 2.1 For $\gamma < -1$ the matrix field A_γ^0 is the classical one in a fixed domain. For $\gamma = -1$, the homogenized matrix A_γ^0 is described in terms of the periodic solution of an elliptic problem posed in the two reference sub-domains of the periodicity cell and prescribing on the interface a conormal derivative proportional to the jump of the solution. Finally in the case $-1 < \gamma < 1$, the matrix field A_γ^0 is the same obtained by Cioranescu and Saint Jean Paulin in [33], for the homogenization of the elliptic problem in the perforated domain $\Omega_{1\varepsilon}$ with a Neumann condition on the boundary of the holes.

Remark 2.2 The boundness (iii) in (2.13) is necessary in order to have *a priori* estimates for the solution of problem (2.8) and (2.9), as shown in Section 3 below.

Remark 2.3 If $f_{i\varepsilon} = f|_{\Omega_{\varepsilon i}}$, with $f \in L^2(\Omega)$, for $i = 1, 2$, then (2.14) holds with $f_i = \theta_i f$.

3 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. At first, we recall some convergence results about the sequence of solutions of problem (2.8) and (2.9). Then we give a characterization of the optimal control w_ε at ε-level in the form of the optimality system and we deduce a uniform estimate for w_ε. Finally we identify the limit of w_ε as the solution of the optimality system related to the homogenized problem with respect to a suitable limit cost functional.

3.1 Asymptotic behavior of the ε-problem

Let A_γ^0 the matrix field defined in the previous section. We will make use of some homogenization results proved in [5] that we recall below, for the reader's convenience.

For any fixed $T > 0$ and $\gamma < 1$ let us consider the following problem:

$$\begin{cases} u_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = g_{1\varepsilon} & in \ \Omega_{1\varepsilon} \times \,]0, T[, \\ u_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = g_{2\varepsilon} & in \ \Omega_{2\varepsilon} \times \,]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & on \ \Gamma^\varepsilon \times \,]0, T[, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & on \ \Gamma^\varepsilon \times \,]0, T[, \\ u_{1\varepsilon} = 0 & on \ \partial\Omega \times \,]0, T[, \\ u_{1\varepsilon}(0) = U_{1\varepsilon}^0 & in \ \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}(0) = U_{2\varepsilon}^0 & in \ \Omega_{2\varepsilon}, \\ u_{1\varepsilon}'(0) = U_{1\varepsilon}^1 & in \ \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}'(0) = U_{2\varepsilon}^1 & in \ \Omega_{2\varepsilon}, \end{cases} \tag{3.1}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, and

$$
\begin{cases}
\text{(i)} & g_\varepsilon = (g_{1\varepsilon}, g_{2\varepsilon}) \in L^2(0, T; L^2(\Omega_{1\varepsilon})) \times L^2(0, T; L^2(\Omega_{2\varepsilon})), \\
\text{(ii)} & U_\varepsilon^0 = (U_{1\varepsilon}^0, U_{2\varepsilon}^0) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \\
\text{(iii)} & U_\varepsilon^1 = (U_{1\varepsilon}^1, U_{2\varepsilon}^1) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}).
\end{cases}
\tag{3.2}
$$

Moreover, let us suppose that

$$
\begin{cases}
\text{(i)} & \widetilde{U_\varepsilon^0} \rightharpoonup U^0 := (U_1^0, U_2^0) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\
& \text{with } U_2^0 \in H_0^1(\Omega), \\
\text{(ii)} & \widetilde{U_\varepsilon^1} \rightharpoonup U^1 := (U_1^1, U_2^1) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\
\text{(iii)} & \|U_\varepsilon^0\|_{H_\gamma^\varepsilon} \le C,
\end{cases}
\tag{3.3}
$$

with C positive constant independent of ε, and

$$
\widetilde{g_\varepsilon} \rightharpoonup g := (g_1, g_2) \quad \text{weakly in } L^2\big(0, T; L^2(\Omega)\big) \times L^2\big(0, T; L^2(\Omega)\big).
\tag{3.4}
$$

Theorem 3.1 ([5]) *Let A_ε and h_ε satisfy* (2.4)-(2.7). *Suppose that* (3.2), (3.3), *and* (3.4) *hold and let u_ε be the solution of problem* (3.1) *and* (3.2), *with $\gamma < 1$. Then there exists an extension operator $P_1^\varepsilon \in \mathcal{L}(L^\infty(0, T; H^k(\Omega_{1\varepsilon})); L^\infty(0, T; H^k(\Omega)))$, for $k = 1, 2$, such that*

$$
\begin{cases}
\text{(i)} & P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 \quad \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)), \\
\text{(ii)} & \widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1 \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\
\text{(iii)} & \widetilde{u_{2\varepsilon}} \rightharpoonup \theta_2 u_1 \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)),
\end{cases}
\tag{3.5}
$$

$$
\begin{cases}
\text{(i)} & P_1^\varepsilon u_{1\varepsilon}' \rightharpoonup u_1' \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\
\text{(ii)} & \widetilde{u_{1\varepsilon}'} \rightharpoonup \theta_1 u_1' \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\
\text{(ii)} & \widetilde{u_{2\varepsilon}'} \rightharpoonup \theta_2 u_1' \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)),
\end{cases}
\tag{3.6}
$$

and

$$
A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} + A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 \quad \text{weakly* in } L^\infty\big(0, T; [L^2(\Omega)]^n\big),
\tag{3.7}
$$

where θ_1 and θ_2 are given by (2.2) *and u_1 is the unique solution in $L^2(0, T; H_0^1(\Omega))$, with u_1' in $L^2(0, T; L^2(\Omega))$, of the following problem:*

$$
\begin{cases}
u_1'' - \operatorname{div}(A_\gamma^0 \nabla u_1) = g_1 + g_2 & \text{in } \Omega \times]0, T[, \\
u_1 = 0 & \text{on } \partial\Omega \times]0, T[, \\
u_1(0) = U_1^0 + U_2^0 & \text{in } \Omega, \\
u_1'(0) = U_1^1 + U_2^1 & \text{in } \Omega.
\end{cases}
\tag{3.8}
$$

Moreover, if $-1 < \gamma < 1$,

$$
\begin{cases}
\text{(i)} & A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A_\gamma^0 \nabla u_1 \quad \text{weakly* in } L^\infty(0, T; [L^2(\Omega)]^n), \\
\text{(ii)} & A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup 0 \quad \text{weakly* in } L^\infty(0, T; [L^2(\Omega)]^n).
\end{cases}
\tag{3.9}
$$

Let us recall the existence and uniqueness result, of the solution of problem (3.1) and (3.2), with $\gamma < 1$ (see for instance [34], Theorem 24.A and [35], Chapter 3, Theorem 8.1,

Remark 8.2). We point out that $H_\gamma^\varepsilon \subseteq L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$ with continuous imbedding so that the triple $(H_\gamma^\varepsilon, L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}), H_\gamma^{\varepsilon\prime})$ is an evolution triple (see Theorem 3.1 in [5] for an in-depth analysis on this aspect).

Theorem 3.2 *Let* $T \in]0, +\infty[$. *Let* W_ε *be defined as in* (2.3), h_ε *and* A_ε *as in* (2.4)-(2.7). *For every* ε, *under assumptions* (3.2), (3.3), *and* (3.4), *problem* (3.1) *admits a unique weak solution* $u_\varepsilon \in W_\varepsilon$. *Moreover, there exists a constant C, independent of* ε, *such that*

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^2(0,T H_\gamma^\varepsilon(\Omega_\varepsilon))} + \|u_\varepsilon'\|_{L^2(0,T L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \\
&\qquad \leq C\big(\|U_\varepsilon^0\|_{H_\gamma^\varepsilon} + \|U_\varepsilon^1\|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{1\varepsilon})} + \|g_\varepsilon\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \big).
\end{aligned}
\tag{3.10}
$$

Let us point out that, for any fixed ε, the solution of problem (3.1) and (3.2) has some further properties (see [35], Chapter 3, Theorem 8.2). In fact, under the same hypotheses of Theorem 3.2 the unique solution u_ε of problem (3.1) and (3.2), with $\gamma < 1$ satisfies

$$
u_\varepsilon \in C\big([0,T]; H_\gamma^\varepsilon\big), \qquad u_\varepsilon' \in C\big([0,T]; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})\big)
\tag{3.11}
$$

and

$$
\begin{aligned}
&\|u_\varepsilon\|_{L^\infty(0,T H_\gamma^\varepsilon(\Omega_\varepsilon))} + \|u_\varepsilon'\|_{L^\infty(0,T L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \\
&\qquad \leq C\big(\|U_\varepsilon^0\|_{H_\gamma^\varepsilon} + \|U_\varepsilon^1\|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{1\varepsilon})} + \|g_\varepsilon\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \big),
\end{aligned}
\tag{3.12}
$$

where C is the same constant as in (3.10).

3.2 The optimality system

The following results give a characterization of the optimal controls for both problem at level ε and homogenized problem (2.24) (see [11], Chapter 4).

Theorem 3.3 *For every* ε, *under assumptions* (2.4)-(2.7) *and* (2.9), *the optimal pair* $(u_\varepsilon, w_\varepsilon)$, *solution of problem* (2.8), (2.9), (2.11), *and* (2.12) *is characterized by the following optimality system*:

$$
\begin{cases}
u_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} + w_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times]0,T[, \\
u_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} + w_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times]0,T[, \\
A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times]0,T[, \\
A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times]0,T[, \\
u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times]0,T[, \\
u_{1\varepsilon}(0) = U_{1\varepsilon}^0 \quad \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}(0) = U_{2\varepsilon}^0 \quad \text{in } \Omega_{2\varepsilon}, \\
u_{1\varepsilon}'(0) = U_{1\varepsilon}^1 \quad \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}'(0) = U_{2\varepsilon}^1 \quad \text{in } \Omega_{2\varepsilon},
\end{cases}
\tag{3.13}
$$

$$
\begin{cases}
p_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla p_{1\varepsilon}) = u_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times]0,T[, \\
p_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla p_{2\varepsilon}) = u_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times]0,T[, \\
A^\varepsilon \nabla p_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla p_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times]0,T[, \\
A^\varepsilon \nabla p_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(p_{1\varepsilon} - p_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times]0,T[, \\
p_{1\varepsilon} = 0 & \text{on } \partial\Omega \times]0,T[, \\
p_{1\varepsilon}(T) = 0 \quad \text{in } \Omega_{1\varepsilon}, \qquad p_{2\varepsilon}(T) = 0 \quad \text{in } \Omega_{2\varepsilon}, \\
p_{1\varepsilon}'(T) = 0 \quad \text{in } \Omega_{1\varepsilon}, \qquad p_{2\varepsilon}'(T) = 0 \quad \text{in } \Omega_{2\varepsilon},
\end{cases}
\tag{3.14}
$$

$$p_\varepsilon = -w_\varepsilon \quad a.e.\ in\ ]0, T[\times \Omega_\varepsilon. \tag{3.15}$$

As previously, we prefer to use the following weak formulation of problems (3.13) and (3.14):

$$\begin{cases}
\text{find } u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\
\langle u''_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u''_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \\
\quad + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \cdot \nabla v_1 \, dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \cdot \nabla v_2 \, dx \\
\quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2) \, d\sigma_x = \int_{\Omega_{1\varepsilon}} (f_{1\varepsilon} + w_{1\varepsilon}) v_1 dx \\
\quad + \int_{\Omega_{2\varepsilon}} (f_{2\varepsilon} + w_{2\varepsilon}) v_2 \, dx \quad \text{for every } (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}) \text{ in } \mathcal{D}'(0, T), \\
u_{1\varepsilon}(0) = U^0_{1\varepsilon} \quad \text{in } \Omega_{1\varepsilon}, \qquad u_{2\varepsilon}(0) = U^0_{2\varepsilon} \quad \text{in } \Omega_{2\varepsilon}, \\
u'_{1\varepsilon}(0) = U^1_{1\varepsilon} \quad \text{in } \Omega_{1\varepsilon}, \qquad u'_{2\varepsilon}(0) = U^1_{2\varepsilon} \quad \text{in } \Omega_{2\varepsilon},
\end{cases} \tag{3.16}$$

and

$$\begin{cases}
\text{find } p_\varepsilon = (p_{1\varepsilon}, p_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\
\langle p''_{1\varepsilon}, v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle p''_{2\varepsilon}, v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \\
\quad + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla p_{1\varepsilon} \cdot \nabla v_1 \, dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla p_{2\varepsilon} \cdot \nabla v_2 \, dx \\
\quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (p_{1\varepsilon} - p_{2\varepsilon})(v_1 - v_2) \, d\sigma_x = \int_{\Omega_{1\varepsilon}} u_{1\varepsilon} v_1 \, dx \, dt \\
\quad + \int_{\Omega_{2\varepsilon}} u_{2\varepsilon} v_2 \, dx \, dt \quad \text{for every } (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}) \text{ in } \mathcal{D}'(0, T), \\
p_{1\varepsilon}(T) = 0 \quad \text{in } \Omega_{1\varepsilon}, \qquad p_{2\varepsilon}(T) = 0 \quad \text{in } \Omega_{2\varepsilon}, \\
p'_{1\varepsilon}(T) = 0 \quad \text{in } \Omega_{1\varepsilon}, \qquad p'_{2\varepsilon}(T) = 0 \quad \text{in } \Omega_{2\varepsilon}.
\end{cases} \tag{3.17}$$

Let us consider the cost functional $J_\infty : L^2(0, T; L^2(\Omega)) \to \mathbb{R}$ defined in the following way:

$$J_\infty(z) := \frac{1}{2} \int_0^T \int_\Omega |u_1(z)|^2 + \frac{1}{2} \int_0^T \int_\Omega |z|^2, \tag{3.18}$$

where for every control $z \in L^2(0, T; L^2(\Omega))$, $u_1(z)$ is the unique solution of problem (2.24). This functional is continuous, strictly convex, and coercive. Hence, by applying the direct method in the calculus of variations, the minimum problem (2.23) admits a unique solution $\bar{w}$ which is the optimal control of problem (2.24) with respect to the cost functional (3.18).

Theorem 3.4 *The optimal pair* $(u_1, \bar{w})$, *solution of problem* (2.23) *and* (2.24) *is characterized by the following optimality system*:

$$\begin{cases}
u''_1 - \operatorname{div}(A^0_\gamma \nabla u_1) = f_1 + f_2 + \bar{w} \quad in\ \Omega \times]0, T[, \\
u_1 = 0 \quad on\ \partial\Omega \times]0, T[, \\
u_1(0) = U^0_1 + U^0_2 \quad in\ \Omega, \\
u'_1(0) = U^1_1 + U^1_2 \quad in\ \Omega,
\end{cases} \tag{3.19}$$

$$\begin{cases}
p''_1 - \operatorname{div}(A^0_\gamma \nabla p_1) = u_1 \quad in\ \Omega \times]0, T[, \\
u_1 = 0 \quad on\ \partial\Omega \times]0, T[, \\
p_1(T) = 0 \quad in\ \Omega, \\
p'_1(T) = 0 \quad in\ \Omega,
\end{cases} \tag{3.20}$$

$$p_1 = -\bar{w}. \tag{3.21}$$

3.3 *A priori* estimates

In this subsection, we deduce some *a priori* norm-estimates either for the sequence of the optimal controls w_ε or for the corresponding solution $u_\varepsilon = u_\varepsilon(w_\varepsilon)$ (resp. $p_\varepsilon = p_\varepsilon(u_\varepsilon)$) of problem (3.16) (resp. (3.17)).

Proposition 3.1 *Let* $(u_\varepsilon, w_\varepsilon) \in (L^2(0, T; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})))^2$ *be the optimal pair, solution of the optimality system* (3.13), (3.14) *and* (3.15). *Under assumptions* (2.4)-(2.7), (2.13) *and* (2.14), *there exists a constant c, independent of ε, such that*

$$\|w_\varepsilon\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \leq c, \tag{3.22}$$

$$\|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \leq c, \tag{3.23}$$

for every ε.

Proof Let us fix ε. Let $u_\varepsilon = u_\varepsilon(w_\varepsilon)$ be the unique solution of problem (3.16) and $p_\varepsilon = p_\varepsilon(u_\varepsilon)$ be the unique solution of the adjoint problem (3.17). Choosing p_ε as test function in (3.16) and u_ε as a test function in (3.17), we have

$$\int_0^T \langle u_{1\varepsilon}''(t,\cdot), p_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt + \int_0^T \langle u_{2\varepsilon}''(t,\cdot), p_{2\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \, dt$$

$$+ \int_0^T \int_{\Omega_{1\varepsilon}} A\nabla_x u_{1\varepsilon} \nabla_x p_{1\varepsilon} + u_{1\varepsilon} p_{1\varepsilon} \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} A\nabla_x u_{2\varepsilon} \nabla_x p_{2\varepsilon} + u_{2\varepsilon} p_{2\varepsilon} \, dx \, dt$$

$$= \int_0^T \int_{\Omega_{1\varepsilon}} f_\varepsilon p_{1\varepsilon} + w_{1\varepsilon} p_{1\varepsilon} \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} f_\varepsilon p_{2\varepsilon} + w_{2\varepsilon} p_{2\varepsilon} \, dx \, dt \tag{3.24}$$

and

$$\int_0^T \langle p_{1\varepsilon}''(t,\cdot), u_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt + \int_0^T \langle p_{2\varepsilon}''(t,\cdot), u_{2\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \, dt$$

$$+ \int_0^T \int_{\Omega_{1\varepsilon}} A\nabla_x p_{1\varepsilon} \nabla_x u_{1\varepsilon} + p_{1\varepsilon} u_{1\varepsilon} \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} A\nabla_x p_{2\varepsilon} \nabla_x u_{2\varepsilon} + p_{2\varepsilon} u_{2\varepsilon} \, dx \, dt$$

$$= \int_0^T \int_{\Omega_{1\varepsilon}} (u_{1\varepsilon})^2 \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} (u_{2\varepsilon})^2 \, dx \, dt. \tag{3.25}$$

Integrating by parts, we get

$$\int_0^T \langle u_{1\varepsilon}''(t,\cdot), p_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt + \int_0^T \langle u_{1\varepsilon}''(t,\cdot), p_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt$$

$$= \langle u_{1\varepsilon}'(T), p_{1\varepsilon}(T) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} - \langle u_{1\varepsilon}'(0), p_{1\varepsilon}(0) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})}$$

$$- \int_0^T (u_{1\varepsilon}'(t,\cdot), p_{1\varepsilon}'(t,\cdot))_{L^2(\Omega_{1\varepsilon})} \, dt + \langle u_{2\varepsilon}'(T), p_{2\varepsilon}(T) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}$$

$$- \langle u_{2\varepsilon}'(0), p_{2\varepsilon}(0) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} - \int_0^T (u_{2\varepsilon}'(t,\cdot), p_{2\varepsilon}'(t,\cdot))_{L^2(\Omega_{2\varepsilon})} \, dt \tag{3.26}$$

and

$$
\int_0^T \langle p_{1\varepsilon}''(t,\cdot), u_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt + \int_0^T \langle p_{1\varepsilon}''(t,\cdot), u_{1\varepsilon}(t,\cdot) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} \, dt
$$

$$
= \langle p_{1\varepsilon}'(T), u_{1\varepsilon}(T) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} - \langle p_{1\varepsilon}'(0), u_{1\varepsilon}(0) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})}
$$

$$
- \int_0^T \left(p_{1\varepsilon}'(t,\cdot), u_{1\varepsilon}'(t,\cdot) \right)_{L^2(\Omega_{1\varepsilon})} \, dt + \langle p_{2\varepsilon}'(T), u_{2\varepsilon}(T) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}
$$

$$
- \langle p_{2\varepsilon}'(0), u_{2\varepsilon}(0) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} - \int_0^T \left(p_{2\varepsilon}'(t,\cdot), u_{2\varepsilon}'(t,\cdot) \right)_{L^2(\Omega_{2\varepsilon})} \, dt. \tag{3.27}
$$

Subtracting (3.24) from (3.25), using (3.26) and (3.27), the symmetry of the matrix A, the initial conditions in (3.16), and the final conditions in (3.17), we obtain

$$
\langle p_{1\varepsilon}'(0), U_{1\varepsilon}^0 \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})} - \langle U_{1\varepsilon}^1, p_{1\varepsilon}(0) \rangle_{(H^1(\Omega_{1\varepsilon}))',H^1(\Omega_{1\varepsilon})}
$$

$$
+ \langle p_{2\varepsilon}'(0), U_{2\varepsilon}^0 \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} - \langle U_{2\varepsilon}^1, p_{2\varepsilon}(0) \rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}
$$

$$
= \int_0^T \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} p_{1\varepsilon} + w_{1\varepsilon} p_{1\varepsilon} - u_{1\varepsilon}^2 \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} p_{2\varepsilon} + w_{2\varepsilon} p_{2\varepsilon} - u_{2\varepsilon}^2 \, dx \, dt.
$$

By virtue of (3.15), as a result we find

$$
\int_0^T \int_{\Omega_{1\varepsilon}} u_{1\varepsilon}^2 + w_{1\varepsilon}^2 \, dx \, dt + \int_0^T \int_{\Omega_{2\varepsilon}} u_{2\varepsilon}^2 + w_{2\varepsilon}^2 \, dx \, dt
$$

$$
= - \int_0^T \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} w_{1\varepsilon} \, dx \, dt - \int_0^T \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} w_{2\varepsilon} \, dx \, dt
$$

$$
+ \int_{\Omega_{1\varepsilon}} U_{1\varepsilon}^1 p_{1\varepsilon}(0) - U_{1\varepsilon}^0 p_{1\varepsilon}'(0) \, dx + \int_{\Omega_{2\varepsilon}} U_{2\varepsilon}^1 p_{2\varepsilon}(0) - U_{2\varepsilon}^0 p_{2\varepsilon}'(0) \, dx, \tag{3.28}
$$

from which, by the Cauchy-Schwarz inequality, it follows that

$$
\| u_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}^2 + \| w_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}^2
$$

$$
\leq \| w_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \| f_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}
$$

$$
+ \| U_\varepsilon^0 \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})} \| p_\varepsilon'(0,\cdot) \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})}
$$

$$
+ \| U_\varepsilon^1 \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})} \| p_\varepsilon(0,\cdot) \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})}. \tag{3.29}
$$

On the other hand, to estimate $\| p_\varepsilon(0,\cdot) \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})}$ and $\| p_\varepsilon'(0,\cdot) \|_{L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})}$ let us apply Theorem 3.2 with p_ε instead of u_ε. Then by translation we get

$$
\| p_\varepsilon(0,\cdot) \|_{H^1(\Omega_\varepsilon)} + \| p_\varepsilon'(0,\cdot) \|_{L^2(\Omega_\varepsilon)} \leq C \| u_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}, \tag{3.30}
$$

where C is a constant independent of ε. Finally, combining (3.29) with (3.30) and using (2.13), as a result we find that

$$
\| u_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}^2 + \| w_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}^2
$$

$$
\leq c_3 \| w_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} + c_4 \| u_\varepsilon \|_{L^2(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))}, \tag{3.31}
$$

with constants independent of ε. So estimates (3.22) and (3.23) are proved. $\qquad\square$

As a consequence of (3.22), up to a subsequence still denoted by ε, we have the following convergences:

$$
\begin{aligned}
\widetilde{w_{1\varepsilon}} &\rightharpoonup w_1 \quad \text{in } L^2\big(0,T;L^2(\Omega)\big), \\
\widetilde{w_{2\varepsilon}} &\rightharpoonup w_2 \quad \text{in } L^2\big(0,T;L^2(\Omega)\big).
\end{aligned}
\tag{3.32}
$$

3.4 Conclusions

Let us recall that for the adjoint state at ε-level $p_\varepsilon = (p_{1\varepsilon}, p_{2\varepsilon})$, the following convergences hold:

$$
\begin{aligned}
\widetilde{p_{1\varepsilon}} &\rightharpoonup \theta_1 p_1 \quad \text{weakly* in } L^\infty\big(0,T;L^2(\Omega)\big), \\
\widetilde{p_{2\varepsilon}} &\rightharpoonup \theta_2 p_1 \quad \text{weakly* in } L^\infty\big(0,T;L^2(\Omega)\big).
\end{aligned}
$$

Hence by (3.15) and (3.32) we get

$$
\begin{aligned}
\theta_1 p_1 &= -w_1, \\
\theta_2 p_1 &= -w_2,
\end{aligned}
$$

where θ_1 and θ_2 are given in (2.2). As a consequence

$$
w_2 = \frac{\theta_2}{\theta_1} w_1,
\tag{3.33}
$$

which is (2.22).

Hence we are able to pass to the limit, as ε goes to 0, in the optimality system (3.13)-(3.15) by applying Theorem 3.1 to both problems (3.13) and (3.14) with, respectively, $g_\varepsilon = f_\varepsilon + w_\varepsilon$ and $g_\varepsilon = u_\varepsilon$. Then by (2.14), (3.32), and (3.33) as $\theta_1 + \theta_2 = 1$, we see that the pair $(u_1, \frac{w_1}{\theta_1})$ is such that

$$
\begin{cases}
u_1'' - \operatorname{div}(A_\gamma^0 \nabla u_1) = f_1 + f_2 + \frac{1}{\theta_1} w_1 & \text{in } \Omega \times\,]0,T[, \\
u_1 = 0 \quad \text{on } \partial\Omega \times\,]0,T[, \\
u_1(0) = U_1^0 + U_2^0 \quad \text{in } \Omega, \\
u_1'(0) = U_1^1 + U_2^1 \quad \text{in } \Omega,
\end{cases}
\tag{3.34}
$$

$$
\begin{cases}
p_1'' - \operatorname{div}(A_\gamma^0 \nabla p_1) = \theta_1 u_1 + \theta_2 u_1 = u_1 & \text{in } \Omega \times\,]0,T[, \\
u_1 = 0 \quad \text{on } \partial\Omega \times\,]0,T[, \\
p_1(T) = 0 \quad \text{in } \Omega, \\
p_1'(T) = 0 \quad \text{in } \Omega,
\end{cases}
\tag{3.35}
$$

$$
p_1 = -\frac{1}{\theta_1} w_1.
\tag{3.36}
$$

Finally, by Theorem 3.4 and uniqueness we get $\frac{w_1}{\theta_1} = \bar{w}$ and the convergences (3.5)-(3.7) and (3.32) hold for the whole sequence. Hence Theorem 2.1 is now completely proved.

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