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Some equalities and inequalities for fusion frames

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Abstract

Fusion frames have some properties similar to those of frames in Hilbert spaces, but not all of their properties are similar. Some authors have established some equalities and inequalities for conventional frames. In this paper, we give some equalities and inequalities for fusion frames. Our results generalize and improve the remarkable results which have been obtained by Balan, Casazza and Găvruta etc.

Keywords: Fusion frame, Equality, Frame operator

Background

Frames, which generalize the concept of bases, can take on infinitely many different representations for a given vector (Christensen 2008). Duffin and Schaeffer (1952) introduced the concept of frame to study some deep problems in nonharmonic Fourier series. After the fundamental paper by Daubechies et al. (1986), frame was popularized from then on. Now, frames are useful in some areas such as the signal and image processing, neural networks, data compression and sampling theory, among others. For signal processing frames can provide resilience to additive noise (Daubechies 1992), resilience to quantization (Goyal et al. 1998), numerical stability of reconstruction (Daubechies 1992), and greater freedom to capture signal characteristic (Benedetto and Colella 1995; Benedetto and Pfander 1998; Unser 1995).

Later on, being the generalization of the frames, fusion frames were introduced by Casazza and Kutyniok (2004) and Fornasier (2002) to handle some large systems which are impossible to handle effectively by just a simple frame. The essence of fusion frame is the construction of global frames from local frames in Hilbert space. So the characteristic fusion frame is special suiting for application such as distributing sensing, parallel processing and packet encoding, and so on. Now, many excellent results of conventional frames have been achieved and applied successfully, which properties of the conventional frames may be extended to the fusion frames? It is a tempting subject because of the complexity of the structure of fusion frames compared with conventional frames.

In this paper, we mainly study the equalities and inequalities of fusion frames. On some equalities for conventional frames were first found by Balan et al. (2007) when the authors studied the optimal decomposition of a Parseval frame. Later on, many authors such as Găvruţa (2006) and Zhu and Wu (2010) developed or improved some equalities

or inequalities of the conventional frames on the basis of the work in originally in Balan et al. (2007).

Preliminaries

First we will briefly recall the definitions and basic properties of fusion frames. For more details we refer to Casazza and Kutyniok (2004) and Asgari and Khosravi (2005). Throughout the paper, $\mathcal{H}$ is a Hilbert spaces, and $I = \{1, 2, \ldots, M\}$ is a subset of $\mathbf{N}$, $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

A family of the vector $\Phi = \{\varphi_i\}_{i\in I} \subset \mathcal{H}$ is called a *frame*, if there exist constants $0 < A \leq B < \infty$ such that for any $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2. \tag{1}$$

The constants A and B are known respectively as the lower and upper *frame bounds*.

Definition 1 Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. Then $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ is a *fusion frame*, if there exist constants $0 < C \leq D < \infty$ such that for any $f \in \mathcal{H}$

$$C\|f\|^2 \leq \sum_{i\in I} w_i \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2, \tag{2}$$

where π_{W_i} denotes the orthogonal projection of $\mathcal{H}$ onto W_i. We call C, D the *fusion frame bounds*. The frame $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ is called a *tight fusion frame* if $C = D$, and is called a *Parseval fusion frame* if $C = D = 1$. If we only know that $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ satisfies the upper inequality in (2), then $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ is called a *Bessel fusion sequence* with Bessel bound D.

Let $\mathbf{W}$ be a Bessel fusion sequence for $\mathcal{H}$. The *synthesis operator* $T^* : l^2(I) \to \mathcal{H}$ is defined by

$$T^*(\{f_i\}_{i\in I}) = \sum_{i\in I} w_i \pi_{W_i}(f_i), \quad \forall \{f_i\}_{i\in I} \in l^2(I).$$

The adjoint operator $T : \mathcal{H} \to l^2(I)$ given by $T(f) = \{w_i \pi_{W_i}(f)\}_{i\in I}$ is called the *analysis operator*. In Casazza and Kutyniok (2004) we know that

$$S : \mathcal{H} \to \mathcal{H},\ Sf = \sum_{i\in I} w_i^2 \pi_{W_i} f,$$

which is a bounded, self-adjoint, positive and invertible operator with $CI_{\mathcal{H}} \leq S \leq DI_{\mathcal{H}}$, and satisfies

$$\langle Sf, f\rangle = \sum_{i\in I} w_i^2 \|\pi_{W_i} f\|^2.$$

Then the following standard reconstruction formula takes places for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{i\in I} w_i^2 \pi_{W_i}(S^{-1}f),$$

and

$$\langle S^{-1}f,f\rangle = \sum_{i\in I} w_i^2 \|\pi_{W_i} S^{-1} f\|^2.$$

Casazza and Kutyniok (2004) define the dual fusion frame of fusion frame, which is similar to the canonical dual frame in the classical frame theory.

Definition 2 (*Casazza and Kutyniok* 2004) Let $\{(W_i, w_i)\}_{i\in I}$ be a fusion frame with fusion frame operator S. Then $\{(S^{-1}W_i, w_i)\}_{i\in I}$ is called the *dual fusion frame* of $\{(W_i, w_i)\}_{i\in I}$.

If $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ is a Bessel fusion sequence in $\mathcal{H}$, for every $J \subset I$ we define the operator S_J by

$$S_J f = \sum_{i\in J} w_i^2 \pi_{W_i} f, \tag{3}$$

it is trivial to show that S_J is a self-adjoint, bounded linear operator in $\mathcal{H}$, and denote $J^c = I \setminus J$.

Găvruţa (2007) gives a more general alternate dual reconstruction formula, that is, given a fusion frame $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ with frame operator S and a Bessel sequence $\mathbf{V} = \{(V_i, v_i)\}_{i\in I}$, there is

$$f = \sum_{i\in I} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f, \quad \forall f \in \mathcal{H}.$$

In this case we call $\mathbf{V} = \{(V_i, v_i)\}_{i\in I}$ an *alternate dual fusion frame* of $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$.

In the study of longstanding conjecture of signal processing community: a signal can be reconstructed without information about the phase. Balan et al. (2006) found some new frame equalities. In order to compare with the "Main results" section, we list the important equalities in Balan et al. (2007) as follows.

Theorem 1 (Balan et al. 2007) *Let $\{f_i\}_{i\in I}$ be a Parseval frame for $\mathcal{H}$. Then for any $J \subset I$ and $f \in \mathcal{H}$ we have*

$$\begin{aligned} &\sum_{i\in J} |\langle f, f_i\rangle|^2 - \|\sum_{i\in J} \langle f, f_i\rangle f_i\|^2 \\ &= \sum_{i\in J^c} |\langle f, f_i\rangle|^2 - \|\sum_{i\in J^c} \langle f, f_i\rangle f_i\|^2. \end{aligned} \tag{4}$$

Remark 1 A frame $\{g_i\}_{i\in I}$ is called alternate dual frame of $\{f_i\}_{i\in I}$ and $f = \sum_{i\in I} \langle f, g_i\rangle f_i$, $f \in \mathcal{H}$. Then we get a more general result about the alternate dual frame (Găvruţa 2006).

Theorem 2 *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with an alternate dual frame $\{g_i\}_{i\in I} \subset \mathcal{H}$. Then for any $J \subset I$ and any $f \in \mathcal{H}$ we have*

$$\begin{aligned} &Re\left(\sum_{i\in J}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\|\sum_{i\in J}\langle f,g_i\rangle f_i\|^2\\ &=Re\left(\sum_{i\in J^c}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\|^2. \end{aligned} \tag{5}$$

Zhu and Wu (2010) generalized the equality (5) to a more general form which does not involve the real parts of the complex numbers.

Theorem 3 *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{g_i\}_{i\in I}\subset\mathcal{H}$ is an alternate dual frame of $\{f_i\}_{i\in I}$. Then for any $J\subset I$ and $f\in\mathcal{H}$ we have*

$$\begin{aligned} &\overline{\left(\sum_{i\in J}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\|\sum_{i\in J}\langle f,g_i\rangle f_i\|^2}\\ &=\left(\sum_{i\in J^c}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}\right)-\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\|^2. \end{aligned} \tag{6}$$

Next, we extended this equality to fusion frame.

Main results

Motivated by the work of Balan et al. (2007) and Găvruţa (2006), in this section, we continue this work about fusion frames and get some important equalities and inequalities of these frames in a different case.

Lemma 1 (Zhu and Wu 2010) *Let P and Q be two linear bounded operators on $\mathcal{H}$ such that $P+Q=I_{\mathcal{H}}$. Then $P-P^*P=Q^*-Q^*Q$.*

Now, we present main theorems of this section.

Theorem 4 *Let $\{(W_i,w_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with the fusion frame operator S, $\{(V_i,v_i)\}_{i\in I}$ is the alternate dual fusion frame of $\{(W_i,w_i)\}_{i\in I}$. Then, for any $J\subset I$ and any $f\in\mathcal{H}$,*

$$\begin{aligned} &\sum_{i\in J}v_iw_i\langle S^{-1}\pi_{W_i}(f),\pi_{V_i}f\rangle-\|\sum_{i\in J}v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\|^2\\ &=\sum_{i\in J^c}v_iw_i\langle\pi_{V_i}f,S^{-1}\pi_{W_i}f\rangle-\|\sum_{i\in J^c}v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f\|^2. \end{aligned}$$

Proof For any $J\subset I$, we define a bounded linear operator S_J as

$$S_Jf=\sum_{i\in J}v_iw_i\pi_{V_i}S^{-1}\pi_{W_i}f,\quad\forall f\in\mathcal{H}.$$

Clearly, $S_J + S_{J^c} = I_{\mathcal{H}}$. This, together with Lemma 1, implies that

$$\begin{aligned}
&\sum_{i\in J} v_i w_i \langle S^{-1}\pi_{W_i}(f), \pi_{V_i} f\rangle - \|\sum_{i\in J} v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\|^2 \\
&= \sum_{i\in J} v_i w_i \langle S^{-1}\pi_{W_i}(f), \pi_{V_i} f\rangle - \langle S_J f, S_J f\rangle \\
&= \langle S_J f, f\rangle - \langle S_J^* S_J f, f\rangle \\
&= \langle S_{J^c}^* f, f\rangle - \langle S_{J^c}^* S_{J^c} f, f\rangle \\
&= \langle f, S_{J^c} f\rangle - \langle S_{J^c} f, S_{J^c} f\rangle \\
&= \langle f, \sum_{i\in J^c} v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\rangle \\
&\quad - \|\sum_{i\in J^c} v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\|^2 \\
&= \sum_{i\in J^c} v_i w_i \langle \pi_{V_i} f, S^{-1}\pi_{W_i} f\rangle \\
&\quad - \|\sum_{i\in J^c} v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\|^2.
\end{aligned} \tag{7}$$

In the situation of Parseval fusion frames the equality is of special form. □

Corollary 1 *Let $\{(W_i, w_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$ with the fusion frame operator $S = I_{\mathcal{H}}$, $\{(V_i, v_i)\}_{i\in I}$ is the alternate dual fusion frame of $\{(W_i, w_i)\}_{i\in I}$. Then, for any $J \subset I$ and any $f \in \mathcal{H}$,*

$$\begin{aligned}
&\sum_{i\in J} v_i w_i \langle \pi_{W_i}(f), \pi_{V_i} f\rangle - \|\sum_{i\in J} v_i w_i \pi_{V_i}\pi_{W_i} f\|^2 \\
&\quad = \sum_{i\in J^c} v_i w_i \langle \pi_{V_i} f, \pi_{W_i} f\rangle - \|\sum_{i\in J^c} v_i w_i \pi_{V_i}\pi_{W_i} f\|^2.
\end{aligned}$$

Remark 2 Clearly, when the dual fusion frame of $\{(W_i, w_i)\}_{i\in I}$ is itself, i.e., $\{(V_i, v_i)\}_{i\in I} = \{(W_i, w_i)\}_{i\in I}$, which was obtained Theorem 2.2 in Xiyan et al. (2009) as a particular case from the above result.

In fact, similarly to the proof of Theorem 4, we can give a more general result as follow. Moreover, the result has another proof in Xiao et al. (2014).

Theorem 5 *Let $\{(W_i, w_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with the fusion frame operator S, $\{(V_i, v_i)\}_{i\in I}$ is the alternate dual fusion frame of $\{(W_i, w_i)\}_{i\in I}$. Then, for any $f \in \mathcal{H}$ and any $\{b_i\}_{i\in I} \in l^\infty(I)$,*

$$\begin{aligned}
&\sum_{i\in I} b_i v_i w_i \langle S^{-1}\pi_{W_i} f, \pi_{V_i} f\rangle \\
&\quad - \|\sum_{i\in I} b_i v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\|^2 \\
&= \sum_{i\in I} (1 - \bar{b_i}) v_i w_i \langle \pi_{V_i} f, S^{-1}\pi_{W_i} f\rangle \\
&\quad - \|\sum_{i\in I} (1 - b_i) v_i w_i \pi_{V_i} S^{-1}\pi_{W_i} f\|^2.
\end{aligned} \tag{8}$$

where $\bar{b_i}$ is the conjugata of b_i.

Remark 3 Let $\{(W_i, w_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$ with the fusion frame bound A, and b_i is real for any $i \in I$. In this case, using the Theorem 5, we obtain

$$
\begin{aligned}
&A\sum_{i\in I} b_i v_i w_i \langle S^{-1}\pi_{W_i}f, \pi_{V_i}f\rangle \\
&\quad - \|\sum_{i\in I} b_i v_i w_i \pi_{V_i} S^{-1}\pi_{W_i}f\|^2 \\
&= A\sum_{i\in I}(1-b_i) v_i w_i \langle \pi_{V_i}f, S^{-1}\pi_{W_i}f\rangle \\
&\quad - \|\sum_{i\in I}(1-b_i) v_i w_i \pi_{V_i} S^{-1}\pi_{W_i}f\|^2.
\end{aligned}
$$

Lemma 2 (Găvruţa 2006) *Let P and Q are two self-adjoint bounded linear operators in $\mathcal{H}$ and $P+Q=I_{\mathcal{H}}$. Then we have*

$$\langle Pf,f\rangle + \|Qf\|^2 = \langle Qf,f\rangle + \|Pf\|^2 \geq \frac{3}{4}\langle f,f\rangle.$$

Theorem 6 *Let $\{(W_i,w_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with the fusion frame operator S, $\{(S^{-1}W_i,w_i)\}_{i\in I}$ is the dual fusion frame of $\{(W_i,w_i)\}_{i\in I}$. Then, for any $J\subset I$ and any $f\in\mathcal{H}$, we have*

$$
\begin{aligned}
&\sum_{i\in J} w_i^2\|\pi_{W_i}f\|^2 + \sum_{i\in I} w_i^2\|\pi_{W_i}S^{-1}S_{J^c}f\|^2 \\
&= \sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2 + \sum_{i\in I} w_i^2\|\pi_{W_i}S^{-1}S_{J}f\|^2 \\
&\geq \frac{3}{4}\langle Sf,f\rangle.
\end{aligned}
$$

Proof Applying $S=S_J+S_{J^c}$, we have that $I_{\mathcal{H}} = S^{-\frac{1}{2}}S_JS^{-\frac{1}{2}} + S^{-\frac{1}{2}}S_{J^c}S^{-\frac{1}{2}}$. Combining this with Lemma 2, it follows that

$$
\begin{aligned}
&\langle S^{-\frac{1}{2}}S_JS^{-\frac{1}{2}}f,f\rangle + \|S^{-\frac{1}{2}}S_{J^c}S^{-\frac{1}{2}}f\|^2 \\
&\quad = \langle S^{-\frac{1}{2}}S_{J^c}S^{-\frac{1}{2}}f,f\rangle + \|S^{-\frac{1}{2}}S_JS^{-\frac{1}{2}}f\| \\
&\quad \geq \frac{3}{4}\langle f,f\rangle.
\end{aligned}
\tag{9}
$$

Replacing f by $S^{\frac{1}{2}}f$, one has

$$
\begin{aligned}
&\langle S_Jf,f\rangle + \langle S^{-1}S_{J^c}f, S_{J^c}f\rangle \\
&= \langle S_{J^c}f,f\rangle + \langle S^{-1}S_Jf, S_Jf\rangle \geq \tfrac{3}{4}\langle Sf,f\rangle.
\end{aligned}
$$

Combining this with $\langle S_Jf,f\rangle = \sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2$ and $\langle S^{-1}f,f\rangle = \sum_{i\in I} w_i^2\|\pi_{W_i}S^{-1}f\|^2$, the proof is completed. □

Remark 4 The identity of above was established Theorem 2.1 in Xiyan et al. (2009), but the inequality in this form is a new result.

Corollary 2 *Let $\{(W_i,w_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$ with the fusion frame bound A. Then*

$$
\begin{aligned}
&A\sum_{i\in J} w_i^2\|\pi_{W_i}f\|^2 + \|S_{J^c}f\|^2 \\
&= A\sum_{i\in J^c} w_i^2\|\pi_{W_i}(f)\|^2 + \|S_Jf\|^2 \\
&\geq \tfrac{3}{4}A^2\langle f,f\rangle.
\end{aligned}
$$

In addition, if $\{(W_i, w_i)\}_{i\in I}$ *is a Parseval fusion frame for* $\mathcal{H}$*, then we have*

$$\begin{aligned}&\sum_{i\in I} w_i^2\|\pi_{W_i}f\|^2 + \|S_{J^c}f\|^2\\ &= \sum_{i\in J^c} w_i^2\|\pi_{W_i}(f)\|^2 + \|S_Jf\|^2\\ &\geq \tfrac{3}{4}\langle f,f\rangle.\end{aligned} \tag{10}$$

Proof Since $\{(W_i, w_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$ with the fusion frame bound A, then for any $f \in \mathcal{H}$,

$$\sum_{i\in J} w_i^2\|\pi_{W_i}f\|^2 = \langle Sf,f\rangle = A\|f\|^2,\ \pi_{W_i}S^{-1} = \frac{1}{A}\pi_{W_i},$$

and

$$\sum_{i\in I} w_i^2\|\pi_{W_i}S^{-1}S_Jf\|^2 = \frac{1}{A}\|S_Jf\|^2.$$

It follows from Theorem 6 that, for any $f \in \mathcal{H}$,

$$\begin{aligned}&A\sum_{i\in J} w_i^2\|\pi_{W_i}f\|^2 + \|S_{J^c}f\|^2\\ &= A\sum_{i\in J^c} w_i^2\|\pi_{W_i}(f)\|^2 + \|S_Jf\|^2\\ &\geq \tfrac{3}{4}A^2\langle f,f\rangle.\end{aligned}$$

□

Theorem 7 *Let* $\{(W_i, w_i)\}_{i\in I}$ *be a tight fusion frame for* $\mathcal{H}$ *with the fusion frame bound* A*. Then, for any* $J, E \subset I$ *with* $J \cap E = \emptyset$*, and any* $f \in \mathcal{H}$*, we have*

$$\begin{aligned}&\|S_{J\cup E}f\|^2 - \|S_{J^c\setminus E}f\|^2\\ &= \|S_Jf\|^2 - \|S_{J^c}f\|^2 + 2A\sum_{i\in E} w_i^2\|\pi_{W_i}f\|^2.\end{aligned} \tag{11}$$

Proof Applying Corollary 2 yields that

$$\begin{aligned}&\|S_{J\cup E}f\|^2 - \|S_{J^c\setminus E}f\|^2\\ &\quad= A\sum_{i\in J\cup E} w_i^2\|\pi_{W_i}f\|^2 - A\sum_{i\in J^c\setminus E} w_i^2\|\pi_{W_i}f\|^2\\ &\quad= A\sum_{i\in J} w_i^2\|\pi_{W_i}f\|^2 - A\sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2\\ &\qquad+ 2A\sum_{i\in E} w_i^2\|\pi_{W_i}f\|^2\\ &\quad= \|S_Jf\|^2 - \|S_{J^c}f\|^2 + 2A\sum_{i\in E} w_i^2\|\pi_{W_i}f\|^2.\end{aligned}$$

Similarly Corollary 3.6 in Xiao and Zeng (2010), obtain

Corollary 3 *Let $\{(W_i, w_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$ with the fusion frame bound A. Then, for any $J_i \subset I$, $(i \in N)$, where $N \geq 2$ is a positive integer, with $J_i \cap J_j = \emptyset$, for $i \neq j$, $I = \cup_{i=1}^{N} J_i$. Then for any $f \in \mathcal{H}$, we have*

$$\begin{aligned}&\|S_{(\cup_{i=N_1}^{N_4} J_i)} f\|^2 - \|S_{(\cup_{i=1}^{N_1-1} J_i + \cup_{i=N_4-1}^{N} J_i)} f\|^2 \\ &= \|S_{(\cup_{i=N_2}^{N_3} J_i)} f\|^2 - \|S_{(\cup_{i=N_2}^{N_3} J_i)^c} f\|^2 \\ &+ 2A \sum_{(\cup_{i=N_1}^{N_2-1} + \cup_{i=N_3+1}^{N_4})} w_i^2 \|\pi_{W_i} f\|^2,\end{aligned}$$

where N_i, $(1 \leq i \leq 4)$ are positive integers satisfying $1 \leq N_1 \leq N_2 < N_3 < N_4 \leq N-1$.

Proof Applying (11), replace J and E by $\cup_{i=N_2}^{N_3} J_i$ and $\cup_{i=N_1}^{N_2-1} J_i + \cup_{i=N_3+1}^{N_4} J_i$, the above result hold. □

The inequality (10) in Corollary 2 leads us to introduce some notations $\nu_-(\mathbf{W}, J)$ and $\nu_+(\mathbf{W}, J)$. Let $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ be a Parseval fusion frame. For any $J \subset I$ and $f \in \mathcal{H}$, define

$$\nu_+(\mathbf{W}, J) = \sup_{f\neq 0} \frac{\sum_{i\in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \|\sum_{i\in J} w_i^2 \pi_{w_i} f\|^2}{\|f\|^2},$$

and

$$\nu_-(\mathbf{W}, J) = \inf_{f\neq 0} \frac{\sum_{i\in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \|\sum_{i\in J} w_i^2 \pi_{w_i} f\|^2}{\|f\|^2}.$$

Theorem 8 *$\nu_-(\mathbf{W}, J)$ and $\nu_+(\mathbf{W}, J)$ have the following properties:*

1. $\frac{3}{4} \leq \nu_-(\mathbf{W}, J) \leq \nu_+(\mathbf{W}, J) \leq 1$;
2. $\nu_-(\mathbf{W}, J^c) = \nu_-(\mathbf{W}, J), \nu_+(\mathbf{W}, J^c) = \nu_+(\mathbf{W}, J)$;
3. $\nu_-(\mathbf{W}, J) = \nu_+(\mathbf{W}, J), \nu_-(\mathbf{W}, \emptyset) = \nu_+(\mathbf{W}, \emptyset)$.

Proof By inequality (10), $\frac{3}{4} \leq \nu_-(\mathbf{W}, J)$ holds trivially.

For any $f, g \in \mathcal{H}$ and any $J \subset I$, we have

$$\begin{aligned}\|\sum_{i\in J} w_i^2 \pi_{w_i} f\|^2 &= \sup_{\|g\|=1} |\langle \sum_{i\in J} w_i^2 \pi_{w_i} f, g\rangle|^2 \\ &= \sup_{\|g\|=1} |\sum_{i\in J} w_i^2 \langle \pi_{w_i} f, \pi_{w_i} g\rangle|^2 \\ &\leq \sup_{\|g\|=1} \sum_{i\in J} w_i^2 \|\pi_{w_i} f\|^2 \sum_{i\in J} w_i^2 \|\pi_{w_i} g\|^2 \\ &= \sup_{\|g\|=1} \|g\|^2 \sum_{i\in J} w_i^2 \|\pi_{w_i} f\|^2 \\ &= \sum_{i\in J} w_i^2 \|\pi_{w_i} f\|^2.\end{aligned}$$

Hence,

$$\sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2 + \|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2$$
$$\leq \sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2 + \sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2 \leq \|f\|^2,$$

This implies that $\|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2 \leq \sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2$. That is $\nu_+(\mathbf{W},J) \leq 1$.

(2) and (3) follow directly by inequality (10) in Corollary 2. □

Some results for the Parseval fusion frame were established in Xiyan et al. (2009). For the reader's convenience and our results equivalence, we not only recall its formulation but also provide its proof as follows.

Theorem 9 *Let $\mathbf{W} = \{(W_i, w_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$. Then, for any $J \subset I$ and any $f \in \mathcal{H}$, the following statements are equivalent:*

1. $\nu_-(\mathbf{W},J) = \nu_+(\mathbf{W},J) = 1$;
2. $\sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2 = \|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2$;
3. $\sum_{i\in J^c} w_i^2\|\pi_{w_i}f\|^2 = \|\sum_{i\in J^c} w_i^2\pi_{w_i}f\|^2$;
4. $S_J S_{J^c} f = 0$.

Proof (1)⇒ (2). Since $\mathbf{W}$ is a Parseval fusion frame, then for any $f \in \mathcal{H}$, we have $\sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2 + \sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2 = \|f\|^2$. This implies that

$$\begin{aligned}\sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2 &= \|f\|^2 - \sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2\\ &= \sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2 + \|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2\\ &\quad - \sum_{i\in J^c} w_i^2\|\pi_{W_i}f\|^2\\ &= \|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2.\end{aligned}$$

Applying (10), (3) ⇐ (2) ⇒ (1) hold trivially.

(2)⇔ (4) follows from

$$\sum_{i\in J} w_i^2\|\pi_{w_i}f\|^2 - \|\sum_{i\in J} w_i^2\pi_{w_i}f\|^2$$
$$= \langle S_J f, f\rangle - \langle S_J f, S_J f\rangle = \langle (S_J - S_J^2)f, f\rangle$$
$$= \langle S_J(I - S_J)f, f\rangle = \langle S_J S_{J^c} f, f\rangle.$$

□

Conclusions

In frame theory, fusion frames have some properties similar to those of frames in Hilbert spaces, but not all of their properties are similar. Many excellent results of frames have been achieved and applied successfully, which properties of the frames may be extended

to the fusion frames, which requires a lot of efforts to deal with. In this paper, we extend some equalities and inequalities of the frame to the fusion frames, which generalize and improve the remarkable results which have been obtained.

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Authors' contributions
This work was carried out in collaboration among the authors. All authors have a good contribution to design the study, and to perform the analysis of this research work. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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