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Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials

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Abstract

In this paper, we consider the Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

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1 Introduction

In this paper, we use umbral calculus techniques (see [1, 2]) to obtain several new and interesting identities of Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. To define the umbral calculus, let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$ and Π^* be the vector space of all linear functionals on Π. The action of a linear functional $L \in \Pi^*$ on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$, and linearly extended as $\langle cL + dL'|p(x)\rangle = c\langle L|p(x)\rangle + d\langle L'|p(x)\rangle$, where $c, d \in \mathbb{C}$. Define $\mathcal{H} = \{f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C}\}$ to be the algebra of formal power series in a single variable t. The formal power series $f(t) \in \mathcal{H}$ defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$ for all $n \geq 0$. Thus, we have (see [1, 2])

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad \text{for all } n,k \geq 0, \tag{1.1}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. By (1.1), we get that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ gives a vector space isomorphism from Π^* onto $\mathcal{H}$. Therefore, $\mathcal{H}$ is thought of as a set of both formal power series and linear functionals, which is called the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is defined to be k when $f(t) = \sum_{n\geq k} a_n t^n$ and $a_k \neq 0$. Suppose that $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n,k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$

(see [1, 2]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have that $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle \frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle \frac{x^n}{n!}$. Therefore, $\langle t^k|p(x)\rangle = p^{(k)}(0)$, $\langle 1|p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. So, $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$ for all $k \geq 0$ (see [1, 2]).

Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y) \frac{t^n}{n!} \tag{1.2}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [1, 2]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$. Then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} \left(\ell(\bar{f}(t))\right)^k \middle| x^n \right\rangle \tag{1.3}$$

(see [1, 2]).

Throughout the paper, let $r, s \in \mathbb{Z}_{>0}$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_r)$, $\mathbf{b} = (b_1, b_2, \ldots, b_s)$ with $a_j, b_i \neq 0$ for all i, j. We define the *Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials* $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ (for other Barnes-types, see [3–5]) as

$$P_{r,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \frac{t^n}{n!}, \tag{1.4}$$

where we define

$$P_{r,s}(t) = \prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}} - 1)}\right) \prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}} + 1}\right).$$

For $x = 0$, $D\mathcal{E}_n(\lambda|\mathbf{a};\mathbf{b}) = D\mathcal{E}_n(\lambda, 0|\mathbf{a};\mathbf{b})$ are called the *Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type numbers*.

We recall here that the polynomials $D_{n,\lambda}(x|\mathbf{a})$ given by

$$P_{r,0}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} D_{n,\lambda}(x|\mathbf{a}) \frac{t^n}{n!}$$

are called the *Barnes-type Daehee polynomials* with λ-parameter (see [6, 7]). Also, the polynomials $\mathcal{E}_n(\lambda, x|\mathbf{b})$ given by

$$P_{0,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} \mathcal{E}_n(\lambda, x|\mathbf{b}) \frac{t^n}{n!} \tag{1.5}$$

are called the *Barnes-type degenerate Euler polynomials* which are studied in [8–11]. In the case $x = 0$, we write $\mathcal{E}_n(\lambda|\mathbf{b}) = \mathcal{E}_n(\lambda, 0|\mathbf{b})$, which are called the *Barnes-type degenerate Euler numbers*. Note that $\lim_{\lambda\to 0}\mathcal{E}_n(\lambda, x|\mathbf{b}) = E_n(x|\mathbf{b})$ and $\lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n(\lambda, \lambda x|\mathbf{b}) = (x)_n$, where $(x)_n = \prod_{i=0}^{n-1}(x-i)$ with $(x)_0 = 1$ and $E_n(x|\mathbf{b})$ are the *Barnes-type degenerate Euler polynomials* given by

$$\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right) e^{xt} = \sum_{n\geq 0} E_n(x|\mathbf{b}) \frac{t^n}{n!}.$$

It is immediate from (1.2) and (1.4) to see that $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ is the Sheffer sequence for the pair $g(t) = \prod_{i=1}^{r}(\frac{e^{a_i t}-1}{t})\prod_{i=1}^{s}(\frac{e^{b_i t}+1}{2})$ and $f(t) = \frac{e^{\lambda t}-1}{\lambda}$. Thus,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right), \frac{e^{\lambda t}-1}{\lambda}\right). \tag{1.6}$$

The aim of the present paper is to present several new identities for Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials by the use of umbral calculus. For some of the related works, one is referred to the papers [12–20].

2 Explicit formulas

In this section we suggest several explicit formulas for the Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. To do that, we recall that the Stirling numbers $S_1(n,m)$ of the first kind are defined as $(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \sim (1, e^t - 1)$ or $\frac{1}{j!}(\log(1+t))^j = \sum_{\ell\geq j} S_1(\ell,j)\frac{t^\ell}{\ell!}$. Let $(x|\lambda)_n$ be the *generalized falling factorials* defined by $(x|\lambda)_n = \prod_{i=0}^{n-1}(x - i\lambda)$ with $(x|\lambda)_0 = 1$, namely $(x|\lambda)_n = \lambda^n(x/\lambda)_n$.

Let $BE_n(x|\mathbf{a};\mathbf{b})$ be the *Barnes-type Bernoulli and Euler mixed-type polynomials* given by

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)e^{xt} = \sum_{n\geq 0} BE_n(x|\mathbf{a};\mathbf{b})\frac{t^n}{n!}. \tag{2.1}$$

Note that $BE_n^{r,s}(x)$ denotes the special case $BE_n(x|\underbrace{1,1,\ldots,1}_{r};\underbrace{1,1,\ldots,1}_{s})$ and was treated in [21, 22] by using p-adic integrals on $\mathbb{Z}_p$.

Theorem 2.1 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} BE_m(x|\mathbf{a};\mathbf{b}).$$

Proof By (1.6), we have that

$$\prod_{i=1}^{r}\left(\frac{e^{a_i t}-1}{t}\right)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right)D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right). \tag{2.2}$$

Thus,

$$\begin{aligned} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)x^m \\ &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} BE_m(x|\mathbf{a};\mathbf{b}), \end{aligned}$$

as claimed. □

Theorem 2.2 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\lambda^{\ell-j}D\mathcal{E}_{n-\ell}(\lambda|\mathbf{a};\mathbf{b})\right)x^j.$$

Proof We proceed the proof by applying the conjugate representation: for $s_n(x) \sim (g(t), f(t))$, we have $S_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1}\bar{f}(t)^j | x^n \rangle x^j$. By (1.6), we obtain

$$\begin{aligned}
&\bigl\langle g\bigl(\bar{f}(t)\bigr)^{-1}\bar{f}(t)^j \big| x^n\bigr\rangle \\
&\quad = \biggl\langle P_{r,s}(t)\frac{\log^j(1+\lambda t)}{\lambda^j}\bigg| x^n \biggr\rangle = \lambda^{-j}\biggl\langle P_{r,s}(t)\bigg| j!\sum_{\ell \geq j} S_1(\ell, j)\frac{\lambda^\ell t^\ell}{\ell!} x^n\biggr\rangle \\
&\quad = \lambda^{-j} j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j)\lambda^\ell \bigl\langle P_{r,s}(t) | x^{n-\ell}\bigr\rangle = \lambda^{-j} j! \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j)\lambda^\ell D\mathcal{E}_{n-\ell}(\lambda | \mathbf{a}; \mathbf{b}).
\end{aligned}$$

Therefore, $D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) = \sum_{j=0}^{n} (\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j)\lambda^{\ell-j} D\mathcal{E}_{n-\ell}(\lambda | \mathbf{a}; \mathbf{b}))x^j$, as claimed. □

Theorem 2.3 *For all* $n \geq 1$,

$$D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} BE_{n-\ell}(x | \mathbf{a}; \mathbf{b}),$$

where $B_\ell^{(n)}$ *is the* ℓ*th Bernoulli number of order* n (*see* [23]).

Proof We proceed the proof by using the following transfer formula: for $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$, we have that $q_n(x) = x(\frac{f(t)}{g(t)})^n x^{-1} p_n(x)$ for all $n \geq 1$. So, by the fact that $x^n \sim (1, t)$ and (2.2), we obtain

$$\begin{aligned}
&\prod_{i=1}^{r}\biggl(\frac{e^{a_i t}-1}{t}\biggr)\prod_{i=1}^{s}\biggl(\frac{e^{b_i t}+1}{2}\biggr) D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) \\
&\quad = x\biggl(\frac{\lambda t}{e^{\lambda t}-1}\biggr)^n x^{n-1} = x\sum_{\ell \geq 0} B_\ell^{(n)}\frac{\lambda^\ell t^\ell}{\ell!} x^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)} x^{n-\ell},
\end{aligned}$$

which, by (2.1), implies

$$\begin{aligned}
D\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b}) &= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)} \prod_{i=1}^{r}\biggl(\frac{t}{e^{a_i t}-1}\biggr)\prod_{i=1}^{s}\biggl(\frac{2}{e^{b_i t}+1}\biggr) x^{n-\ell} \\
&= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)} BE_{n-\ell}(x | \mathbf{a}; \mathbf{b}),
\end{aligned}$$

as required. □

In order to state our next theorem, we recall the polynomials $\beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b})$, which are called the *Barnes-type degenerate Bernoulli and Euler mixed-type polynomials*. They are defined as

$$Q_{r,s}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n \geq 0} \beta\mathcal{E}_n(\lambda, x | \mathbf{a}; \mathbf{b})\frac{t^n}{n!}, \tag{2.3}$$

where $Q_{r,s}(t) = \prod_{i=1}^{r}(\frac{t}{(1+\lambda t)^{\frac{a_i}{\lambda}}-1})\prod_{i=1}^{s}(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1})$, for example, see [3].

Theorem 2.4 *For all $n \geq 0$,*

$$D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^{\ell} S_1(\ell+r,r)\beta\mathcal{E}_{n-\ell}(\lambda,x|\mathbf{a};\mathbf{b}).$$

Proof By (1.4), we have

$$\begin{aligned} D\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b}) &= \left\langle \sum_{\ell\geq 0} D\mathcal{E}_\ell(\lambda,y|\mathbf{a};\mathbf{b})\frac{t^\ell}{\ell!} \middle| x^n \right\rangle = \left\langle P_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} | x^n \right\rangle \\ &= \left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} \middle| \frac{\log^r(1+\lambda t)}{\lambda^r t^r} x^n \right\rangle \\ &= \left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} \middle| r! \sum_{\ell\geq 0} \frac{S_1(\ell+r,r)\lambda^\ell t^\ell}{(\ell+r)!} x^n \right\rangle \\ &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r) \left\langle \sum_{m\geq 0} \beta\mathcal{E}_m(\lambda,y|\mathbf{a};\mathbf{b})\frac{t^m}{m!} \middle| x^{n-\ell} \right\rangle, \end{aligned}$$

which, by (2.3), implies $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\beta\mathcal{E}_{n-\ell}(\lambda,x|\mathbf{a};\mathbf{b})$, as required. □

In order to present our next theorem, we recall the polynomials $\beta_n(\lambda,x|\mathbf{a})$, which are called the *Barnes-type degenerate Bernoulli polynomials*. They are given by

$$Q_{r,0}(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} \beta_n(\lambda,x|\mathbf{a})\frac{t^n}{n!}, \tag{2.4}$$

for example, see [8, 9, 23].

Theorem 2.5 *For all $n \geq 0$,*

$$\begin{aligned} D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell} \frac{\binom{n}{\ell}\binom{n-\ell}{m}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\mathcal{E}_{n-\ell-m}(\lambda|\mathbf{b})\beta_m(\lambda,x|\mathbf{a}) \\ &= \sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell} \frac{\binom{n}{\ell}\binom{n-\ell}{m}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\beta_{n-\ell-m}(\lambda|\mathbf{a})\mathcal{E}_m(\lambda,x|\mathbf{b}). \end{aligned}$$

Proof By the proof of Theorem 2.4, we have

$$\begin{aligned} D\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\left\langle Q_{r,s}(t)(1+\lambda t)^{\frac{y}{\lambda}} | x^{n-\ell}\right\rangle \\ &= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\left\langle Q_{0,s}(t) | Q_{r,0}(t)(1+\lambda t)^{\frac{y}{\lambda}} x^{n-\ell}\right\rangle. \end{aligned}$$

Thus, by (1.5) and (2.4), we obtain

$$\begin{aligned} &D\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b}) \\ &\quad= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^\ell S_1(\ell+r,r)\left\langle Q_{0,s}(t) \middle| \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda,y|\mathbf{a}) x^{n-\ell-m} \right\rangle \end{aligned}$$

$$= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^{\ell} S_1(\ell+r,r) \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda, y|\mathbf{a}) \langle Q_{0,s}(t)|x^{n-\ell-m}\rangle$$

$$= \sum_{\ell=0}^{n} \frac{\binom{n}{\ell}}{\binom{\ell+r}{r}} \lambda^{\ell} S_1(\ell+r,r) \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \beta_m(\lambda, y|\mathbf{a}) \mathcal{E}_{n-\ell-m}(\lambda|\mathbf{b}),$$

which completes the proof of the first formula.

The second formula can be obtained by using very similar techniques. □

3 Recurrence relations

In this section, we present several recurrence relations for Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials. Our first recurrence is based on the polynomials $(x|\lambda)_n$.

Theorem 3.1 *For all* $n \geq 0$,

$$D\mathcal{E}_n(\lambda, x+y|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n} \binom{n}{j} D\mathcal{E}_j(\lambda, x|\mathbf{a};\mathbf{b})(y|\lambda)_{n-j}.$$

Proof Let $p_n(x) = \prod_{i=1}^{r}(\frac{e^{a_i t}-1}{t})\prod_{i=1}^{s}(\frac{e^{b_i t}+1}{2})D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$. By (2.2) we have that $p_n(x) = (x|\lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$, which leads to the required recurrence. □

The second recurrence is obtained from the fact that $f(t)s_n(x) = ns_{n-1}(x)$ for all $s_n(x) \sim (g(t), f(t))$ (see [1, 2]).

Theorem 3.2 *For all* $n \geq 1$,

$$D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\lambda D\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}).$$

Proof By (1.6) and $f(t)s_n(x) = ns_{n-1}(x)$ whenever $s_n(x) \sim (g(t), f(t))$, we have

$$\frac{e^{\lambda t}-1}{\lambda} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = nD\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}),$$

which implies $D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\lambda D\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b})$, as required. □

The next result gives an explicit formula for $\frac{d}{dx}D\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b})$.

Theorem 3.3 *For all* $n \geq 1$,

$$\frac{d}{dx} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n! \sum_{\ell=0}^{n-1} \frac{(-\lambda)^{n-\ell-1}}{\ell!(n-\ell)} D\mathcal{E}_\ell(\lambda, x|\mathbf{a};\mathbf{b}).$$

Proof It is well known that for $s_n(x) \sim (g(t), f(t))$, $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle \bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$ (see [1, 2]). In our case, by (1.6), we have

$$\langle \bar{f}(t)|x^{n-\ell}\rangle = \left\langle \frac{1}{\lambda}\log(1+\lambda t) \middle| x^{n-\ell} \right\rangle$$

$$= \lambda^{-1} \left\langle \sum_{m\geq 1} \frac{(-1)^{m-1}(m-1)!\lambda^m t^m}{m!} \middle| x^{n-\ell} \right\rangle$$

$$= \lambda^{-1}(-1)^{n-\ell-1}\lambda^{n-\ell}(n-\ell-1)!$$
$$= (-\lambda)^{n-\ell-1}(n-\ell-1)!.$$

Thus $\frac{d}{dx}D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-\ell-1}}{\ell!(n-\ell)}D\mathcal{E}_\ell(\lambda,x|\mathbf{a};\mathbf{b})$, as required. □

Another recurrence relation can be stated as follows.

Theorem 3.4 *For all $n \geq 1$,*

$$\begin{aligned} &D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) \\ &\quad= \left(x-\sum_{i=1}^r a_i - \sum_{j=1}^s b_j\right)D\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};\mathbf{b}) + \frac{r}{n}\sum_{\ell=0}^n\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,x-\lambda|\mathbf{a};\mathbf{b}) \\ &\qquad - \frac{1}{n}\sum_{i=1}^r a_i\sum_{\ell=0}^n\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,x-\lambda|a_i,a_1,\ldots,a_r;\mathbf{b}) \\ &\qquad + \frac{1}{2}\sum_{j=1}^s b_j D\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};b_j,b_1,\ldots,b_s), \end{aligned}$$

where $\mathfrak{b}_n$ is the nth Bernoulli number of the second kind, which is defined by $\frac{t}{\log(1+t)} = \sum_{n\geq 0}\mathfrak{b}_n\frac{t^n}{n!}$.

Proof Let $n \geq 1$. Then

$$\begin{aligned} &D\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b}) \\ &\quad= \left\langle \sum_{\ell\geq 0} D\mathcal{E}_\ell(\lambda,y|\mathbf{a};\mathbf{b})\frac{t^\ell}{\ell!}\middle| x^n\right\rangle \\ &\quad= \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^n\right\rangle = \left\langle \frac{d}{dt}\left(P_{r,s}(t)(1+\lambda t)^{y/\lambda}\right)\middle| x^{n-1}\right\rangle \\ &\quad= \left\langle \frac{d}{dt}\prod_{i=1}^r\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle && (3.1) \\ &\qquad + \left\langle \prod_{i=1}^r\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\frac{d}{dt}\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle && (3.2) \\ &\qquad + \left\langle P_{r,s}(t)\frac{d}{dt}(1+\lambda t)^{y/\lambda}\middle| x^{n-1}\right\rangle. && (3.3) \end{aligned}$$

By (1.6), the term in (3.3) equals

$$y\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\right\rangle = yD\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b}). \tag{3.4}$$

For the term in (3.2), we observe that

$$\frac{d}{dt}\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right) = \prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{\frac{b_i}{\lambda}}+1}\right)\sum_{i=1}^s\left(\frac{-b_i}{1+\lambda t}+\frac{b_i}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right).$$

So the term in (3.2) is

$$\begin{aligned}&-\sum_{j=1}^{s} b_j\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\rangle+\frac{1}{2}\sum_{j=1}^{s} b_j\left\langle P_{r,s}(t)\frac{2(1+\lambda t)^{(y-\lambda)/\lambda}}{(1+\lambda t)^{b_j/\lambda}+1}\middle|x^{n-1}\right\rangle\\&=-\sum_{j=1}^{s} b_j D\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{1}{2}\sum_{j=1}^{s} b_j D\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};b_j,b_1,\ldots,b_s).\end{aligned}\tag{3.5}$$

For the term in (3.1), we note that

$$\begin{aligned}&(1+\lambda t)\frac{d}{dt}\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\\&=\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{\frac{a_i}{\lambda}}-1)}\right)\left(-\sum_{i=1}^{r}a_i+\frac{1}{t}\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)}-\frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\right),\end{aligned}$$

where $\frac{\lambda t}{\log(1+\lambda t)}-\frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}$ has order at least 1. Thus, the term in (3.1) equals

$$\begin{aligned}&-\sum_{i=1}^{r}a_i\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}|x^{n-1}\rangle\\&\quad+\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\frac{1}{t}\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)}-\frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n-1}\right\rangle\\&=-\sum_{i=1}^{r}a_i D\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})\\&\quad+\frac{1}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{i=1}^{r}\left(\frac{\lambda t}{\log(1+\lambda t)}-\frac{a_i t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n}\right\rangle\\&=-\sum_{i=1}^{r}a_i D\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})\\&\quad+\frac{r}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{\ell\geq 0}^{r}\mathfrak{b}_\ell\frac{\lambda^\ell t^\ell}{\ell!}x^{n}\right\rangle\\&\quad-\frac{1}{n}\sum_{i=1}^{r}a_i\left\langle\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}P_{r,s}(t)(1+\lambda t)^{(y-\lambda)/\lambda}\middle|\sum_{\ell\geq 0}^{r}\mathfrak{b}_\ell\frac{\lambda^\ell t^\ell}{\ell!}x^{n}\right\rangle,\end{aligned}$$

which is equal to

$$\begin{aligned}&-\sum_{i=1}^{r}a_i D\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{r}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})\\&\quad-\frac{1}{n}\sum_{i=1}^{r}a_i\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^\ell\mathfrak{b}_\ell D\mathcal{E}_{n-\ell}(\lambda,y-\lambda|a_i,a_1,\ldots,a_r;\mathbf{b}).\end{aligned}\tag{3.6}$$

By using (3.4), (3.5) and (3.6) instead of (3.3), (3.2) and (3.1), respectively, we complete the proof. □

Theorem 3.5 *For all* $n \geq 0$,

$$\begin{aligned} &D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) \\ &\quad = xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - \sum_{i=1}^{r} a_i \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} BE_m(x-\lambda|\mathbf{a};\mathbf{b}) \\ &\qquad - \sum_{m=0}^{n}\sum_{\ell=0}^{m} S_1(n,m)\lambda^{n-m}\binom{n}{m}\left(\frac{B_{\ell+1}}{\ell+1}\sum_{i=1}^{r} a_i^{\ell+1} + \frac{E_\ell(1)}{2}\sum_{j=1}^{s} b_j^{\ell+1}\right) \\ &\qquad \times BE_{m-\ell}(x-\lambda|\mathbf{a};\mathbf{b}), \end{aligned}$$

where B_ℓ *is the* ℓ*th Bernoulli number and* $E_\ell(1)$ *is the* ℓ*th Euler polynomial evaluated at* 1.

Proof It is well known that for $s_n(x) \sim (g(t), f(t))$, $s_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)} s_n(x)$ (see [1, 2]). In our case, by (1.6), we have

$$D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) = xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - e^{-\lambda t}\frac{g'(t)}{g(t)} D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}),$$

and by Theorem 2.1, we obtain

$$\begin{aligned} D\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) &= xD\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) \\ &\quad - \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} e^{-\lambda t}\frac{g'(t)}{g(t)} BE_m(x|\mathbf{a};\mathbf{b}). \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \frac{g'(t)}{g(t)} = \bigl(\log g(t)\bigr)' &= \sum_{i=1}^{r}\frac{a_i e^{a_i t}}{e^{a_i t}-1} - \frac{r}{t} + \sum_{j=1}^{s}\frac{b_j e^{b_j t}}{e^{b_j t}+1} \\ &= \sum_{i=1}^{r} a_i + \frac{1}{t}\sum_{i=1}^{r}\left(\frac{a_i t}{e^{a_i t}-1} - 1\right) + \frac{1}{2}\sum_{j=1}^{s}\frac{2b_j e^{b_j t}}{e^{b_j t}+1} \\ &= \sum_{i=1}^{r} a_i + \frac{1}{t}\sum_{i=1}^{r}\sum_{\ell\geq 0}\beta_\ell a_i^\ell\frac{t^\ell}{\ell!} + \frac{1}{2}\sum_{j=1}^{s}\sum_{\ell\geq 0}E_\ell(1)b_j^{\ell+1}\frac{t^\ell}{\ell!} \\ &= \sum_{i=1}^{r} a_i + \sum_{\ell\geq 0}\frac{\beta_{\ell+1}}{(\ell+1)!}\sum_{i=1}^{r} a_i^{\ell+1}t^\ell + \frac{1}{2}\sum_{\ell\geq 0}\frac{E_\ell(1)}{\ell!}\sum_{j=1}^{s} b_j^{\ell+1}t^\ell. \end{aligned}$$

So

$$\begin{aligned} \frac{g'(t)}{g(t)} BE_m(x|\mathbf{a};\mathbf{b}) &= \sum_{i=1}^{r} a_i BE_m(x|\mathbf{a};\mathbf{b}) + \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\beta_{\ell+1}}{\ell+1}\sum_{i=1}^{r} a_i^{\ell+1} BE_{m-\ell}(x|\mathbf{a};\mathbf{b}) \\ &\quad + \frac{1}{2}\sum_{\ell=0}^{m}\binom{m}{\ell}E_\ell(1)\sum_{j=1}^{s} b_j^{\ell+1} BE_{m-\ell}(x|\mathbf{a};\mathbf{b}). \end{aligned}$$

Hence, by substituting into (3.7), we complete the proof. □

4 Relations with other families of polynomials

In this section, we establish a connection between Barnes-type Daehee with λ-parameter and degenerate Euler mixed-type polynomials and several known families of polynomials.

Theorem 4.1 *For all $n \geq 0$,*

$$D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} \binom{n}{m} D\mathcal{E}_{n-m}(\lambda|\mathbf{a};\mathbf{b})(x|\lambda)_m.$$

Proof Note that $(x|\lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$. Let $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m}(x|\lambda)_m$. By (1.3) and (1.6), we have

$$\begin{aligned} c_{n,m} &= \frac{1}{m!}\langle P_{r,s}(t)|t^m x^n\rangle = \binom{n}{m}\langle P_{r,s}(t)|x^{n-m}\rangle \\ &= \binom{n}{m} D\mathcal{E}_{n-m}(\lambda|\mathbf{a};\mathbf{b}), \end{aligned}$$

which completes the proof. □

For the following, we note that $B_n^{(\alpha)}(x) \sim (\frac{(e^t-1)^\alpha}{t^\alpha}, t)$.

Theorem 4.2 *For all $n \geq 0$, the polynomial $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ is given by*

$$\sum_{m=0}^{n} \left(\sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \sum_{q=0}^{n-\ell-k} \sum_{p=0}^{q} \frac{\binom{n}{\ell}\binom{n-\ell}{k}\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}} a_{\ell,k,q,p} D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b}) \right) B_m^{(\alpha)}(x),$$

where $a_{\ell,k,q,p} = S_1(\ell, m)S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha, \alpha)\lambda^{k+\ell+p-m} b_\ell^{(\alpha)}$ and $b_\ell^{(\alpha)}$ is the ℓth Bernoulli number of the second kind of order α given by $(\frac{t}{\log(1+t)})^\alpha = \sum_{\ell\geq 0} b_\ell^{(\alpha)} \frac{t^\ell}{k!}$.

Proof Let $D\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m} B_m^{(\alpha)}(x)$. By (1.3) and (1.6), we have

$$\begin{aligned} c_{n,m} &= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^\alpha \middle| \left(\log(1+\lambda t)\right)^m x^n \right\rangle \\ &= \frac{1}{\lambda^m}\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \middle| \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^\alpha x^{n-\ell}\right\rangle \\ &= \frac{1}{\lambda^m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k} S_1(\ell,m)\lambda^{\ell+k} b_k^{(\alpha)} \left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha \middle| x^{n-\ell-k}\right\rangle. \end{aligned}$$

One can show that

$$\begin{aligned} \left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^\alpha &= \left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}{t}\right)^\alpha \\ &= \sum_{q\geq 0}\sum_{p=0}^{q}\binom{q+\alpha}{\alpha}^{-1} S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p \frac{t^q}{q!}, \end{aligned}$$

where $S_2(n,m)$ is the Stirling number of the second kind. Thus,

$$\begin{aligned}&\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\Big| x^{n-\ell-k}\right\rangle\\&\quad=\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\langle P_{r,s}(t)|x^{n-\ell-k-q}\rangle,\end{aligned}$$

where $\langle P_{r,s}(t)|x^{n-\ell-k-q}\rangle = D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})$. Hence,

$$c_{n,m}=\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n}{\ell}\binom{n-\ell}{k}\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}a_{\ell,k,q,p}D\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b}),$$

which completes the proof. □

By similar techniques as in the proof of the last theorem, we can express our polynomials $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ in terms of the degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda,x)$ of order α. These polynomials are the Sheffer sequence which is given by $\beta_n^{(\alpha)}(\lambda,x)\sim((\frac{\lambda(e^t-1)}{e^{\lambda t}-1})^{\alpha},\frac{e^{\lambda t}-1}{\lambda})$.

Theorem 4.3 *For all $n\geq 0$, the polynomial $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ is given by*

$$\sum_{m=0}^{n}\binom{n}{m}c_{n,m}\beta_m^{(\alpha)}(\lambda,x),$$

where $c_{n,m}=\sum_{q=0}^{n-m}\sum_{p=0}^{q}\frac{\binom{n-m}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p D\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b})$.

Now we are interested in expressing our polynomials in terms of $H_n^{(\alpha)}(x|\mu)$ which are called the Frobenius-Euler polynomials of order α. Note that $H_n^{(\alpha)}(x|\mu)\sim((\frac{e^t-\mu}{1-\mu})^{\alpha},t)$ (see [10, 24]).

Theorem 4.4 *For all $n\geq 0$,*

$$D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\sum_{m=0}^{n}\left(\frac{a_{n,m}}{(1-\mu)^{\alpha}\lambda^m}\right)H_m^{(\alpha)}(x|\mu),$$

where

$$a_{n,m}=\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{p=0}^{\alpha}\binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p}S_1(\ell,m)\lambda^{\ell}(-\mu)^{\alpha-p}D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k}.$$

Proof Let $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\sum_{m=0}^{n}c_{n,m}H_m^{(\alpha)}(x|\mu)$. By (1.3) and (1.6), we have

$$\begin{aligned}c_{n,m}&=\frac{1}{m!(1-\mu)^{\alpha}\lambda^m}\langle P_{r,s}(t)\big((1+\lambda t)^{1/\lambda}-\mu\big)^{\alpha}|\big(\log(1+\lambda t)\big)^m x^n\rangle\\&=\frac{1}{m!(1-\mu)^{\alpha}\lambda^m}\left\langle P_{r,s}(t)\big((1+\lambda t)^{1/\lambda}-\mu\big)^{\alpha}\Big| m!\sum_{\ell\geq m}S_1(\ell,m)\frac{\lambda^{\ell}}{\ell!}t^{\ell}x^n\right\rangle\end{aligned}$$

$$= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \binom{n}{\ell} S_1(\ell,m)\lambda^{\ell} \big\langle \big((1+\lambda t)^{1/\lambda} - \mu\big)^{\alpha} | P_{r,s}(t) x^{n-\ell} \big\rangle$$

$$= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{k} S_1(\ell,m)\lambda^{\ell} D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b}) w_{n,\ell,k},$$

where

$$w_{n,\ell,k} = \big\langle \big((1+\lambda t)^{1/\lambda} - \mu\big)^{\alpha} | x^{n-\ell-k} \big\rangle$$

$$= \left\langle \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (1+\lambda t)^{p/\lambda} \middle| x^{n-\ell-k} \right\rangle$$

$$= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} \left\langle \sum_{q\geq 0} (p|\lambda)_q \frac{t^q}{q!} \middle| x^{n-\ell-k} \right\rangle$$

$$= \sum_{p=0}^{\alpha} \binom{\alpha}{p} (-\mu)^{\alpha-p} (p|\lambda)_{n-\ell-k}.$$

Thus, the constants $c_{n,m}$ are given by

$$\frac{1}{(1-\mu)^{\alpha}\lambda^{m}} \sum_{\ell=m}^{n} \sum_{k=0}^{n-\ell} \sum_{p=0}^{\alpha} \binom{n}{\ell} \binom{n-\ell}{k} \binom{\alpha}{p} S_1(\ell,m)\lambda^{\ell} (-\mu)^{\alpha-p} D\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b}) (p|\lambda)_{n-\ell-k},$$

which completes the proof. □

Now we are interested in expressing our polynomials in terms of $\mathcal{E}_n^{(\alpha)}(\lambda,x)$ which are called the degenerate Euler polynomials of order α. Note that

$$\mathcal{E}_n^{(\alpha)}(\lambda,x) \sim \left(\left(\frac{e^t+1}{2}\right)^{\alpha}, \frac{e^{\lambda t}-1}{\lambda} \right)$$

(see [10]). Using similar techniques as in the proof of the above theorem, we obtain the following relation.

Theorem 4.5 *For all $n \geq 0$, the polynomial $D\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ is given by*

$$\frac{1}{2^{\alpha}} \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{q=0}^{n-m} \sum_{p=0}^{\alpha} \binom{n-m}{q} \binom{\alpha}{p} (p|\lambda)_q D\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b}) \right) \mathcal{E}_m^{(\alpha)}(\lambda,x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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