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Approximation of the 2D incompressible electrohydrodynamics system by the artificial compressibility method

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Abstract

This paper is devoted to the investigation of the 2D incompressible electrohydrodynamics system by the artificial compressibility method. We first introduce a family of perturbed compressible electrohydrodynamics, which approximate the incompressible electrohydrodynamics as the compressibility parameter is sent to zero. Then we prove the unique existence and convergence of solutions for the perturbed compressible electrohydrodynamics system to the solutions of the incompressible electrohydrodynamics system.

Keywords: artificial compressibility method; electrohydrodynamics system; approximation

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded smooth domain. We investigate the artificial compressibility approximation of the following electrohydrodynamics system [1]:

$$\begin{cases} a_1(v_t + v_{x_k}\vec{v}_k) - a_2\triangle v - QE + \nabla p = f, \\ Q_t - a_3\triangle Q + a_4 Q + \nabla \cdot (Qv) = 0, \\ -\triangle\phi = Q, \\ \nabla \cdot v = 0, \\ v = 0, \quad Q = \tilde{Q}, \quad \phi = \tilde{\phi}, \quad x \in \partial\Omega, \quad t > 0, \\ v(x,0) = v_0, \quad Q(x,0) = Q_0, \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here $v \in \mathbb{R}^2$ denotes the velocity vector field, $p \in \mathbb{R}$ denotes the pressure of the fluid, Q and E denote the charge density and the electric field, respectively, $k > 0$ is the constant coefficient of diffusion. We can simply denote the body forces of the electric origin by the form QE because the electric permeability and density of mass are constant. We suppose that the magnetic effects are negligible, thus $\nabla \times E = 0$, and the electric field is defined by $E = -\nabla\phi$.

The form of $\tilde{Q}$ is crucial for the nature of the flow. $\tilde{Q} = 0$ stands for no injection of free charges from the outside. Namely, the flow is quite simple. The stationary problem has one and only one solution and, if the electrostatic field is sufficiently small, there exists a

Liapunov function. On the contrary, the flow is richer and more complex when $\tilde{Q} \neq 0$. In particular, the solution of the stationary problem is non-unique in certain cases [2].

Let $\tilde{\phi}_0$ and $\tilde{Q}_0$ be the solutions of the following problems:

$$-a_3\triangle\tilde{Q}_0 + a_4\tilde{Q}_0 = 0, \quad \text{in } \Omega, \quad \tilde{Q}_0 = \tilde{Q} \quad \text{on } \partial\Omega, \tag{1.2}$$

$$-\triangle\tilde{\phi}_0 = \tilde{Q}_0, \quad \text{in } \Omega, \quad \tilde{\phi}_0 = \tilde{\phi} \quad \text{on } \partial\Omega. \tag{1.3}$$

We assume $\tilde{Q}, \tilde{\phi} \in C^\infty(\partial\Omega)$, thus we have $\tilde{Q}_0, \tilde{\phi}_0 \in C^\infty(\bar{\Omega})$ by standard results of regularity.

Defining $\varphi = \phi - \tilde{\phi}_0$, and $q = Q - \tilde{Q}_0$ we can rewrite the problem equations (1.1) as follows:

$$\begin{cases} a_1\frac{\partial v}{\partial t} + a_1\mathcal{B}(v,v) + a_2\mathcal{A}_0 v + \mathcal{N}(q+\tilde{Q}_0, \varphi+\tilde{\phi}_0) + \nabla p = f, \\ \frac{\partial q}{\partial t} + a_3\mathcal{A}_1 q + a_4 q - \mathcal{M}(q+\tilde{Q}_0, v) = 0, \\ \mathcal{A}_1\varphi = q, \\ \operatorname{div} v = 0, \\ v = 0, \quad q = 0, \quad \varphi = 0, \quad x \in \partial\Omega, \qquad t > 0, \\ v(x,0) = v_0, \quad q(x,0) = q_0, \quad x \in \Omega. \end{cases} \tag{1.4}$$

Here, for simplicity, we have made a detailed explanation of '$\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}(u,v)$, $\mathcal{N}(u,v)$, $\mathcal{M}(q,v)$' in the section of Preliminaries.

The electrohydrodynamics system has been studied largely in the past years, see for example [1, 3–10].

The artificial compressibility approximation method was introduced by Chorin [11, 12], Temam [13–15]. Their aim was to deal with the difficulties induced by the incompressibility constraint '$\operatorname{div} u = 0$' in the numerical approximations to the Navier-Stokes equations. As '$\operatorname{div} u = 0$' can never hold exactly, discretization errors accumulate at each iteration. Then after a significant amount of error accumulation, the approximate algorithm breaks down. To overcome this difficulty, Chorin and Temam introduced the following perturbed compressible Navier-Stokes equations:

$$\begin{aligned} &\frac{\partial u^\epsilon}{\partial t} + \nabla p^\epsilon = \nu\triangle u^\epsilon - \left(u^\epsilon \cdot \nabla\right)u^\epsilon - \frac{1}{2}\left(\operatorname{div} u^\epsilon\right)u^\epsilon, \\ &\epsilon\frac{\partial p^\epsilon}{\partial t} + \operatorname{div} u^\epsilon = 0. \end{aligned}$$

In [15], Temam also discussed the convergence of the approximations to the incompressible Navier-Stokes equations on bounded domains by using the classical Sobolev compactness embedding theorems and the classical Lions method [16] of fractional derivatives to recover the compactness in time.

Following the ideas of Chorin and Temam, the artificial compressibility approximation for some other models was also studied recently. For example, in [17, 18], Zhao *et al.* studied the artificial compressibility approximation for the incompressible convective Brinkman-Forchheimer equations and the non-Newtonian fluid equations, respectively. In [19–22], Donatelli *et al.* investigated the artificial compressibility approximation for the Navier-Stokes equations, MHD equations and the Navier-Stokes-Fourier equations on unbounded domains or the whole space $\mathbb{R}^3$.

In this paper, we consider the approximation of system (1.4) by the artificial compressibility method. Consider the following family of perturbed compressible systems depending on the parameter $\epsilon \in (0,1]$:

$$a_1\frac{\partial v_\epsilon}{\partial t} + a_1\mathcal{B}(v_\epsilon, v_\epsilon) + \frac{1}{2}a_1(\operatorname{div} v_\epsilon)v_\epsilon + a_2\mathcal{A}_0 v_\epsilon + \mathcal{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0) - \frac{1}{2}\nabla(q_\epsilon + \tilde{Q}_0)(\varphi_\epsilon + \tilde{\phi}_0) + \nabla p_\epsilon = f, \tag{1.5}$$

$$\frac{\partial q_\epsilon}{\partial t} + a_3\mathcal{A}_1 q_\epsilon + a_4 q_\epsilon - \mathcal{M}(q_\epsilon + \tilde{Q}_0, v_\epsilon) + \frac{1}{2}(\operatorname{div} v_\epsilon)(q_\epsilon + \tilde{Q}_0) = 0, \tag{1.6}$$

$$\mathcal{A}_1\varphi_\epsilon = q_\epsilon, \tag{1.7}$$

$$\epsilon\frac{\partial p_\epsilon}{\partial t} + \operatorname{div} v_\epsilon = 0, \tag{1.8}$$

with the initial and boundary value conditions

$$v_\epsilon = 0, \quad q_\epsilon = 0, \quad \varphi_\epsilon = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{1.9}$$

$$v_\epsilon(x,0) = v_0, \quad q_\epsilon(x,0) = q_0, \quad \varphi_\epsilon(x,0) = \varphi_0, \quad p_\epsilon(x,0) = p_0. \tag{1.10}$$

Here, for simplicity, we give a detailed explanation of the terms '$\frac{1}{2}(\operatorname{div} v_\epsilon)v_\epsilon$, $\frac{1}{2}\nabla(q_\epsilon + \tilde{Q}_0)(\varphi_\epsilon + \tilde{\phi}_0)$' and '$\frac{1}{2}(\operatorname{div} v_\epsilon)(q_\epsilon + \tilde{Q}_0)$' in Remark 3.1.

It is obvious that the perturbed compressible electrohydrodynamics system (1.5)-(1.8) recover the incompressible one (1.4) when $\epsilon = 0$. So it is natural to ask whether system (1.5)-(1.8) possesses a solution, and whether it converges to the solution $\{v, q, \varphi, p\}$ of the incompressible electrohydrodynamics system (1.4) as $\epsilon \to 0^+$? In the next parts of the paper, we shall give positive answers to these questions.

At last we mention that, from the point of view of physics, one can formally derive incompressible fluid models from compressible ones. The incompressible limit of the compressible fluid equations is a significant issue in the mathematical theory of fluid dynamics. In the past years, some nice works on this subject have been performed; see, for example, Jiang *et al.* [23, 24], Lions and Masmoudi [25], Temam [15] and Feireisl *et al.* [26, 27].

Now let us talk about the organization of this paper. In Section 2, we provide some mathematical tools needed in the sequel and recall the existence of weak solutions. In Section 3, we prove the unique existence of solution for equations (1.5)-(1.10), while in Section 4 we show how the solutions of the perturbed problems converge to the solutions of the incompressible electrohydrodynamics system (1.4) as $\epsilon \to 0^+$.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ is an open bounded smooth domain, firstly, we define

$$\mathbb{H} = \{u \in \mathbb{L}_0^2(\Omega), \operatorname{div} u = 0\}, \quad \text{with norm } \|\cdot\| = \|\cdot\|_{\mathbb{L}^2} \text{ (the usual } \mathbb{L}^2 \text{ norm)},$$

$$\mathbb{V} = \{u \in \mathbb{H}_0^1(\Omega), \operatorname{div} u = 0\}, \quad \text{with norm } \|\cdot\|_{\mathbb{V}} = \|\nabla\cdot\|, \text{and the dual space } \mathbb{V}^*,$$

$$\mathcal{V} = \{\phi \in \mathcal{C}_0^\infty(\Omega)\times\mathcal{C}_0^\infty(\Omega) : \phi = (\phi_1,\phi_2), \operatorname{div}\phi = 0\},$$

$$\mathcal{W} = \{\varphi \in \mathcal{C}_0^\infty(\Omega) : \operatorname{div}\varphi = 0\},$$

H, V denote separately the closure space of $\mathcal{V}$ in $\mathbb{L}^2$ norm and in $\mathbb{H}_0^1$ norm.

Secondly, we give a definition of some operators.

Define the linear 'Stokes operator' $\mathcal{A} = -\triangle$ from $\mathbb{V}$ to $\mathbb{V}^*$ by

$$\langle \mathcal{A}u, v\rangle_{\mathbb{V}^*,\mathbb{V}} = \sum_{i,j=1}^{2} \int_\Omega \partial_j u_i \partial_j v_i \, dx, \quad \forall u, v \in \mathbb{V}.$$

We define the bilinear operator $\mathcal{B}(u, v)$ from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}^*$ as

$$\big\langle \mathcal{B}(u, v), w\big\rangle_{\mathbb{V}^*,\mathbb{V}} = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad \forall u, v, w \in \mathbb{V},$$

we define the bilinear operator $\mathcal{N}(u, v)$ from $H_0^1 \times H_0^1$ into $\mathbb{V}^*$ as

$$\big\langle \mathcal{N}(q, \varphi), \beta\big\rangle_{\mathbb{V}^*,\mathbb{V}} = \sum_{i}^{2} \int_\Omega q \frac{\partial \varphi}{\partial x_i} \beta_i \, dx, \quad \forall q, \varphi \in H_0^1, \beta \in \mathbb{V},$$

we define the bilinear operator $\mathcal{M}(q, v)$ from $H_0^1 \times \mathbb{V}$ into H^{-1} as

$$\big\langle \mathcal{M}(q, v), \xi\big\rangle_{H^{-1},H_0^1} = \sum_{i}^{2} \int_\Omega q \frac{\partial \xi}{\partial x_i} v_i \, dx, \quad \forall q, \xi \in H_0^1, v \in \mathbb{V}.$$

Note that (see, *e.g.*, [13, 15, 28])

$$\mathcal{A}_0 u = -\Delta u \quad \text{for } u \in D(\mathcal{A}_0) = \big\{u \in \mathbb{H}_0^2(\Omega), \operatorname{div} u = 0\big\}.$$

By the Hilbert-Schmidt theorem, one can deduce that $\mathcal{A}_0$ has a sequence of orthonormal eigenfunctions w_j, belonging to $C_p^\infty(\Omega)$ with zero mean in Ω. Since $\mathcal{A}_0$ is a self-adjoint positive operator with compact inverse, $\{\omega_j\}_{j=1}^\infty$ forms a basis of the space $\mathbb{H}$. Moreover, $\{\omega_j\}_{j=1}^\infty$ also forms a basis of the space $D(\mathcal{A}_0^{s/2}) = \{\mathbb{H}_0^s(\Omega), \operatorname{div} u = 0\}$, for any positive integer s [28]. In a similar way $\mathcal{A}_1 : H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$ is the isomorphism given by

$$\langle \mathcal{A}_1 q, \xi\rangle = \big((q, \xi)\big) \quad \text{for all } \xi \in H_0^1(\Omega).$$

We will denote by $\mathcal{A}_1$ the corresponding operator defined in $D(\mathcal{A}_1) = H_0^1(\Omega) \cap H^2(\Omega)$ with range in $L^2(\Omega)$. As usual with problems involving the Navier-Stokes equations, we set

$$b(u, v, w) = \int_\Omega u_i D_i v_j w_j \, dx,$$
$$n(q, \varphi, v) = \int_\Omega q D_i \varphi v_i \, dx.$$

Problem 2.1 Let $T > 0$. For any given v_0 and q_0 with

$$v_0(x) \in H, \tag{2.1}$$
$$q_0(x) \in L^2(\Omega), \tag{2.2}$$

to find v, q, φ satisfying

$$v \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{2.3}$$

$$q \in L^2\big(0,T;H_0^1(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big), \tag{2.4}$$

$$\varphi \in L^2\big(0,T;H_0^1(\Omega)\big), \tag{2.5}$$

$$a_1\frac{\partial v}{\partial t} + a_1\mathcal{B}(v,v) + a_2\mathcal{A}_0 v + \mathcal{N}(q+\tilde{Q}_0, \varphi+\tilde{\phi}_0) = f, \tag{2.6}$$

$$\frac{\partial q}{\partial t} + a_3\mathcal{A}_1 q + a_4 q - \mathcal{M}(q+\tilde{Q}_0, v) = 0, \tag{2.7}$$

$$\mathcal{A}_1\varphi = q, \tag{2.8}$$

$$v(x,0) = v_0, \quad q(x,0) = q_0. \tag{2.9}$$

Lemma 2.1 ([1]) *Let* $a_i > 0$, $i = 1,2,3,4$, *for* $T > 0$, *Problem* 2.1 *admits unique solution.*

Remark 2.1 Let $a_i > 0$, $i = 1,2,3,4$, for any given v_0 and q_0 satisfying (2.1) and (2.2). Then, for each solution $\{v,q,\varphi\}$ obtained by Lemma 2.1, there exists a unique pressure p corresponding to v, and for each $t \in [0,T]$, $p \in L^2(\Omega)$. Moreover, $\nabla p \in \mathcal{C}([0,T];\mathbb{H}^{-1}(\Omega))$ and hence

$$p \in \mathcal{C}\big([0,T];L^2(\Omega)\big), \tag{2.10}$$

which satisfies in the distribution sense that

$$a_1\frac{\partial v}{\partial t} + a_1\mathcal{B}(v,v) + a_2\mathcal{A}_0 v + \mathcal{N}(q+\tilde{Q}_0, \varphi+\tilde{\phi}_0) + \nabla p = f, \tag{2.11}$$

$$\frac{\partial q}{\partial t} + a_3\mathcal{A}_1 q + a_4 q - \mathcal{M}(q+\tilde{Q}_0, v) = 0, \tag{2.12}$$

$$\mathcal{A}_1\varphi = q. \tag{2.13}$$

Therefore, for $\forall\psi(t) \in \mathcal{C}_c^\infty(0,T)$,

$$\begin{aligned} &-a_1\int_0^T \langle v,\omega\rangle\psi'(t)\,dt + a_1\int_0^T b\big(v,v,\omega\psi(t)\big)\,dt \\ &\qquad + a_2\int_0^T \big(\big(v,\omega\psi(t)\big)\big)\,dt + \int_0^T n\big(q+\tilde{Q}_0, \varphi+\tilde{\phi}_0, \omega\psi(t)\big)\,dt \\ &\quad = \int_0^T \big(f,\omega\psi(t)\big)\,dt + (v_0,\omega)\psi(0), \quad \forall\omega \in V, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &-\int_0^T \big\langle q(t),w\big\rangle\psi'(t)\,dt + a_3\int_0^T \big(\big(q,w\psi(t)\big)\big)\,dt \\ &\qquad + a_4\int_0^T \big(q,w\psi(t)\big)\,dt - \int_0^T n\big(q+\tilde{Q}_0, w\psi(t), v\big)\,dt \\ &\quad = (q_0,w)\psi(0), \quad \forall w \in H_0^1. \end{aligned} \tag{2.15}$$

Remark 2.2 We will prove that if $\{v, q, \varphi\}$ is a solution of Problem 2.1, then

$$v' \in L^1(0, T; V'), \quad q' \in L^1(0, T; H^{-1}(\Omega)). \tag{2.16}$$

By the classical embedding theorem (see [15]), we can conclude that $v \in \mathcal{C}(0, T; H)$, $q \in \mathcal{C}(0, T; L^2)$ and so the initial condition $v_0 \in H$, $q_0 \in L^2$ makes sense.

We need the compactness theorem [15] involving the fractional derivative.

Assume X_0, X, X_1 are Hilbert spaces with

$$X_0 \hookrightarrow X \hookrightarrow X_1, \tag{2.17}$$

the embedding being continuous, and

$$X_0 \hookrightarrow X \quad \text{is compact.} \tag{2.18}$$

Let $\psi(t)$ be a function from $\mathbb{R}$ to X_1; we denote by $\hat{\psi}(\tau)$ its Fourier transform,

$$\hat{\psi}(\tau) = \int_{-\infty}^{+\infty} \exp(-2\pi i t\tau)\psi(t)\,dt.$$

The derivative in t of order γ is the inverse Fourier transform of $(2i\pi\tau)^\gamma \hat{\psi}(\tau)$, that is,

$$\widehat{D_t^\gamma \psi(t)} = (2i\pi\tau)^\gamma \hat{\psi}(\tau).$$

For given $\gamma > 0$, define the space

$$M^\gamma = M^\gamma(\mathbb{R}; X_0, X_1) = \{\psi \in L^2(\mathbb{R}; X_0), D_t^\gamma \psi \in L^2(\mathbb{R}; X_1)\}.$$

Then M^γ is a Hilbert space with the norm

$$\|\psi\|_{M^\gamma} = \{\|\psi\|^2_{L^2(\mathbb{R};X_0)} + \||\tau|^\gamma \hat{\psi}(\tau)\|^2_{L^2(\mathbb{R};X_1)}\}^{\frac{1}{2}}.$$

For any set $K \subset \mathbb{R}$, the subspace M_K^γ of M^γ is defined as the set of functions $u \in M^\gamma$ with support contained in K:

$$M_K^\gamma = M_K^\gamma(\mathbb{R}; X_0, X_1) = \{\psi \in M^\gamma, \operatorname{supp}\psi \subseteq K\}.$$

Lemma 2.2 ([13]) *Assume X_0, X, X_1 are Hilbert spaces satisfying* (2.10) *and* (2.11). *Then, for any bounded set K and $\forall\gamma > 0$, we have the following compact embedding*:

$$M_K^\gamma(\mathbb{R}; X_0, X_1) \hookrightarrow L^2(\mathbb{R}; X).$$

Lemma 2.3 *Let $q, \varphi \in H_0^1(\Omega)$ and $v \in \mathbb{V}$, then*

$$n(q, q, v) = 0, \tag{2.19}$$

$$n(q, \varphi, v) = -n(\varphi, q, v). \tag{2.20}$$

Lemma 2.4 *When $n = 2$, if $q \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, then $q \in L^4(0, T; L^4(\Omega))$.*

3 Unique existence of solutions for the compressible electrohydrodynamics

In this section, we first give a description of the perturbed compressible electrohydrodynamics and then prove its unique existence of solutions.

For given $\varepsilon \in (0,1]$, we consider the following initial boundary value problem:

$$\begin{cases} a_1 \frac{\partial v_\epsilon}{\partial t} + a_1 \mathcal{B}(v_\epsilon, v_\epsilon) + \frac{1}{2} a_1 (\operatorname{div} v_\epsilon) v_\epsilon + a_2 \mathcal{A}_0 v_\epsilon + \mathcal{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0) \\ \quad - \frac{1}{2} \nabla (q_\epsilon + \tilde{Q}_0)(\varphi_\epsilon + \tilde{\phi}_0) + \nabla p_\epsilon = f, \quad \text{in } \Omega \times (0,T), \\ \frac{\partial q_\epsilon}{\partial t} + a_3 \mathcal{A}_1 q_\epsilon + a_4 q_\epsilon - \mathcal{M}(q_\epsilon + \tilde{Q}_0, v_\epsilon) \\ \quad + \frac{1}{2} (\operatorname{div} v_\epsilon)(q_\epsilon + \tilde{Q}_0) = 0, \quad \text{in } \Omega \times (0,T), \\ \mathcal{A}_1 \varphi_\epsilon = q_\epsilon, \quad \text{in } \Omega \times (0,T), \\ \epsilon \frac{\partial p_\epsilon}{\partial t} + \operatorname{div} v_\epsilon = 0, \quad \text{in } \Omega \times (0,T), \end{cases} \tag{3.1}$$

supplemented with the initial boundary value conditions

$$v_\epsilon = 0, \quad q_\epsilon = 0, \quad \varphi_\epsilon = 0, \quad x \in \partial\Omega, \tag{3.2}$$

$$v_\epsilon(x,0) = v_0, \quad q_\epsilon(x,0) = q_0, \quad \varphi_\epsilon(x,0) = \varphi_0, \quad p_\epsilon(x,0) = p_0, \quad x \in \Omega, \tag{3.3}$$

where v_0, q_0 satisfy (2.1) and (2.2), the function p_0 is independent of ϵ and

$$p_0 \in L^2(\Omega). \tag{3.4}$$

Remark 3.1 The terms $\frac{1}{2}(\operatorname{div} v_\epsilon)v_\epsilon$, $\frac{1}{2}\nabla(q_\epsilon + \tilde{Q}_0)(\varphi_\epsilon + \tilde{\phi}_0)$ and $\frac{1}{2}(\operatorname{div} v_\epsilon)(q_\epsilon + \tilde{Q}_0)$ in (3.1) are the stabilization terms corresponding to the substitution of the trilinear form $\hat{b}$ for the form b and the trilinear form $\hat{n}$ for the form n, where the trilinear form $\hat{b}$, $\hat{n}$ is given by

$$\hat{b}(u,v,w) = \frac{1}{2}\big[b(u,v,w) - b(u,w,v)\big], \quad \forall u,v,w \in \mathbb{H}_0^1(\Omega), \tag{3.5}$$

$$\hat{n}(q,\varphi,v) = \frac{1}{2}\big[n(q,\varphi,v) - n(\varphi,q,v)\big], \quad \forall v \in \mathbb{H}_0^1(\Omega), \quad q,\varphi \in H_0^1(\Omega). \tag{3.6}$$

Note that if $\operatorname{div} u \neq 0$, then $b(u,u,u) \neq 0$, $n(q,q,v) \neq 0$. But $\hat{b}(u,v,v) = 0$, $\hat{n}(q,q,v) = 0$, for $\forall u, v \in \mathbb{H}_0^1(\Omega)$, $q \in H_0^1(\Omega)$.

Let us assume that $\{v_\epsilon, \varphi_\epsilon, q_\epsilon, p_\epsilon\}$ is the classical solution of system (3.1), that is, $v_\epsilon \in \mathcal{C}^2(\overline{Q})$, $\varphi_\epsilon \in \mathcal{C}^2(\overline{Q})$, $q_\epsilon \in \mathcal{C}^2(\overline{Q})$, $p_\epsilon \in \mathcal{C}^1(\overline{Q})$. Then we have $\phi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega)$ and $\varrho \in \mathcal{C}_0^\infty(\Omega)$. Multiplying equation $(3.1)_1$ by ϕ, $(3.1)_2$-$(3.1)_4$ by ϱ, we obtain

$$\begin{aligned} & a_1 \frac{d}{dt}(v_\epsilon, \phi) + a_2\big((v_\epsilon, \phi)\big) + a_1 \hat{b}(v_\epsilon, v_\epsilon, \phi) + \hat{n}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \phi) + (\nabla p_\epsilon, \phi) = (f, \phi), \\ & \frac{d}{dt}(q_\epsilon, \varrho) + a_3\big((q_\epsilon, \varrho)\big) + a_4(q_\epsilon, \varrho) - \hat{n}(q_\epsilon + \tilde{Q}_0, \varrho, v_\epsilon) = 0, \\ & \big((\varphi_\epsilon, \varrho)\big) = (q_\epsilon, \varrho), \\ & \epsilon \frac{d}{dt}(p_\epsilon, \varrho) + (\operatorname{div} v_\epsilon, \varrho) = 0. \end{aligned}$$

Let $\hat{b}$, $\hat{n}$ be defined by (3.5), (3.6) and set $\hat{B}(u) = \hat{B}(u,u)$, $\hat{N}(q) = \hat{N}(q,q)$ via

$$\langle \hat{B}(u), v\rangle = \hat{b}(u,u,v), \quad \forall u, v \in \mathbb{H}_0^1(\Omega), \tag{3.7}$$

$$\langle \hat{N}(q), v\rangle = \hat{n}(q,q,v), \quad \forall v \in \mathbb{H}_0^1(\Omega), q \in H_0^1(\Omega). \tag{3.8}$$

Then like $B(u)$, $N(q)$, the operators $\hat{B}(u)$, $\hat{N}(q)$ are continuous on $\mathbb{H}_0^1(\Omega)$, $H_0^1(\Omega)$, respectively.

Lemma 3.1 *If $v \in L^2(0,T;\mathbb{H}_0^1(\Omega)) \cap L^\infty(0,T;\mathbb{H})$, $q \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ then the function $t \to \hat{B}(v(t))$ belongs to $L^{4/3}(0,T,\mathbb{H}^{-1}(\Omega))$, $t \to \hat{N}(q(t),\varphi(t))$ belongs to $L^{4/3}(0,T,\mathbb{H}^{-1}(\Omega))$, $t \to \hat{M}(q(t),v(t))$ belongs to $L^{4/3}(0,T,H^{-1}(\Omega))$.*

Proof For almost all $t \in (0,T)$, $\hat{B}(v(t))$, $\hat{N}(q(t))$ are the elements of $\mathbb{H}^{-1}(\Omega)$, $\hat{M}(q(t),v(t))$ is an element of $H^{-1}(\Omega)$, and the measurability of the functions $t \to \hat{B}(v(t))$, $t \to \hat{N}(q(t))$, $t \to \hat{M}(q(t),v(t))$ is easy to check. Now for $\forall \phi \in \mathbb{H}_0^1(\Omega)$, $\forall \psi \in H_0^1(\Omega)$ by the Hölder inequality and

$$\|u\|_{\mathbb{L}^4} \le C|u|^{1/2}\|u\|^{1/2}, \quad \forall u \in \mathbb{R}^2,$$

we have

$$\begin{aligned} \bigl|\langle \hat{B}(v(t)),\phi\rangle\bigr| &= \left|\frac{1}{2}\bigl[b(v(t),v(t),\phi) - b(v(t),\phi,v(t))\bigr]\right| \\ &\le \frac{1}{2}\sum_{i,j=1}^{2}\left(\int_\Omega \left|v_i\frac{\partial v_j}{\partial x_i}\phi_j\right| dx + \int_\Omega \left|v_i\frac{\partial \phi_j}{\partial x_i}v_j\right| dx\right) \\ &\le C\bigl(\|v\|_{\mathbb{L}^4(\Omega)}\|v\|\|\phi\|_{\mathbb{L}^4(\Omega)} + \|v\|^2_{\mathbb{L}^4(\Omega)}\|\phi\|\bigr) \\ &\le C\bigl(|v|^{\frac{1}{2}}\|v\|^{\frac{3}{2}} + |v|\|v\|\bigr)\|\phi\|, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \bigl|\langle \hat{N}(q(t),\varphi(t)),\phi\rangle\bigr| &= \left|\frac{1}{2}\bigl[n(q(t),\varphi(t),\phi) - n(\varphi(t),q(t),\phi(t))\bigr]\right| \\ &\le \frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega \left|q\frac{\partial \varphi}{\partial x}\phi_i\right| dx + \int_\Omega \left|\varphi\frac{\partial q}{\partial x}\phi_i\right| dx\right) \\ &\le C\bigl(\|q\|_{\mathbb{L}^4(\Omega)}\|\varphi\|_{\mathbb{L}^4(\Omega)}|\phi| + \|\varphi\|_{\mathbb{L}^4(\Omega)}\|q\|\|\phi\|_{\mathbb{L}^4(\Omega)}\bigr) \\ &\le C\bigl(|q|^{\frac{1}{2}}\|q\|^{\frac{1}{2}}|\varphi|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} + |\varphi|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|q\|\bigr)\|\phi\|, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\bigl|\langle \hat{M}(q(t),v(t)),\psi\rangle\bigr| \\ &\quad = \left|\frac{1}{2}\bigl[n(q(t),\psi(t),v(t)) - n(\psi(t),q(t),v(t))\bigr]\right| \\ &\quad \le \frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega \left|q\frac{\partial \varphi}{\partial x}\phi_i\right| dx + \int_\Omega \left|\varphi\frac{\partial q}{\partial x}\phi_i\right| dx\right) \end{aligned}$$

$$\leq C\big(\|q\|_{\mathbb{L}^4(\Omega)}\|v\|_{\mathbb{L}^4(\Omega)}\|\psi\| + \|\psi\|_{\mathbb{L}^4(\Omega)}\|q\|\|v\|_{\mathbb{L}^4(\Omega)}\big)$$
$$\leq C\big(|q|^{\frac{1}{2}}\|q\|^{\frac{1}{2}}|v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}} + |v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|q\|\big)\|\psi\|, \tag{3.11}$$

(3.9), (3.10), and (3.11) imply that

$$\|\hat{B}(v(t))\|_{\mathbb{H}^{-1}(\Omega)} \leq C\big(|v|^{\frac{1}{2}}\|v\|^{\frac{3}{2}} + |v|\|v\|\big), \tag{3.12}$$
$$\|\hat{N}(q(t),\varphi(t))\|_{\mathbb{H}^{-1}(\Omega)} \leq C\big(|q|^{\frac{1}{2}}\|q\|^{\frac{1}{2}}|\varphi|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} + |\varphi|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|q\|\big), \tag{3.13}$$
$$\|\hat{M}(q(t),v(t))\|_{H^{-1}(\Omega)} \leq C\big(|q|^{\frac{1}{2}}\|q\|^{\frac{1}{2}}|v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}} + |v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|q\|\big). \tag{3.14}$$

Therefore, the lemma follows from (3.12)-(3.14). □

Now if $v_\epsilon \in L^2(0,T;\mathbb{H}_0^1(\Omega)) \cap L^\infty(0,T;\mathbb{H})$, $q_\epsilon \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\varphi_\epsilon \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, and $p_\epsilon \in L^2(0,T;L^2(\Omega))$ satisfy (3.1)-(3.2) in the distribution sense, then

$$a_1\frac{d}{dt}(v_\epsilon,\phi) = \langle f - a_1\hat{B}(v_\epsilon,v_\epsilon) - a_2\mathcal{A}_0 v_\epsilon - \hat{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0) - \nabla p_\epsilon, \phi\rangle, \tag{3.15}$$
$$\frac{d}{dt}(q_\epsilon,\psi) = -\langle a_3\mathcal{A}_1 q_\epsilon + a_4 q_\epsilon - \hat{M}(q_\epsilon + \tilde{Q}_0, v_\epsilon), \psi\rangle. \tag{3.16}$$

Since $p_\epsilon \in L^2(0,T;L^2(\Omega))$, we see from Lemma 3.1 that

$$v_\epsilon' = \frac{1}{a_1}\big(f - a_1\hat{B}(v_\epsilon,v_\epsilon) - a_2\mathcal{A}_0 v_\epsilon - \hat{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0) - \nabla p_\epsilon\big)$$
$$\in L^{4/3}\big(0,T;\mathbb{H}^{-1}\big), \tag{3.17}$$
$$q_\epsilon' = -\big(a_3\mathcal{A}_1 q_\epsilon + a_4 q_\epsilon - \hat{M}(q_\epsilon + \tilde{Q}_0, v_\epsilon)\big) \in L^{4/3}\big(0,T;H^{-1}\big); \tag{3.18}$$

similarly,

$$p_\epsilon' \in L^2\big(0,T;L^2(\Omega)\big). \tag{3.19}$$

The above analysis leads to the following weak formulation of the problem described by (3.1).

Problem 3.1 Let $\epsilon \in (0,1]$ be fixed. For any given v_0, q_0 and p_0 satisfying (2.1), (2.2), and (3.4), find $\{v_\epsilon, q_\epsilon, \varphi_\epsilon, p_\epsilon\}$ such that

$$v_\epsilon \in L^2\big(0,T;\mathbb{H}_0^1(\Omega)\big) \cap L^\infty(0,T;\mathbb{H}), \quad v_\epsilon' \in L^{4/3}\big(0,T;\mathbb{H}^{-1}(\Omega)\big), \tag{3.20}$$
$$q_\epsilon \in L^2\big(0,T;H_0^1(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big), \quad q_\epsilon' \in L^{4/3}\big(0,T;H^{-1}(\Omega)\big), \tag{3.21}$$
$$\varphi_\epsilon \in L^2\big(0,T;H_0^1(\Omega)\big); \tag{3.22}$$
$$p_\epsilon \in L^2\big(0,T;L^2(\Omega)\big), \quad p_\epsilon' \in L^2\big(0,T;L^2(\Omega)\big), \tag{3.23}$$
$$a_1 v_\epsilon' + a_1\mathcal{B}(v_\epsilon,v_\epsilon) + \frac{1}{2}a_1(\operatorname{div} v_\epsilon)v_\epsilon + a_2\mathcal{A}_0 v_\epsilon + \mathcal{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0)$$
$$-\frac{1}{2}\nabla(q_\epsilon + \tilde{Q}_0)(\varphi_\epsilon + \tilde{\phi}_0) + \nabla p_\epsilon = f, \tag{3.24}$$

$$q'_\epsilon + a_3\mathcal{A}_1 q_\epsilon + a_4 q_\epsilon - \mathcal{M}(q_\epsilon + \tilde{Q}_0, v_\epsilon) + \frac{1}{2}(\operatorname{div} v_\epsilon)(q_\epsilon + \tilde{Q}_0) = 0, \tag{3.25}$$

$$\mathcal{A}_1\varphi_\epsilon = q_\epsilon, \tag{3.26}$$

$$\epsilon p'_\epsilon + \operatorname{div} v_\epsilon = 0, \tag{3.27}$$

$$v_\epsilon(x,0) = v_0, \quad q_\epsilon(x,0) = q_0, \quad \varphi_\epsilon(x,0) = \varphi_0, \quad p_\epsilon(x,0) = p_0, \quad x \in \Omega. \tag{3.28}$$

Theorem 3.1 *Let* $\epsilon \in (0,1]$ *be fixed. For any given* v_0, q_0, *and* p_0 *satisfying* (2.1), (2.2), *and* (3.4). *Then there exists a unique solution* $\{v_\epsilon, q_\epsilon, \varphi_\epsilon, p_\epsilon\}$ *of Problem* 3.1.

Proof (i) In order to apply the Galerkin procedure, we consider a basis of $\mathbb{H}_0^1(\Omega)$ constituted of elements ω_i of $\mathcal{D}(\Omega)$, a basis of $H_0^1(\Omega)$ constituted of elements w_i of $\mathcal{C}_0^\infty(\Omega)$, and a basis of $L^2(\Omega)$ constituted of elements γ_i of $\mathcal{C}_0^\infty(\Omega)$.

For each m, we define an approximate solution $v_{\epsilon m}$, $q_{\epsilon m}$, $\varphi_{\epsilon m}$, $p_{\epsilon m}$ of Problem 3.1 by

$$\begin{aligned} &v_{\epsilon m}(t) = \sum_{i=1}^m g_{im}(t)\omega_i, \quad \varphi_{\epsilon m}(t) = \sum_{i=1}^m h_{im}(t) w_i, \\ &p_{\epsilon m}(t) = \sum_{j=1}^m \xi_{jm}(t)\gamma_j, \quad q_{\epsilon m}(t) = \mathcal{A}_1\varphi_{\epsilon m}(t), \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &a_1\big(v'_{\epsilon m}, \omega_k\big) + a_2\big((v_{\epsilon m}, \omega_k)\big) + a_1\hat{b}(v_{\epsilon m}, v_{\epsilon m}, \omega_k) + \hat{n}(q_{\epsilon m} + \tilde{Q}_0, \varphi_{\epsilon m} + \tilde{\phi}_0, \omega_k) \\ &\quad + (\nabla p_{\epsilon m}, \omega_k) = (f, \omega_k), \end{aligned} \tag{3.30}$$

$$\big(q'_{\epsilon m}, w_k\big) + a_3\big((q_{\epsilon m}, w_k)\big) + a_4(q_{\epsilon m}, w_k) - \hat{n}(q_{\epsilon m} + \tilde{Q}_0, w_k, v_{\epsilon m}) = 0, \tag{3.31}$$

$$\big((\varphi_{\epsilon m}, w_k)\big) = (q_{\epsilon m}, w_k), \tag{3.32}$$

$$\epsilon\big(p'_{\epsilon m}, \gamma_l\big) + (\operatorname{div} v_{\epsilon m}, \gamma_l) = 0, \quad k, l = 1, \ldots, m. \tag{3.33}$$

Moreover, this differential system is required to satisfy the initial conditions

$$v_{\epsilon m}(0) = v_{0m}, \quad q_{\epsilon m}(0) = q_{0m}, \quad \varphi_{\epsilon m}(0) = \varphi_{0m}, \quad p_{\epsilon m}(0) = p_{0m}. \tag{3.34}$$

Here v_{0m} (or q_{0m} or φ_{0m} or p_{0m}) is the orthogonal projection of v_0 (or q_0 or φ_0 or p_0) onto the space spanned by $\omega_1, \ldots, \omega_m$ (or $w_1, \ldots, w_m$ or $w_1, \ldots, w_m$ or $\gamma_1, \ldots, \gamma_m$) in $\mathbb{H}_0^1$ (resp. H_0^1 or H_0^1 or $L^2(\Omega)$).

Equations (3.30)-(3.33) form a nonlinear differential system for the functions $g_{1m}, \ldots, g_{mm}, h_{1m}, \ldots, h_{mm}, \xi_{1m}, \ldots, \xi_{mm}$. By the standard theory of ODE, we have the existence of a solution defined at least on some interval $[0, t_m)$, $0 < t_m \leq T$. And the following *a priori* estimates show that in fact $t_m = T$.

(ii) If we multiply (3.30) by $g_{km}(t)$, multiply (3.31)-(3.32) by $h_{km}(t)$, multiply (3.33) by $\xi_{lm}(t)$, we have

$$\begin{aligned} &a_1\big(v'_{\epsilon m}, v_{\epsilon m}\big) + a_2\big((v_{\epsilon m}, v_{\epsilon m})\big) + a_1\hat{b}(v_{\epsilon m}, v_{\epsilon m}, v_{\epsilon m}) + \hat{n}(q_{\epsilon m} + \tilde{Q}_0, \varphi_{\epsilon m} + \tilde{\phi}_0, v_{\epsilon m}) \\ &\quad + (\nabla p_{\epsilon m}, v_{\epsilon m}) = (f, v_{\epsilon m}), \end{aligned} \tag{3.35}$$

$$\big(q'_{\epsilon m},\varphi_{\epsilon m}\big)+a_3\big((q_{\epsilon m},\varphi_{\epsilon m})\big)+a_4(q_{\epsilon m},\varphi_{\epsilon m})-\hat{n}(q_{\epsilon m}+\tilde{Q}_0,\varphi_{\epsilon m},v_{\epsilon m})=0, \tag{3.36}$$

$$\big((\varphi_{\epsilon m},\varphi_{\epsilon m})\big)=(q_{\epsilon m},\varphi_{\epsilon m}), \tag{3.37}$$

$$\epsilon\big(p'_{\epsilon m},p_{\epsilon m}\big)+(\operatorname{div}v_{\epsilon m},p_{\epsilon m})=0,\quad k,l=1,\dots,m. \tag{3.38}$$

Due to (3.5), $\hat{b}(v_{\epsilon m},v_{\epsilon m},v_{\epsilon m})=0$, and since $v_{\epsilon m}|_{\partial\Omega}=0$, we have

$$(\nabla p_{\epsilon m},v_{\epsilon m})+(\operatorname{div}v_{\epsilon m},p_{\epsilon m})=0.$$

Then add all these equations for $k,l=1,\dots,m$, and as a result

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big[a_1\big|v_{\epsilon m}(t)\big|^2+\big\|\varphi_{\epsilon m}(t)\big\|^2+\epsilon\big|p_{\epsilon m}(t)\big|^2\big]+a_2\|v_{\epsilon m(t)}\|^2+a_3\big|q_{\epsilon m}(t)\big|^2+a_4\big\|\varphi_{\epsilon m}(t)\big\|^2\\
&\quad=-\hat{n}\big(q_{\epsilon m}(t),\tilde{\phi}_0,v_{\epsilon m}(t)\big)-\hat{n}\big(\tilde{Q}_0,\tilde{\phi}_0,v_{\epsilon m}(t)\big)+\big(f,v_{\epsilon m}(t)\big)\\
&\quad\le C\bigg[\delta\big(\big|q_{\epsilon m}(t)\big|^2+\big\|q_{\epsilon m}(t)\big\|^2\big)+\frac{2}{\delta}\big|v_{\epsilon m}(t)\big|^2+1\bigg].
\end{aligned} \tag{3.39}$$

Taking δ sufficiently small and by integration of (3.39) from 0 to s one shows that

$$\begin{aligned}
&a_1\big|v_{\epsilon m}(s)\big|^2+\big\|\varphi_{\epsilon m}(s)\big\|^2+\epsilon\big|p_{\epsilon m}(s)\big|^2+2a_2\int_0^s\big\|v_{\epsilon m}(t)\big\|^2\,dt\\
&\qquad+2a_3\int_0^s\big|q_{\epsilon m}(t)\big|^2\,dt+2a_4\int_0^s\big\|\varphi_{\epsilon m}(t)\big\|^2\,dt\\
&\quad\le|v_0|^2+\|\varphi_0\|^2+\epsilon|p_0|^2+\frac{2C}{\delta}\int_0^s\big|v_{\epsilon m}(t)\big|^2\,dt+sC\\
&\quad\le|v_0|^2+\|\varphi_0\|^2+\epsilon|p_0|^2+\frac{2C}{\lambda_1\delta}\int_0^s\big\|v_{\epsilon m}(t)\big\|^2\,dt+sC.
\end{aligned}$$

Hence $t_m=T$, and

$$\sup_{s\in[0,T]}\big\{a_1\big|v_{\epsilon m}(s)\big|^2+\big\|\varphi_{\epsilon m}(s)\big\|^2+\epsilon\big|p_{\epsilon m}(s)\big|^2\big\}\le d_1, \tag{3.40}$$

$$d_1=a_1|v_0|^2+\|\varphi_0\|^2+|p_0|^2+sC. \tag{3.41}$$

Integrating (3.39) in t from 0 to T, we have

$$\int_0^T\big\|v_{\epsilon m}(t)\big\|^2\,dt\le\frac{d_1}{2a_2-\frac{2C}{\lambda_1\delta}}, \tag{3.42}$$

$$\int_0^T\big|q_{\epsilon m}(t)\big|^2\,dt\le\frac{d_1}{2a_3}, \tag{3.43}$$

$$\int_0^T\big\|\varphi_{\epsilon m}(t)\big\|^2\,dt\le\frac{d_1}{2a_4}. \tag{3.44}$$

Putting $w_k=q_{\epsilon m}$ in equation (3.31), we find, using $n(q,q,v)=0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|q_{\epsilon m}|^2+a_3\|q_{\epsilon m}\|^2+a_4|q_{\epsilon m}|^2&=\hat{n}(\tilde{Q}_0,q_{\epsilon m},v_{\epsilon m})\\
&\le C\bigg[\delta\big(|q_{\epsilon m}|^2+\|q_{\epsilon m}\|^2\big)+\frac{2}{\delta}|v_{\epsilon m}|^2+1\bigg].
\end{aligned} \tag{3.45}$$

Taking δ sufficiently small and by integration of (3.45) from 0 to s one shows that

$$\begin{aligned}
&\left|q_{\epsilon m}(s)\right|^2 + 2a_3\int_0^s \left\|q_{\epsilon m}(s)\right\|^2 dt + 2a_4\int_0^s \left|q_{\epsilon m}(s)\right|^2 dt \\
&\quad\le \frac{2C}{\delta}\int_0^s \left|v_{\epsilon m}(s)\right|^2 dt + |q_0|^2 + Cs \\
&\quad\le \frac{2C}{\lambda_1\delta}\int_0^s \left\|v_{\epsilon m}(s)\right\|^2 dt + |q_0|^2 + Cs \triangleq d_2.
\end{aligned} \tag{3.46}$$

Then we can obtain

$$\sup_{s\in[0,T]}\left|q_{\epsilon m}(s)\right|^2 \le d_2. \tag{3.47}$$

Integrating (3.45) in t from 0 to T, we have

$$\int_0^T \left\|q_{\epsilon m}(t)\right\|^2 dt \le \frac{d_2}{2a_3}. \tag{3.48}$$

(iii) In order to pass to the limit in the nonlinear term we need an estimate of the fractional derivative in time of $v_{\epsilon m}$ and $q_{\epsilon m}$.

Setting

$$\begin{aligned}
&\phi_{\epsilon m}(t) = f - a_2\mathcal{A}_0 v_{\epsilon m} - \hat{B}(v_{\epsilon m}) - \hat{N}(q_{\epsilon m} + \tilde{Q}_0, \varphi_{\epsilon m} + \tilde{\phi}_0), \\
&\psi_{\epsilon m}(t) = -a_3\mathcal{A}_1 q_{\epsilon m} - a_4 q_{\epsilon m} - \hat{M}(q_{\epsilon m} + \tilde{Q}_0, v_{\epsilon m}).
\end{aligned}$$

We write equations (3.30)-(3.33) as

$$\begin{aligned}
&a_1\left(v'_{\epsilon m}(t), \omega_k\right) + \left(\nabla p_{\epsilon m}(t), \omega_k\right) = \left(\phi_{\epsilon m}(t), \omega_k\right), \quad k = 1,\ldots,m, \\
&\left(q'_{\epsilon m}(t), w_k\right) = \left(\psi_{\epsilon m}(t), w_k\right), \quad k = 1,\ldots,m, \\
&\epsilon\left(p'_{\epsilon m}(t), \gamma_l\right) + \left(\operatorname{div} v_{\epsilon m}(t), \gamma_l\right) = 0, \quad l = 1,\ldots,m, \\
&\varphi_{\epsilon m}(\cdot,t) = \mathcal{A}_1^{-1} q_{\epsilon m}(\cdot,t).
\end{aligned}$$

As done several times before, we extend all functions by 0 outside the interval $[0,T]$ and consider the Fourier transform of the different equations.

The following relations then hold on $\mathbb{R}$:

$$\begin{aligned}
&a_1\frac{d}{dt}\left(\tilde{v}_{\epsilon m}(t), \omega_k\right) + \left(\nabla\tilde{p}_{\epsilon m}(t), \omega_k\right) \\
&\quad= \left(\tilde{\phi}_{\epsilon m}(t), \omega_k\right) + (v_{0m}, \omega_k)\delta_{(0)} - \left(v_{\epsilon m}(T), \omega_k\right)\delta_{(T)}, \\
&\frac{d}{dt}\left(\tilde{q}_{\epsilon m}(t), w_k\right) = \left(\tilde{\psi}_{\epsilon m}(t), w_k\right) + (q_{0m}, w_k)\delta_{(0)} - \left(q_{\epsilon m}(T), w_k\right)\delta_{(T)}, \\
&\epsilon\frac{d}{dt}\left(\tilde{p}_{\epsilon m}(t), \gamma_l\right) + \left(\operatorname{div}\tilde{v}_{\epsilon m}(t), \gamma_l\right) = \epsilon(p_{0m}, \gamma_l)\delta_{(0)} - \epsilon\left(p_{\epsilon m}(T), \gamma_l\right)\delta_{(T)}, \\
&\tilde{\varphi}_{\epsilon m}(\cdot,t) = \mathcal{A}_1^{-1}\tilde{q}_{\epsilon m}(\cdot,t).
\end{aligned}$$

After taking Fourier transforms, as a result

$$\begin{aligned}
&2a_1 i\pi\tau\big(\hat{v}_{\epsilon m}(\tau),\omega_k\big)+\big(\nabla\hat{p}_{\epsilon m}(\tau),\omega_k\big)\\
&\quad=\big\langle\hat{\phi}_{\epsilon m}(\tau),\omega_k\big\rangle+(v_{0m},\omega_k)-\big(v_{\epsilon m}(T),\omega_k\big)\exp(-2i\pi\tau T),\\
&2i\pi\tau\big(\hat{q}_{\epsilon m}(\tau),w_k\big)=\big\langle\hat{\psi}_{\epsilon m}(\tau),w_k\big\rangle+(q_{0m},w_k)-\big(q_{\epsilon m}(T),w_k\big)\exp(-2i\pi\tau T),\\
&2i\pi\tau\epsilon\big(\hat{p}_{\epsilon m}(\tau),\gamma_l\big)+\big(\operatorname{div}\hat{v}_{\epsilon m}(\tau),\gamma_l\big)=\epsilon(p_{0m},\gamma_l)-\epsilon\big(p_{\epsilon m}(T),\gamma_l\big)\exp(-2i\pi\tau T).
\end{aligned}$$

We multiply the first of the last three equations by $\hat{g}_{km}(\tau)$ ($\hat{g}_{km}$ = Fourier transform of $\tilde{g}_{km}$) and the second by $\hat{h}_{km}(\tau)$ ($\hat{h}_{km}$ = Fourier transform of $\tilde{h}_{km}$) and the third by $\hat{\xi}_{lm}(\tau)$ ($\hat{\xi}_{lm}$ = Fourier transform of $\tilde{\xi}_{lm}$), and then add these relations for $k=1,\dots,m$, $l=1,\dots,m$, we obtain

$$\begin{aligned}
&2i\pi\tau\big\{a_1\big|\hat{v}_{\epsilon m}(\tau)\big|^2+\big\|\hat{\varphi}_{\epsilon m}(\tau)\big\|^2+\epsilon\big|\hat{p}_{\epsilon m}(\tau)\big|^2\big\}\\
&\qquad+\big(\nabla\hat{p}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big)+\big(\operatorname{div}\hat{v}_{\epsilon m}(\tau),\hat{p}_{\epsilon m}(\tau)\big)\\
&\quad=\big\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big\rangle\\
&\qquad+\big\langle\hat{\psi}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau)\big\rangle+\big(v_{0m},\hat{v}_{\epsilon m}(\tau)\big)+\big(q_{0m},\hat{\varphi}_{\epsilon m}(\tau)\big)+\epsilon\big(p_{0m},\hat{p}_{\epsilon m}(\tau)\big)\\
&\qquad-\big\{\big(v_{\epsilon m}(T),\hat{v}_{\epsilon m}(\tau)\big)+\big(q_{\epsilon m}(T),\hat{\varphi}_{\epsilon m}(\tau)\big)\\
&\qquad+\epsilon\big(p_{\epsilon m}(T),\hat{p}_{\epsilon m}(\tau)\big)\big\}\exp(-2i\pi\tau T).
\end{aligned}\tag{3.49}$$

The term $(\nabla\hat{p}_{\epsilon m},\hat{v}_{\epsilon m})+(\operatorname{div}\hat{v}_{\epsilon m},\hat{p}_{\epsilon m})=0$, and we have $\hat{v}_{\epsilon m}(\tau)|_{\partial\Omega}=0$.

We deduce from (3.49) that

$$\begin{aligned}
&2\pi|\tau|\big\{a_1\big|\hat{v}_{\epsilon m}(\tau)\big|^2+\big\|\hat{\varphi}_{\epsilon m}(\tau)\big\|^2+\epsilon\big|\hat{p}_{\epsilon m}(\tau)\big|^2\big\}\\
&\quad\le\big|\big\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big\rangle\big|+\big|\big\langle\hat{\psi}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau)\big\rangle\big|+|v_{0m}|\big|\hat{v}_{\epsilon m}(\tau)\big|+|q_{0m}|\big|\hat{\varphi}_{\epsilon m}(\tau)\big|\\
&\qquad+\epsilon|p_{0m}|\big|\hat{p}_{\epsilon m}(\tau)\big|+\big|v_{\epsilon m}(T)\big|\big|\hat{v}_{\epsilon m}(\tau)\big|+\big|q_{\epsilon m}(T)\big|\big|\hat{\varphi}_{\epsilon m}(\tau)\big|+\epsilon\big|p_{\epsilon m}(T)\big|\big|\hat{p}_{\epsilon m}(\tau)\big|\\
&\quad\le\big|\big\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big\rangle\big|+\big|\big\langle\hat{\psi}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau)\big\rangle\big|+2\sqrt{d_1}\big(\big|\hat{v}_{\epsilon m}(\tau)\big|+\big|\hat{\varphi}_{\epsilon m}(\tau)\big|+\epsilon\big|\hat{p}_{\epsilon m}(\tau)\big|\big)\\
&\quad\le\big|\big\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big\rangle\big|+\big|\big\langle\hat{\psi}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau)\big\rangle\big|+2c_1\sqrt{d_1}\big(\big|\hat{v}_{\epsilon m}(\tau)\big|+\big|\hat{\varphi}_{\epsilon m}(\tau)\big|+\epsilon\big|\hat{p}_{\epsilon m}(\tau)\big|\big).
\end{aligned}$$

We next estimate the term $|\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\rangle|$ and $|\langle\hat{\psi}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau)\rangle|$. In fact,

$$\begin{aligned}
&\big|\big\langle\hat{\phi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big\rangle\big|\\
&\quad=\big|\big\langle\hat{f}(\tau)-a_2\mathcal{A}_0\hat{v}_{\epsilon m}(\tau)-\hat{B}\big(\hat{v}_{\epsilon m}(\tau)\big)-\hat{N}\big(\hat{q}_{\epsilon m}(\tau)+\tilde{Q}_0,\hat{\varphi}_{\epsilon m}(\tau)+\tilde{\phi}_0\big),\hat{v}_{\epsilon m}(\tau)\big\rangle\big|\\
&\quad\le\big|\big(\hat{f}(\tau),\hat{v}_{\epsilon m}(\tau)\big)\big|+a_2\big\|\hat{v}_{\epsilon m}(\tau)\big\|^2+\hat{b}\big(\hat{v}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big)\\
&\qquad+\hat{n}\big(\hat{q}_{\epsilon m}(\tau)+\hat{Q}_0,\hat{\varphi}_{\epsilon m}(\tau)+\hat{\phi}_0,\hat{v}_{\epsilon m}(\tau)\big)\\
&\quad\le\big|\hat{f}(\tau)\big|^2+\big|\hat{v}_{\epsilon m}(\tau)\big|^2+a_2\big\|\hat{v}_{\epsilon m}(\tau)\big\|^2+\hat{n}\big(\hat{q}_{\epsilon m}(\tau),\hat{\varphi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big)\\
&\qquad+\hat{n}\big(\hat{q}_{\epsilon m}(\tau),\hat{\phi}_0,\hat{v}_{\epsilon m}(\tau)\big)\\
&\qquad+\hat{n}\big(\hat{Q}_0,\hat{\varphi}_{\epsilon m}(\tau),\hat{v}_{\epsilon m}(\tau)\big)+\hat{n}\big(\hat{Q}_0,\hat{\phi}_0,\hat{v}_{\epsilon m}(\tau)\big)\\
&\quad\le\big|\hat{f}(\tau)\big|^2+\big|\hat{v}_{\epsilon m}(\tau)\big|^2+a_2\big\|\hat{v}_{\epsilon m}(\tau)\big\|^2+C\big(\big\|\hat{q}_{\epsilon m}(\tau)\big\|_{L^4}^4+\big\|\hat{\varphi}_{\epsilon m}(\tau)\big\|^2+\big\|\hat{v}_{\epsilon m}(\tau)\big\|_{L^4}^4
\end{aligned}$$

$$+ \left|\hat{q}_{\epsilon m}(\tau)\right|^2 + \left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2\big), \tag{3.50}$$

$$\begin{aligned}
&\left|\langle\hat{\psi}_{\epsilon m}(\tau), \hat{\varphi}_{\epsilon m}(\tau)\rangle\right| \\
&\quad = \left|\langle -a_3\mathcal{A}_1\hat{q}_{\epsilon m}(\tau) - a_4\hat{q}_{\epsilon m}(\tau) - \hat{M}\big(\hat{q}_{\epsilon m}(\tau) + \tilde{Q}_0, \hat{v}_{\epsilon m}(\tau)\big), \hat{\varphi}_{\epsilon m}(\tau)\rangle\right| \\
&\quad \le a_3\left|\hat{q}_{\epsilon m}(\tau)\right|^2 + a_4\left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2 + \hat{n}\big(\hat{q}_{\epsilon m}(\tau), \hat{\varphi}_{\epsilon m}(\tau), \hat{v}_{\epsilon m}(\tau)\big) + \hat{n}\big(\hat{Q}_0, \hat{\varphi}_{\epsilon m}(\tau), \hat{v}_{\epsilon m}(\tau)\big) \\
&\quad \le a_3\left|\hat{q}_{\epsilon m}(\tau)\right|^2 + a_4\left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2 \\
&\qquad + C\big(\left\|\hat{q}_{\epsilon m}(\tau)\right\|_{L^4}^4 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2 + \left\|\hat{v}_{\epsilon m}(\tau)\right\|_{L^4}^4 + \left|\hat{v}_{\epsilon m}(\tau)\right|^2\big).
\end{aligned} \tag{3.51}$$

Combining (3.49) and (3.50)-(3.51), we get

$$\begin{aligned}
&2\pi|\tau|\big\{a_1\left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2 + \left|\hat{p}_{\epsilon m}(\tau)\right|^2\big\} \\
&\quad \le \left|\hat{f}(\tau)\right|^2 + \left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{v}_{\epsilon m}(\tau)\right\|^2 + \left|\hat{q}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2 \\
&\qquad + C\big(\left\|\hat{q}_{\epsilon m}(\tau)\right\|_{L^4}^4 + \left\|\hat{v}_{\epsilon m}(\tau)\right\|_{L^4}^4 + 2\sqrt{d_1}\big(\left|\hat{v}_{\epsilon m}(\tau)\right| + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\| + \epsilon\left|\hat{p}_{\epsilon m}(\tau)\right|\big)\big).
\end{aligned} \tag{3.52}$$

For some fixed $\gamma \in (0, \frac{1}{4})$, we have $|\tau|^{2\gamma} \le (2\gamma + 1)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}$, $\forall \tau \in \mathbb{R}$. Thus

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\big\{a_1\left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2\big\}\,d\tau \\
&\quad \le (2\gamma + 1)\int_{-\infty}^{+\infty} \frac{a_1|\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{\varphi}_{\epsilon m}(\tau)\|^2}{1 + |\tau|^{1-2\gamma}}\,d\tau \\
&\qquad + (2\gamma + 1)\int_{-\infty}^{+\infty} \frac{|\tau|(a_1|\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{\varphi}_{\epsilon m}(\tau)\|^2)}{1 + |\tau|^{1-2\gamma}}\,d\tau \\
&\quad \le (2\gamma + 1)\int_{-\infty}^{+\infty} \big\{a_1\left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2\big\}\,d\tau \\
&\qquad + (2\gamma + 1)\int_{-\infty}^{+\infty} \frac{|\tau|(a_1|\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{\varphi}_{\epsilon m}(\tau)\|^2)}{1 + |\tau|^{1-2\gamma}}\,d\tau.
\end{aligned} \tag{3.53}$$

By the Parseval equality and the Poincaré inequality, we have

$$\begin{aligned}
&(2\gamma + 1)\int_{-\infty}^{+\infty} \big\{a_1\left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2\big\}\,d\tau \\
&\quad = (2\gamma + 1)\int_{-\infty}^{+\infty} \big\{a_1\left|\tilde{v}_{\epsilon m}(t)\right|^2 + \left\|\tilde{\varphi}_{\epsilon m}(t)\right\|^2\big\}\,dt \\
&\quad = (2\gamma + 1)\int_0^T \big\{a_1\left|v_{\epsilon m}(t)\right|^2 + \left\|\varphi_{\epsilon m}(t)\right\|^2\big\}\,dt \\
&\quad \le C(\lambda_1)(2\gamma + 1)\int_0^T \big\{a_1\left\|v_{\epsilon m}(t)\right\|^2 + \left\|\varphi_{\epsilon m}(t)\right\|^2\big\}\,dt \\
&\quad \le C(\lambda_1, a_1, a_2, a_4, d_1, \gamma)
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
&(2\gamma + 1)\int_{-\infty}^{+\infty} \frac{|\tau|(a_1|\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{\varphi}_{\epsilon m}(\tau)\|^2)}{1 + |\tau|^{1-2\gamma}}\,d\tau \\
&\quad \le \frac{2\gamma + 1}{2\pi}\int_{-\infty}^{+\infty} \big(\big(\left|\hat{f}(\tau)\right|^2 + \left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{v}_{\epsilon m}(\tau)\right\|^2 + \left|\hat{q}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ C\big(\|\hat{q}_{\epsilon m}(\tau)\|_{L^4}^4 + \|\hat{v}_{\epsilon m}(\tau)\|_{L^4}^4)\big)/\big(1 + |\tau|^{1-2\gamma}\big)\big)\,d\tau \\
&+ \frac{2\gamma+1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sqrt{d_1}(|\hat{v}_{\epsilon m}(\tau)| + \|\hat{\varphi}_{\epsilon m}(\tau)\| + \epsilon|\hat{p}_{\epsilon m}(\tau)|)}{1 + |\tau|^{1-2\gamma}}\,d\tau \\
\le{}& \frac{(2\gamma+1)}{2\pi}\int_{-\infty}^{+\infty} \big(|\hat{f}(\tau)|^2 + |\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{v}_{\epsilon m}(\tau)\|^2 + |\hat{q}_{\epsilon m}(\tau)|^2 \\
&+ \|\hat{\varphi}_{\epsilon m}(\tau)\|^2 + C\big(\|\hat{q}_{\epsilon m}(\tau)\|_{L^4}^4 + \|\hat{v}_{\epsilon m}(\tau)\|_{L^4}^4\big)\big)\,d\tau \\
&+ \frac{(2\gamma+1)\sqrt{d_1}}{\pi}\int_{-\infty}^{+\infty} \frac{|\hat{v}_{\epsilon m}(\tau)| + \|\hat{\varphi}_{\epsilon m}(\tau)\| + \epsilon|\hat{p}_{\epsilon m}(\tau)|}{1 + |\tau|^{1-2\gamma}}\,d\tau \\
\le{}& C\int_0^T \big(|f(t)|^2 + |v_{\epsilon m}(t)|^2 + \|v_{\epsilon m}(t)\|^2 + |q_{\epsilon m}(t)|^2 + \|\varphi_{\epsilon m}(t)\|^2 \\
&+ \|q_{\epsilon m}(t)\|_{L^4}^4 + \|v_{\epsilon m}(t)\|_{L^4}^4\big)\,dt \\
&+ C\bigg(\int_{-\infty}^{+\infty} \frac{d\tau}{(1 + |\tau|^{1-2\gamma})^2}\bigg)^{\frac{1}{2}} \bigg(\int_0^T \big(|v_{\epsilon m}(t)|^2 + \|\varphi_{\epsilon m}(t)\|^2 + \epsilon|p_{\epsilon m}(t)|^2\big)\,dt\bigg)^{\frac{1}{2}} \\
\le{}& C(\pi,\gamma,d_1,a_1,a_2,a_4,T).
\end{aligned}
\tag{3.55}
$$

Here we have also used the convergence of the infinite integral

$$
\bigg(\int_{-\infty}^{+\infty} \frac{d\tau}{(1 + |\tau|^{1-2\gamma})^2}\bigg)^{\frac{1}{2}}, \quad \text{for some } \gamma \in \bigg(0, \frac{1}{4}\bigg).
$$

We can conclude that

$$
\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\big(|\hat{v}_{\epsilon m}(\tau)|^2 + \|\hat{\varphi}_{\epsilon m}(\tau)\|^2\big)\,d\tau \le C, \quad \text{for some } \gamma \in \bigg(0, \frac{1}{4}\bigg). \tag{3.56}
$$

(iv) We want to pass to the limit as $m \to \infty$ in (3.30)-(3.33) using the estimates (3.40), (3.42)-(3.44) and (3.55), we recall that at the present time $\epsilon \in (0,1]$ is fixed, and we are only concerned with a passage to the limit as $m \to \infty$. There exist a sequence $m' \to \infty$ and some $\{v_\epsilon, q_\epsilon, \varphi_\epsilon, p_\epsilon\}$ such that

$$
\begin{aligned}
&v_{\epsilon m'} \rightharpoonup v_\epsilon \quad \text{in } L^2\big(0,T;\mathbb{H}_0^1(\Omega)\big) \text{ weakly}, && (3.57)\\
&v_{\epsilon m'} \to v_\epsilon \quad \text{in } L^\infty\big(0,T;\mathbb{L}^2(\Omega)\big) \text{ weak-star}, && (3.58)\\
&v_{\epsilon m'} \to v_\epsilon \quad \text{in } L^2\big(0,T;\mathbb{L}^2(\Omega)\big) \text{ strongly}, && (3.59)\\
&q_{\epsilon m'} \rightharpoonup q_\epsilon \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big) \text{ weakly}, && (3.60)\\
&q_{\epsilon m'} \to q_\epsilon \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big) \text{ weak-star}, && (3.61)\\
&q_{\epsilon m'} \to q_\epsilon \quad \text{in } L^2\big(0,T;L^2(\Omega)\big) \text{ strongly}, && (3.62)\\
&\varphi_{\epsilon m'} \rightharpoonup \varphi_\epsilon \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big) \text{ weak-star}, && (3.63)\\
&\varphi_{\epsilon m'} \to \varphi_\epsilon \quad \text{in } L^2\big(0,T;H^2(\Omega)\big) \text{ strongly}, && (3.64)\\
&p_{\epsilon m'} \to p_\epsilon \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big) \text{ weak-star}. && (3.65)
\end{aligned}
$$

Take $\psi(t) \in \mathcal{C}_c^\infty([0,T])$ with $\psi(T) = 0$. We multiply (3.30) (resp. (3.31), (3.33)) by $\psi(t)$, integrate over $[0,T]$, and then integrate the first term by parts:

$$\begin{aligned} -a_1 \int_0^T \big(v_{\epsilon m'}(t), \omega_k \psi'(t)\big)\,dt + a_2 \int_0^T \big(\big(v_{\epsilon m'}(t), \omega_k \psi'(t)\big)\big)\,dt \\ + a_1 \int_0^T \hat{b}\big(v_{\epsilon m'}(t), v_{\epsilon m'}(t), \omega_k \psi(t)\big)\,dt \\ + \int_0^T \hat{n}\big(q_{\epsilon m'} + \tilde{Q}_0, \varphi_{\epsilon m'} + \tilde{\phi}_0, \omega_k \psi(t)\big)\,dt + \int_0^T \big(\nabla p_{\epsilon m'}, \omega_k \psi(t)\big)\,dt \\ = \int_0^T \big(f(t), \omega_k \psi(t)\big)\,dt + (v_{0m'}, \omega_k)\psi(0), \end{aligned} \tag{3.66}$$

$$\begin{aligned} -\int_0^T \big(q_{\epsilon m'}(t), w_k \psi'(t)\big)\,dt + a_3 \int_0^T \big(\big(q_{\epsilon m'}(t), w_k \psi(t)\big)\big)\,dt + a_4 \int_0^T \big(q_{\epsilon m'}(t), w_k \psi(t)\big)\,dt \\ - \int_0^T \hat{n}\big(q_{\epsilon m'} + \tilde{Q}_0, \omega_k \psi(t), v_{\epsilon m'}\big)\,dt = (q_{0m'}, w_k)\psi(0), \end{aligned} \tag{3.67}$$

$$\begin{aligned} -\epsilon \int_0^T \big(p_{\epsilon m'}(t), \gamma_l \psi'(t)\big)\,dt + \int_0^T \big(\operatorname{div} v_{\epsilon m'}(t), \gamma_l \psi(t)\big)\,dt \\ = \epsilon (p_{0m'}, \gamma_l)\psi(0), \quad 1 \le k, l \le m. \end{aligned} \tag{3.68}$$

We look at the convergence of the nonlinear terms in (3.66) and (3.67). Firstly

$$\begin{aligned} &\left| \int_0^T \hat{b}\big(v_{\epsilon m'}(t), v_{\epsilon m'}(t), \omega_k \psi(t)\big)\,dt - \int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega_k \psi(t)\big)\,dt \right| \\ &\quad \le \frac{1}{2} \left| \int_0^T b\big(v_{\epsilon m'}(t), v_{\epsilon m'}(t), \omega_k \psi(t)\big)\,dt - \int_0^T b\big(v_\epsilon(t), v_\epsilon(t), \omega_k \psi(t)\big)\,dt \right| \\ &\qquad + \frac{1}{2} \left| \int_0^T b\big(v_{\epsilon m'}(t), \omega_k \psi(t), v_{\epsilon m'}(t)\big)\,dt - \int_0^T b\big(v_\epsilon(t), \omega_k \psi(t), v_\epsilon(t)\big)\,dt \right| \\ &\quad \le \frac{1}{2} \left| \int_0^T b\big(v_{\epsilon m'}(t) - v_\epsilon(t), v_{\epsilon m'}(t), \omega_k \psi(t)\big)\,dt \right| \\ &\qquad + \frac{1}{2} \left| \int_0^T b\big(v_\epsilon(t), v_{\epsilon m'}(t) - v_\epsilon(t), \omega_k \psi(t)\big)\,dt \right| \\ &\qquad + \frac{1}{2} \left| \int_0^T b\big(v_{\epsilon m'}(t) - v_\epsilon(t), \omega_k \psi(t), v_{\epsilon m'}(t)\big)\,dt \right| \\ &\qquad + \frac{1}{2} \left| \int_0^T b\big(v_\epsilon(t), \omega_k \psi(t), v_{\epsilon m'}(t) - v_\epsilon(t)\big)\,dt \right| \\ &\quad \triangleq b_1 + b_2 + b_3 + b_4, \end{aligned} \tag{3.69}$$

where

$$\begin{aligned} b_1 &= \frac{1}{2} \left| \int_0^T b\big(v_{\epsilon m'}(t) - v_\epsilon(t), v_{\epsilon m'}(t), \omega_k \psi(t)\big)\,dt \right| \\ &\le \frac{1}{2} \sup_{t \in (0,T)} \big|\psi(t)\big| \sup_{x \in \Omega} |\omega_k| \int_0^T \big|v_{\epsilon m'}(t) - v_\epsilon(t)\big| \big|\nabla v_{\epsilon m'}(t)\big|\,dt \end{aligned}$$

$$\le C\|v_{\epsilon m'}(t)-v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\|v_{\epsilon m'}(t)\|_{L^2(0,T;\mathbb{H}_0^1(\Omega))}\to 0\quad \text{as } m'\to\infty, \tag{3.70}$$

$$\begin{aligned}
b_2 &= \frac{1}{2}\left|\int_0^T b\big(v_\epsilon(t), v_{\epsilon m'}(t)-v_\epsilon(t), \omega_k\psi(t)\big)\,dt\right| \\
&= \frac{1}{2}\left|\int_0^T \operatorname{div} v_\epsilon\big(v_{\epsilon m'}(t)-v_\epsilon(t)\big)\omega_k\psi(t)\,dt\right| \\
&\quad + \frac{1}{2}\left|\int_0^T b\big(v_\epsilon(t), \omega_k\psi(t), v_{\epsilon m'}(t)-v_\epsilon(t)\big)\,dt\right| \\
&\le \frac{1}{2}\sup_{t\in(0,T)}\big|\psi(t)\big|\sup_{x\in\Omega}|\omega_k|\int_0^T \big|v_{\epsilon m'}(t)-v_\epsilon(t)\big|\big|\nabla v_\epsilon(t)\big|\,dt \\
&\quad + \frac{1}{2}\sup_{t\in(0,T)}\big|\psi(t)\big|\sup_{x\in\Omega}|\nabla\omega_k|\int_0^T \big|v_\epsilon(t)\big|\big|v_{\epsilon m'}(t)-v_\epsilon(t)\big|\,dt \\
&\le C\|v_{\epsilon m'}(t)-v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\big(\|v_\epsilon(t)\|_{L^2(0,T;\mathbb{H}_0^1(\Omega))} \\
&\quad + \|v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\big)\to 0\quad \text{as } m'\to 0,
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
b_3 &= \frac{1}{2}\left|\int_0^T b\big(v_{\epsilon m'}(t)-v_\epsilon(t), \omega_k\psi(t), v_{\epsilon m'}(t)\big)\,dt\right| \\
&\le \frac{1}{2}\sup_{t\in(0,T)}\big|\psi(t)\big|\sup_{x\in\Omega}|\nabla\omega_k|\int_0^T \big|v_{\epsilon m'}(t)-v_\epsilon(t)\big|\big|v_{\epsilon m'}(t)\big|\,dt \\
&\le C\|v_{\epsilon m'}(t)-v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\|v_{\epsilon m'}(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\to 0\quad \text{as } m'\to\infty,
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
b_4 &= \frac{1}{2}\left|\int_0^T b\big(v_\epsilon(t), \omega_k\psi(t), v_{\epsilon m'}(t)-v_\epsilon(t)\big)\,dt\right| \\
&= \frac{1}{2}\sup_{t\in(0,T)}\big|\psi(t)\big|\sup_{x\in\Omega}|\nabla\omega_k|\int_0^T \big|v_{\epsilon m'}(t)-v_\epsilon(t)\big|\big|v_\epsilon(t)\big|\,dt \\
&\le C\|v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\|v_{\epsilon m'}(t)-v_\epsilon(t)\|_{L^2(0,T;\mathbb{L}^2(\Omega))}\to 0\quad \text{as } m'\to\infty.
\end{aligned} \tag{3.73}$$

It then follows from (3.69)-(3.73) that

$$\int_0^T \hat{b}\big(v_{\epsilon m'}(t), v_{\epsilon m'}(t), \omega_k\psi(t)\big)\,dt \to \int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega_k\psi(t)\big)\,dt. \tag{3.74}$$

Similarly, we can conclude that

$$\begin{aligned}
&\int_0^T \hat{n}\big(q_{\epsilon m'}(t)+\tilde{Q}_0, \varphi_{\epsilon m'}(t)+\tilde{\phi}_0, \omega_k\psi(t)\big)\,dt \\
&\quad \to \int_0^T \hat{n}\big(q_\epsilon(t)+\tilde{Q}_0, \varphi_\epsilon(t)+\tilde{\phi}_0, \omega_k\psi(t)\big)\,dt,
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
&\int_0^T \hat{n}\big(q_{\epsilon m'}(t)+\tilde{Q}_0, \omega_k\psi(t), v_{\epsilon m'}(t)\big)\,dt \\
&\quad \to \int_0^T \hat{n}\big(q_\epsilon(t)+\tilde{Q}_0, \omega_k\psi(t), v_\epsilon(t)\big)\,dt.
\end{aligned} \tag{3.76}$$

Taking the limit of (3.66)-(3.68) as $m' \to \infty$, we find that

$$
\begin{aligned}
&-a_1 \int_0^T \big(v_\epsilon(t), \omega_k \psi'(t)\big)\,dt + a_2 \int_0^T \big(\big(v_\epsilon(t), \omega_k \psi(t)\big)\big)\,dt \\
&\quad + a_1 \int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega_k \psi(t)\big)\,dt \\
&\quad + \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \omega_k \psi(t)\big)\,dt + \int_0^T \big(\nabla p_\epsilon, \omega_k \psi(t)\big)\,dt \\
&= \int_0^T \big(f(t), \omega_k \psi(t)\big)\,dt + (v_0, \omega_k)\psi(0), \qquad (3.77) \\
&- \int_0^T \big(q_\epsilon(t), w_k \psi'(t)\big)\,dt + a_3 \int_0^T \big(\big(q_\epsilon(t), w_k \psi(t)\big)\big)\,dt + a_4 \int_0^T \big(q_\epsilon(t), w_k \psi(t)\big)\,dt \\
&\quad - \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \omega_k \psi(t), v_\epsilon\big)\,dt = (q_0, w_k)\psi(0), \qquad (3.78) \\
&-\epsilon \int_0^T \big(p_\epsilon(t), \gamma_l \psi'(t)\big)\,dt + \int_0^T \big(\operatorname{div} v_\epsilon(t), \gamma_l \psi(t)\big)\,dt \\
&= \epsilon(p_0, \gamma_l)\psi(0), \quad 1 \le k, l \le m. \qquad (3.79)
\end{aligned}
$$

Thus, (3.77) (resp. (3.78) or (3.79)) holds for $\omega =$ (resp. w or γ) any finite linear combination of ω_k (resp. w_k or γ_l). And by a continuity argument, (3.77) is still valid for any $\omega \in \mathbb{H}_0^1(\Omega)$, (3.78) is still valid for any $w \in H_0^1(\Omega)$, and (3.79) is still valid for any $\gamma \in L^2(\Omega)$. Hence,

$$
\begin{aligned}
&-a_1 \int_0^T \big(v_\epsilon(t), \omega \psi'(t)\big)\,dt + a_2 \int_0^T \big(\big(v_\epsilon(t), \omega \psi(t)\big)\big)\,dt \\
&\quad + a_1 \int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega \psi(t)\big)\,dt \\
&\quad + \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \omega \psi(t)\big)\,dt + \int_0^T \big(\nabla p_\epsilon, \omega \psi(t)\big)\,dt \\
&= \int_0^T \big(f(t), \omega \psi(t)\big)\,dt + (v_0, \omega)\psi(0), \qquad (3.80) \\
&- \int_0^T \big(q_\epsilon(t), w \psi'(t)\big)\,dt + a_3 \int_0^T \big(\big(q_\epsilon(t), w \psi(t)\big)\big)\,dt + a_4 \int_0^T \big(q_\epsilon(t), w \psi(t)\big)\,dt \\
&\quad - \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \omega \psi(t), v_\epsilon\big)\,dt = (q_0, w)\psi(0), \qquad (3.81) \\
&-\epsilon \int_0^T \big(p_\epsilon(t), \gamma \psi'(t)\big)\,dt + \int_0^T \big(\operatorname{div} v_\epsilon(t), \gamma \psi(t)\big)\,dt \\
&= \epsilon(p_0, \gamma)\psi(0), \quad 1 \le k, l \le m. \qquad (3.82)
\end{aligned}
$$

Equations (3.80)-(3.82) show that $\{v_\epsilon, q_\epsilon, \varphi_\epsilon, p_\epsilon\}$ satisfy (3.24)-(3.27) in the sense of distributions.

It remains to prove that v_ϵ, q_ϵ, φ_ϵ and p_ϵ satisfy (3.28). To this end, we take $\psi(t) \in C_c^\infty([0, T])$ with $\psi(T) = 0$ and use $\psi(t)$ to multiply (3.24)-(3.25), (3.27), respectively, then integrate the resulting equalities over $[0, T]$ and then use integrating by parts for the first term to obtain

$$\begin{aligned} &-a_1 \int_0^T \big(v_\epsilon(t), \omega\psi'(t)\big)\,dt + a_2 \int_0^T \big(\big(v_\epsilon(t), \omega\psi(t)\big)\big)\,dt \\ &\quad + a_1 \int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega\psi(t)\big)\,dt \\ &\quad + \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \omega\psi(t)\big)\,dt + \int_0^T \big(\nabla p_\epsilon, \omega\psi(t)\big)\,dt \\ &= \int_0^T \big(f(t), \omega\psi(t)\big)\,dt + \big(v_\epsilon(0), \omega\big)\psi(0), \end{aligned} \tag{3.83}$$

$$\begin{aligned} &-\int_0^T \big(q_\epsilon(t), w\psi'(t)\big)\,dt + a_3 \int_0^T \big(\big(q_\epsilon(t), w\psi(t)\big)\big)\,dt + a_4 \int_0^T \big(q_\epsilon(t), w\psi(t)\big)\,dt \\ &\quad - \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \omega\psi(t), v_\epsilon\big)\,dt = \big(q_\epsilon(0), w\big)\psi(0), \end{aligned} \tag{3.84}$$

$$\begin{aligned} &-\epsilon \int_0^T \big(p_\epsilon(t), \gamma\psi'(t)\big)\,dt + \int_0^T \big(\operatorname{div} v_\epsilon(t), \gamma\psi(t)\big)\,dt \\ &= \epsilon\big(p_\epsilon(0), \gamma\big)\psi(0), \quad 1 \le k, l \le m. \end{aligned} \tag{3.85}$$

By comparing (3.80) with (3.83), (3.81) with (3.84), (3.82) with (3.85), we obtain

$$\begin{aligned} &\big(v_\epsilon(0) - u_0, \omega\big)\psi(0) = 0 \quad \forall \omega \in \mathbb{H}_0^1(\Omega), \\ &\big(q_\epsilon(0) - q_0, w\big)\psi(0) = 0 \quad \forall w \in H_0^1(\Omega), \\ &\big(p_\epsilon(0) - p_0, \gamma\big)\psi(0) = 0 \quad \forall \gamma \in L^2(\Omega). \end{aligned}$$

We can choose $\psi(0) \neq 0$ and obtain

$$\begin{aligned} &\big(v_\epsilon(0) - v_0, \omega\big) = 0 \quad \forall \omega \in \mathbb{H}_0^1(\Omega), \\ &\big(q_\epsilon(0) - q_0, w\big) = 0 \quad \forall w \in H_0^1(\Omega), \\ &\big(p_\epsilon(0) - p_0, \gamma\big) = 0 \quad \forall \gamma \in L^2(\Omega). \end{aligned}$$

Thus, (3.28) holds.

Next, we will prove the solution is unique. For this purpose we drop the indices ϵ and denote by $\{v_1, q_1, \varphi_1, p_1\}$, $\{v_2, q_2, \varphi_2, p_2\}$ two solutions of Problem 3.1 and then set

$$\begin{aligned} &u = v_2 - v_1, \quad r = q_2 - q_1, \quad \psi = \varphi_2 - \varphi_1, \quad p = p_2 - p_1, \\ &Q_i = q_i + \tilde{Q}_0, \quad \phi_i = \varphi_i + \tilde{\phi}_0, \quad i = 1, 2. \end{aligned}$$

Then

$$a_1 u' + a_2 \mathcal{A}_0 u + \nabla p = -a_1 \hat{B}(v_2, v_2) + a_1 \hat{B}(v_1, v_1) + \hat{N}(Q_1, \phi_1) - \hat{N}(Q_2, \phi_2), \tag{3.86}$$

$$r' = -a_3 \mathcal{A}_1 r - a_4 r + \hat{M}(Q_2, v_2) - \hat{M}(Q_1, v_1), \tag{3.87}$$

$$\epsilon p' + \operatorname{div} u = 0, \tag{3.88}$$

$$\mathcal{A}_1 \psi = r. \tag{3.89}$$

Taking the scalar of (3.86) with u, (3.87) with ψ and (3.88) with p, and adding these equations and using (3.89), we find

$$\begin{aligned}
&\frac{d}{dt}\left[a_1|u(t)|^2 + \|\psi(t)\|^2 + \epsilon|p(t)|^2\right] + 2a_2\|u(t)\|^2 + 2a_3|r(t)|^2 + 2a_4\|\psi(t)\|^2 \\
&\quad = -2a_1\hat{b}(u, v_1, u) + 2\hat{n}(r, \psi, v_1) - 2\hat{n}(r, \phi_1, u) \\
&\quad \le a_1\left[\|u\||u|\|v_1\| + |u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}\|v_1\|^{\frac{1}{2}}|v_1|^{\frac{1}{2}}\right] \\
&\qquad + |r|^{\frac{3}{2}}|v_1|^{\frac{1}{2}}\|v_1\|^{\frac{1}{2}}\|\psi\|^{\frac{1}{2}} + \|\psi\|^{\frac{1}{2}}|\psi|^{\frac{1}{2}}|r||v_1|^{\frac{1}{2}}\|v_1\|^{\frac{1}{2}} \\
&\qquad + |r||u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|\phi_1\|^{\frac{1}{2}}\|\phi_1\|_{H^2}^{\frac{1}{2}} + |\phi_1|^{\frac{1}{2}}\|\phi_1\|^{\frac{1}{2}}|r||u|^{\frac{1}{2}} + \|u\|^{\frac{1}{2}} \\
&\quad \le a_2\|u\|^2 + C_1(a_2)\left(|u|^2\|v_1\|^2 + |u|^2\|v_1\|^2|v_1|^2\right) + a_3|r|^2 \\
&\qquad + C_2\left(|v_1|^2\|v_1\|^2\|\psi\|^2 + \|\psi\||\psi||v_1|\|v_1\|\right) \\
&\qquad + a_3|r|^2 + a_2\|u\|^2 + C_3(a_3, a_2)\left(|u|^2\|\phi_1\|^2\|\phi_1\|_{H^2}^2 + |u|^2|\phi_1|^2\|\phi_1\|^2\right).
\end{aligned} \tag{3.90}$$

Then we can conclude from that equation

$$\begin{aligned}
&\frac{d}{dt}\left[a_1|u(t)|^2 + \|\psi(t)\|^2 + \epsilon|p(t)|^2\right] \\
&\quad \le C_1(a_2)\left(|u|^2\|v_1\|^2 + |u|^2\|v_1\|^2|v_1|^2\right) + C_2(a_3)\left(|v_1|^2\|v_1\|^2\|\psi\|^2 + \|\psi\||\psi||v_1|\|v_1\|\right) \\
&\qquad + C_3(a_3, a_2)\left(|u|^2\|\phi_1\|^2\|\phi_1\|_{H^2}^2 + |u|^2|\phi_1|^2\|\phi_1\|^2\right) \\
&\quad \le C_1(a_2)\left(|u|^2\|v_1\|^2 + |u|^2\|v_1\|^2|v_1|^2\right) + C_2(a_3)\left(|v_1|^2\|v_1\|^2\|\psi\|^2 + \|\psi\|^2|v_1|\|v_1\|\right) \\
&\qquad + C_3(a_3, a_2)\left(|u|^2\|\phi_1\|^2\|\phi_1\|_{H^2}^2 + |u|^2|\phi_1|^2\|\phi_1\|^2\right).
\end{aligned} \tag{3.91}$$

From (3.91), we have

$$\frac{d}{dt}\left[a_1|u(t)|^2 + \|\psi(t)\|^2 + \epsilon|p(t)|^2\right] \le h(t)\left[a_1|u(t)|^2 + \|\psi(t)\|^2\right], \tag{3.92}$$

where

$$\begin{aligned}
h(t) = {} & \frac{C_1(a_2)}{a_1}\left[\|v_1(t)\|^2 + |v_1(t)|^2\|v_1(t)\|^2\right] \\
& + C_2(a_3)\left[|v_1(t)|^2\|v_1(t)\|^2 + |v_1(t)|\|v_1(t)\|\right] \\
& + \frac{C_3(a_3, a_2)}{a_1}\left[\|\phi_1(t)\|^2\|\phi_1(t)\|_{H^2}^2 + |\phi_1(t)|^2\|\phi(t)\|^2\right].
\end{aligned} \tag{3.93}$$

The function $t \to h(t)$ belongs to $L^1(0, T)$ in view of equations (3.40), (3.42) and (3.47)-(3.48). Moreover, we infer $\psi(0) = 0$ from $r(0) = 0$ and $\mathcal{A}_1\psi = r$. We apply the Gronwall inequality to equation (3.92), we have

$$|u(t)| = 0, \quad \|\psi(t)\| = 0, \quad \text{for all } t \in [0, T].$$

Thus, $v_1 = v_2$, $q_1 = q_2$ and $\varphi_1 = \varphi_2$. The proof is complete. □

4 Convergence of solutions for the compressible electrohydrodynamics system to the incompressible electrohydrodynamics system

In this section, we show how the solutions of the perturbed problems converge to the solutions of the incompressible electrohydrodynamics system.

Theorem 4.1 *For* $\epsilon \in (0,1]$, v_0, q_0 *and* p_0 *be given satisfying* (2.1), (2.2), *and* (3.4), *the solutions* $\{v_\epsilon, q_\epsilon, \varphi_\epsilon, p_\epsilon\}$ *of Problem* 3.1 *converge to the solutions* $\{v, q, \varphi, p\}$ *of Problem* 2.1 *as* $\epsilon \to 0^+$ *in the following sense*:

$$v_\epsilon \to v \quad \text{in } L^\infty(0,T;\mathbb{L}^2(\Omega)) \text{ weak-star,} \tag{4.1}$$

$$q_\epsilon \to q \quad \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star,} \tag{4.2}$$

$$\varphi_\epsilon \to \varphi \quad \text{in } L^\infty(0,T;H^1(\Omega)) \text{ weak-star,} \tag{4.3}$$

$$v_\epsilon \to v \quad \text{in } L^2(0,T;\mathbb{H}_0^1(\Omega)) \text{ weakly,} \tag{4.4}$$

$$q_\epsilon \to q \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ weakly,} \tag{4.5}$$

$$v_\epsilon \to v \quad \text{in } L^2(0,T;\mathbb{L}^2(\Omega)) \text{ strongly,} \tag{4.6}$$

$$q_\epsilon \to q \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ strongly,} \tag{4.7}$$

$$\varphi_\epsilon \to \varphi \quad \text{in } L^2(0,T;H^2(\Omega)) \text{ strongly,} \tag{4.8}$$

$$\nabla p_\epsilon \to \nabla p \quad \text{in } L^{4/3}(0,T;\mathbb{H}^{-1}) \text{ weakly.} \tag{4.9}$$

Proof By (3.40), (3.42), (3.44), (3.48), (3.53)-(3.55), we have for all $\epsilon \in (0,1]$

$$\|v_\epsilon\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} \le \liminf_{m\to\infty} \|v_{\epsilon m}\|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} \le \frac{\sqrt{d_1}}{a_1}, \tag{4.10}$$

$$\|\varphi_\epsilon\|_{L^\infty(0,T;H^1(\Omega))} \le \liminf_{m\to\infty} \|\varphi_{\epsilon m}\|_{L^\infty(0,T;H^1(\Omega))} \le \sqrt{d_1}, \tag{4.11}$$

$$\|q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \le \liminf_{m\to\infty} \|q_{\epsilon m}\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{d_2}, \tag{4.12}$$

$$\|v_\epsilon\|_{L^2(0,T;\mathbb{H}_0^1(\Omega))} \le \liminf_{m\to\infty} \|v_{\epsilon m}\|_{L^2(0,T;\mathbb{H}_0^1(\Omega))} \le \sqrt{\frac{d_1}{2a_2 - \frac{2C}{\lambda_1\delta}}}, \tag{4.13}$$

$$\|q_\epsilon\|_{L^2(0,T;H_0^1(\Omega))} \le \liminf_{m\to\infty} \|q_{\epsilon m}\|_{L^2(0,T;H_0^1(\Omega))} \le \sqrt{\frac{d_2}{2a_3}}, \tag{4.14}$$

$$\sqrt{\epsilon}\|p_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \le \liminf_{m\to\infty} \sqrt{\epsilon}\|p_{\epsilon m}\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{d_1}, \tag{4.15}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\left(\left|\hat{v}_\epsilon(\tau)\right|^2 + \left\|\hat{\varphi}_\epsilon(\tau)\right\|^2\right) d\tau \\ &\quad \le \liminf_{m\to\infty} \int_{-\infty}^{+\infty} |\tau|^{2\gamma}\left(\left|\hat{v}_{\epsilon m}(\tau)\right|^2 + \left\|\hat{\varphi}_{\epsilon m}(\tau)\right\|^2\right) d\tau \le C.\end{aligned} \tag{4.16}$$

By virtue of (4.10)-(4.16), there is a sequence $\{\epsilon_m\} \subset (0,1]$ ($\epsilon_m \to 0^+$ as $m \to \infty$) and $v_* \in L^\infty(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathbb{H}_0^1(\Omega))$, $q_* \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, $\varphi_* \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and $p_* \in L^\infty(0,T;L^2(\Omega))$ such that when $\epsilon_m \to 0^+$

$$v_{\epsilon_m} \to v_* \quad \text{in } L^\infty(0,T;\mathbb{L}^2(\Omega)) \text{ weak-star,} \tag{4.17}$$

$$q_{\epsilon_m} \to q_* \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big) \text{ weak-star,} \tag{4.18}$$
$$\varphi_{\epsilon_m} \to \varphi_* \quad \text{in } L^\infty\big(0,T;H^1(\Omega)\big) \text{ weak-star,} \tag{4.19}$$
$$v_{\epsilon_m} \to v_* \quad \text{in } L^2\big(0,T;\mathbb{H}_0^1(\Omega)\big) \text{ weakly,} \tag{4.20}$$
$$q_{\epsilon_m} \to q_* \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big) \text{ weakly,} \tag{4.21}$$
$$v_{\epsilon_m} \to v_* \quad \text{in } L^2\big(0,T;\mathbb{L}^2(\Omega)\big) \text{ strongly,} \tag{4.22}$$
$$q_{\epsilon_m} \to q_* \quad \text{in } L^2\big(0,T;L^2(\Omega)\big) \text{ strongly,} \tag{4.23}$$
$$\varphi_{\epsilon_m} \to \varphi_* \quad \text{in } L^2\big(0,T;H^2(\Omega)\big) \text{ strongly,} \tag{4.24}$$
$$\sqrt{\epsilon_m} p_{\epsilon_m} \to \chi \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big) \text{ weak-star.} \tag{4.25}$$

Passing to the limit in (3.27) in the distribution sense as $\epsilon_m \to 0^+$, hence

$$\epsilon_m \frac{d}{dt}(p_{\epsilon_m}, q) = \sqrt{\epsilon_m}\frac{d}{dt}(\sqrt{\epsilon_m} p_{\epsilon_m}, q) \to 0, \quad \forall q \in L^2(\Omega). \tag{4.26}$$

Equations (3.27) and (4.26) give

$$(\operatorname{div} v_*, q) = 0, \quad \forall q \in L^2(\Omega). \tag{4.27}$$

This implies that $\operatorname{div} v_* = 0$ and thus

$$v_* \in L^\infty\big(0,T;\mathbb{L}^2(\Omega)\big) \cap L^2\big(0,T;\mathbb{H}_0^1(\Omega)\big), \tag{4.28}$$
$$q_* \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big), \tag{4.29}$$
$$\varphi_* \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big). \tag{4.30}$$

Now let ω be an element of $\mathcal{V}$ and w be an element of $\mathcal{W}$. Equations (3.24) and (3.25) then give

$$a_1 \frac{d}{dt}\langle v_\epsilon, \omega\rangle + a_1 \hat{b}(v_\epsilon, v_\epsilon, \omega) - a_2(\triangle v_\epsilon, \omega) + \hat{n}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \omega) = (f, \omega), \tag{4.31}$$
$$\frac{d}{dt}\langle q_\epsilon, w\rangle - a_3(\triangle q_\epsilon, w) + a_4(q_\epsilon, w) - \hat{n}(q_\epsilon + \tilde{Q}_0, w, v_\epsilon) = 0. \tag{4.32}$$

We have

$$(\nabla p_\epsilon, \omega) = (p_\epsilon, \operatorname{div}\omega) = 0.$$

If ψ is a continuously differentiable scalar function on $[0,T]$ with $\psi(T) = 0$, we can multiply (4.31), (4.32) by $\psi(t)$ integrate in t. And then we integrate by parts, to obtain

$$\begin{aligned} &-a_1 \int_0^T \big(v_\epsilon, \omega\psi'(t)\big)\,dt + a_1 \int_0^T \hat{b}\big(v_\epsilon, v_\epsilon, \omega\psi(t)\big)\,dt + a_2 \int_0^T \big((v_\epsilon, \omega\psi(t))\big)\,dt \\ &\quad + \int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0, \omega\psi(t)\big)\,dt = \int_0^T \big(f, \omega\psi(t)\big)\,dt + (v_0, \omega)\psi(0), \end{aligned} \tag{4.33}$$

$$
\begin{aligned}
&-\int_0^T \big(q_\epsilon(t), w\psi'(t)\big)\,dt + a_3\int_0^T \big(\big(q_\epsilon, w\psi(t)\big)\big)\,dt + a_4\int_0^T \big(q_\epsilon, w\psi(t)\big)\,dt \\
&\quad -\int_0^T \hat{n}\big(q_\epsilon + \tilde{Q}_0, w\psi(t), v_\epsilon\big)\,dt \\
&= (q_0, w)\psi(0).
\end{aligned}
\tag{4.34}
$$

Using the convergent relations (4.17)-(4.25) and similar derivations to (3.74)-(3.76)

$$
\int_0^T \hat{b}\big(v_\epsilon(t), v_\epsilon(t), \omega\psi(t)\big)\,dt \to \int_0^T \hat{b}\big(v_*(t), v_*(t), \omega\psi(t)\big)\,dt, \tag{4.35}
$$

$$
\begin{aligned}
&\int_0^T \hat{n}\big(q_\epsilon(t) + \tilde{Q}_0, \varphi_\epsilon(t) + \tilde{\phi}_0, \omega\psi(t)\big)\,dt \\
&\quad \to \int_0^T \hat{n}\big(q_*(t) + \tilde{Q}_0, \varphi_*(t) + \tilde{\phi}_0, \omega\psi(t)\big)\,dt,
\end{aligned}
\tag{4.36}
$$

$$
\int_0^T \hat{n}\big(q_\epsilon(t) + \tilde{Q}_0, \omega\psi(t), v_\epsilon(t)\big)\,dt \to \int_0^T \hat{n}\big(q_*(t) + \tilde{Q}_0, \omega\psi(t), v_*(t)\big)\,dt. \tag{4.37}
$$

We can pass to the limit in (4.33) and (4.34), we obtain

$$
\begin{aligned}
&-a_1\int_0^T \big(v_*, \omega\psi'(t)\big)\,dt + a_1\int_0^T \hat{b}\big(v_*, v_*, \omega\psi(t)\big)\,dt + a_2\int_0^T \big(\big(v_*, \omega\psi(t)\big)\big)\,dt \\
&\quad + \int_0^T \hat{n}\big(q_* + \tilde{Q}_0, \varphi_* + \tilde{\phi}_0, \omega\psi(t)\big)\,dt = \int_0^T \big(f, \omega\psi(t)\big)\,dt + (v_0, \omega)\psi(0),
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
&-\int_0^T \big(q_*(t), w\psi'(t)\big)\,dt + a_3\int_0^T \big(\big(q_*, w\psi(t)\big)\big)\,dt + a_4\int_0^T \big(q_*, w\psi(t)\big)\,dt \\
&\quad -\int_0^T \hat{n}\big(q_* + \tilde{Q}_0, w\psi(t), v_*\big)\,dt \\
&= (q_0, w)\psi(0).
\end{aligned}
\tag{4.39}
$$

Equations (4.38)-(4.39) are the same as (2.14)-(2.15) because $\operatorname{div} v_* = 0$ and $\hat{b}(v_*(t), v_*(t), \omega\psi(t)) = b(v_*(t), v_*(t), \omega\psi(t))$, $\hat{n}(q_* + \tilde{Q}_0, \varphi_* + \tilde{\phi}_0, \omega\psi(t)) = n(q_* + \tilde{Q}_0, \varphi_* + \tilde{\phi}_0, \omega\psi(t))$, $\hat{n}(q_* + \tilde{Q}_0, \omega\psi(t), v_*) = n(q_* + \tilde{Q}_0, \omega\psi(t), v_*)$. Then we can easily show that $\{v_*, q_*, \varphi_*\}$ is a solution of Problem 2.1. We next prove the convergent relations (4.1)-(4.9). Equation (4.1) follows easily from (4.10), (4.2) follows easily from (4.12), (4.3) follows easily from (4.11), (4.4) follows easily from (4.13), (4.5) follows easily from (4.14). Equation (4.6) can be deduced from (4.16)-(4.17), (4.20) and the compact embedding theorem (see, *e.g.*, [29]), simply, (4.7) can be deduced from (4.16), (4.18), (4.21) and the compact embedding theorem, (4.8) can be deduced from (4.16), (4.19) and the compact embedding theorem. It remains to prove (4.9). For this purpose we write (3.24) as

$$
\nabla p_{\epsilon m} = f - a_1 v'_\epsilon - a_1\hat{B}(v_\epsilon) - a_2\mathcal{A}_0 v_\epsilon - \hat{N}(q_\epsilon + \tilde{Q}_0, \varphi_\epsilon + \tilde{\phi}_0). \tag{4.40}
$$

The convergent results for v_{ϵ_m} show that the right-hand side of (4.40) converges weakly,

$$
\begin{aligned}
&f - a_1 v'_* - a_1\hat{B}(v_*) - a_2\mathcal{A}_0 v_* - \hat{N}(q_* + \tilde{Q}_0, \varphi_* + \tilde{\phi}_0) \\
&\quad \text{in } L^{4/3}\big(0, T; \mathbb{H}^{-1}\big) \text{ as } \epsilon_m \to 0^+.
\end{aligned}
\tag{4.41}
$$

We find that (4.41) is exactly ∇p. Hence,

$$\nabla p_\epsilon \to \nabla p \quad \text{in } L^{4/3}\big(0, T; \mathbb{H}^{-1}\big) \text{ weakly.} \tag{4.42}$$

That is, (4.9) is proved. The proof of Theorem 4.1 is complete. □

Competing interests
This paper has no financial or non-financial competing interests.

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The authors contributed to the manuscripts equally.

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