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Nonlocal problems for Langevin-type differential equations with two fractional-order derivatives

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Abstract

In this paper, we intend to study nonlocal problems for Langevin-type differential equations with two fractional derivatives of orders $\alpha, \beta \in (1,2)$. By using Laplace transform methods, formula of solutions involving Mittag-Leffler functions $A_{\alpha,\beta}(w)$, $\alpha, \beta \in (1,2)$, $w \in R$, and nonlocal terms of such equations are presented by studying the corresponding linear Langevin-type equations with two fractional derivatives. Meanwhile, existence results of solutions are established by utilizing boundedness, continuity, monotonicity, nonnegative of Mittag-Leffler function $A_{\alpha,\beta}(w)$, $\alpha, \beta \in (1,2)$, $w \in R$, and fixed point methods. Finally, two examples are presented to illustrate our theoretical results.

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Keywords: Langevin-type differential equations; fractional derivative; Mittag-Leffler function; nonlocal problem; existence

1 Introduction

With the fractional-order derivatives being applied in more and more fields, the basic theory of fractional differential equations (FDEs for short) also has a good development. The qualitative properties of solutions to Langevin-type FDEs were studied in the significant monographs [1–6] and recently have also been discussed in the literature [7–17] via fixed point methods.

Recently, Wang *et al.* [14] and Zhao [17] studied impulsive problems for Langevin-type equations with Caputo fractional derivatives of the form

$$^{c}D_t^{\beta}\left(^{c}D_t^{\alpha} + \mu\right)x(t) = g(t), \qquad \mu > 0, \alpha, \beta \in (0,1). \tag{1}$$

Definition 1.1 The Caputo derivative of order p for a function $v : [0,\infty) \to R$ can be written as

$$^{c}D_t^{p} v(t) = {}^{L}D_t^{p}\left(v(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} v^{(k)}(0)\right), \qquad t > 0, n-1 < p < n,$$

where ${}^{\mathrm{L}}D_t^p v$ denotes the Riemann-Liouville derivative of order p with the lower limit zero for a function v, that is,

$$ {}^{\mathrm{L}}D_t^p v(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t \frac{v(s)}{(t-s)^{p+1-n}}\, ds, \quad t > 0, n-1 < p < n. $$

For some suitable function $g : I \to R$, [14] applied Laplace transform methods to derive the general solution of (1) as follows:

$$ x(t) = A_\alpha\bigl(-t^\alpha \mu\bigr)b - \frac{1}{\mu}\bigl[1 - A_\alpha\bigl(-t^\alpha \mu\bigr)\bigr]a + \int_0^t (t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-(t-\tau)^\alpha \mu\bigr) g(\tau)\, d\tau, $$

where $I := [0,1]$, a, b are constants, $A_{\alpha,\beta}(w) := \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k+\beta)}$, $\alpha, \beta \in (0,1)$ for $w \in R$ is the Mittag-Leffler function, and $A_{\alpha,1}(w) := A_\alpha(w)$. Then by applying the properties (continuity, explicit boundedness, monotonicity, and nonnegativity) of $A_{\alpha,\beta}(w)$, $\alpha, \beta \in (0,1)$, for $w < 0$ and fixed point theorems, we can deduce some interesting existence results for a nonlinear problem.

However, we will less study the theory of Langevin-type differential equation with two Caputo fractional derivatives ${}^{\mathrm{c}}D_t^\beta({}^{\mathrm{c}}D_t^\alpha + \mu)x(t)$, $\alpha, \beta \in (1,2)$, and the properties of $A_{\alpha,\beta}(w)$, $\alpha, \beta \in (1,2)$, for $w < 0$ and $w > 0$.

Let $f : I \times R \to R$ be a given real-value function. We consider nonlocal problems for Langevin-type equations with two Caputo fractional derivatives of the form

$$ \begin{cases} {}^{\mathrm{c}}D_t^\beta({}^{\mathrm{c}}D_t^\alpha + \mu)x(t) = f(t, x(t)), & t \in I := [0,1], 1 < \alpha, \beta < 2, \\ x(0) = \sum_{i=1}^m a_i x(t_i), \quad x'(0) = b, & [{}^{\mathrm{c}}D_t^\alpha x(t)]_{t=0} = c, \quad [{}^{\mathrm{c}}D_t^\alpha x(t)]'_{t=0} = d. \end{cases} \tag{2} $$

Here ${}^{\mathrm{c}}D_t^\alpha$ and ${}^{\mathrm{c}}D_t^\beta$ denote the Caputo fractional derivatives of orders $\alpha, \beta \in (1,2)$ with the lower limit zero, respectively, and the constants $a_i \in R$, $i = 1, \ldots, m$, $b, c, d \in R$, $\mu \in R \setminus \{0\}$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$.

In the beginning of the paper, we first apply the Laplace transform method to derive that the linear Cauchy problem

$$ \begin{cases} {}^{\mathrm{c}}D_t^\beta({}^{\mathrm{c}}D_t^\alpha + \mu)x(t) = g(t), & t \in I := [0,1], \mu \in R \setminus \{0\}, \\ x(0) = a, \quad x'(0) = b, & [{}^{\mathrm{c}}D_t^\alpha x(t)]_{t=0} = c, \quad [{}^{\mathrm{c}}D_t^\alpha x(t)]'_{t=0} = d, \end{cases} \tag{3} $$

has the unique solution

$$ \begin{aligned} x(t) &= aA_\alpha\bigl(-\mu t^\alpha\bigr) + btA_{\alpha,2}\bigl(-\mu t^\alpha\bigr) + (a\mu + c)t^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr) \\ &\quad + (b\mu + d)t^{\alpha+1} A_{\alpha,\alpha+2}\bigl(-\mu t^\alpha\bigr) + \int_0^t (t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr) g(\tau)\, d\tau, \end{aligned} \tag{4} $$

where $g : I \to R$ is a linear function.

Then we go on studying the asymptotic behavior, boundedness, and continuity of the Mittag-Leffler functions $A_{\alpha,\beta}(w)$, $\alpha \in (1,2]$, $\beta > 0$, $w \in R$ (see Lemmas 3.2, 3.3, and 3.4), which will further use to study nonlinear problems. In fact, the properties of Mittag-Leffler functions $A_{\alpha,\beta}(w)$, $\alpha \in (1,2]$, $\beta > 0$, $w \in R$, presented in this paper can also used for other possible problems. Finally, we apply fixed point methods to derive the existence of solution to (2) under Lipschitz and growth conditions on the nonlinear term.

2 Preliminaries

Let $C(I,R)$ denote the Banach space of all continuous functions from I into R with the standard norm $\|u\|_\infty = \sup\{|u(t)| : 0 \leq t \leq 1\}$ for $u \in C(I,R)$. We denote by $L_q(I,R)$ the Banach space of all Lebesgue-measurable functions $v : I \to R$ with the norm $\|v\|_{L^q(I)} < \infty$ defined as

$$\|v\|_{L^q(I)} = \begin{cases} (\int_I |v(t)|^q \, dt)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \inf_{\mu(\Delta)=0}\{\sup_{t\in I-\Delta} |v(t)|\}, & q = \infty, \end{cases}$$

where $\mu(\Delta)$ is the Lebesgue measure of Δ.

Before we study (2), we first introduce the following linear fractional initial value problem.

Lemma 2.1 *Let $g : I \to R$ be continuous. Then fractional initial value problem* (3) *has the unique solution* (4).

Proof From (5.3.3) of [2] we know that the Laplace transform of the Caputo fractional derivative ${}^{\mathrm{c}}D_t^\beta x(s)$ satisfies the following equation:

$$\left(\mathcal{L}{}^{\mathrm{c}}D_t^\beta x\right)(s) = s^\beta (\mathcal{L}x)(s) - \sum_{j=0}^{l-1} s^{\beta-j-1} x^{(j)}(0). \tag{5}$$

Thus, we apply the Laplace transform to the first equation in (3) via (5) and derive

$$\begin{aligned} &s^\beta\left(s^\alpha \mathcal{L}x(s) - s^{\alpha-1}x(0) - s^{\alpha-2}x'(0)\right) - s^{\beta-1}\left[{}^{\mathrm{c}}D_t^\alpha x(t)\right]_{t=0} - s^{\beta-2}\left[{}^{\mathrm{c}}D_t^\alpha x(t)\right]'_{t=0} \\ &\quad + \mu\left(s^\beta \mathcal{L}x(s) - s^{\beta-1}x(0) - s^{\beta-2}x'(0)\right) = \mathcal{L}g(s). \end{aligned}$$

Furthermore, we obtain

$$\mathcal{L}x(s) = \frac{s^{-\beta}[\mathcal{L}g(s) + a(s^{\beta+\alpha-1} + \mu s^{\beta-1}) + b(s^{\beta+\alpha-2} + \mu s^{\beta-2}) + cs^{\beta-1} + ds^{\beta-2}]}{s^\alpha + \mu}. \tag{6}$$

Next we find $x(t)$. According to (1.10.9) of [2], we know that

$$w(t) = t^{\beta-1} A_{\alpha,\beta}(-\mu t^\alpha) \quad \Longrightarrow \quad \mathcal{L}w(s) = \tfrac{s^{\alpha-\beta}}{s^\alpha+\mu}. \tag{7}$$

From equality (7), by applying inverse Laplace transforms to (6) we get the unique solution of (3):

$$\begin{aligned} x(t) &= aA_\alpha\left(-\mu t^\alpha\right) + btA_{\alpha,2}\left(-\mu t^\alpha\right) + (a\mu + c)t^\alpha A_{\alpha,\alpha+1}\left(-\mu t^\alpha\right) \\ &\quad + (b\mu + d)t^{\alpha+1} A_{\alpha,\alpha+2}\left(-\mu t^\alpha\right) + \int_0^t (t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\left(-\mu(t-\tau)^\alpha\right) g(\tau)\, d\tau. \end{aligned}$$

The proof of the lemma is completed. □

At the end of this section, we introduce the following fixed point theorems.

Lemma 2.2 *Let W be a closed convex and nonempty subset of a Banach space X. Let G, H be two operators such that* (i) $G(x) + H(y) \in W$ *for* $x, y \in W$; (ii) *G is compact and continuous*; (iii) *H is a contraction mapping. Then there exists $z \in W$ such that $z = Gz + Hz$.*

Lemma 2.3 *Let $(X, \|\cdot\|)$ be a Banach space, and $B : X \to X$ be a compact operator. Assume that $L : X \to X$ is a bounded linear operator such that 1 is not an eigenvalue of L and $\lim_{\|x\|\to\infty} \frac{\|Bx-Lx\|}{\|x\|} = 0$. Then B has a fixed point in X.*

3 Properties of Mittag-Leffler functions $A_{\alpha,\beta}$, $\alpha \in (1,2]$, $\beta > 0$

In order to give some results on the asymptotic behavior of Mittag-Leffler functions $A_{\alpha,\beta}$, $\alpha \in (1,2]$, $\beta > 0$, we need the following lemma.

Lemma 3.1 (see [18]) *Let $\alpha \in (0,2)$ and $\beta \in R$ be arbitrary. Then for $l = [\frac{\beta}{\alpha}]$, the following asymptotic expansions hold*:

(i)

$$A_{\alpha,\beta}(w) = \frac{1}{\alpha} w^{\frac{1-\beta}{\alpha}} \exp\bigl(w^{\frac{1}{\alpha}}\bigr) - \sum_{k=1}^{l} \frac{w^{-k}}{\Gamma(\beta-\alpha k)} + O\bigl(w^{-1-l}\bigr) \quad \text{as } w \to \infty;$$

(ii)

$$A_{\alpha,\beta}(w) = -\sum_{k=1}^{l} \frac{w^{-k}}{\Gamma(\beta-\alpha k)} + O\bigl(|w|^{-1-l}\bigr) \quad \text{as } w \to -\infty.$$

Define

$$\begin{aligned} A_{\alpha,\beta}(t,\mu) &= t^{\beta-1} A_{\alpha,\beta}\bigl(-\mu t^{\alpha}\bigr), \quad \alpha,\beta,\mu > 0, \\ Q_{\alpha,\beta}(r,\mu) &= \frac{r^{\alpha-\beta}}{\pi} \frac{r^{\alpha}\sin(\pi\beta) + \mu\sin(\pi(\beta-\alpha))}{r^{2\alpha} + 2\mu\cos(\pi\alpha) r^{\alpha} + \mu^2}, \\ Q(r,w) &= \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} \exp\bigl(-r^{\frac{1}{\alpha}}\bigr) \frac{r\sin(\pi(1-\beta)) - w\sin(\pi(1-\beta+\alpha))}{r^2 - 2rw\cos(\pi\alpha) + w^2}. \end{aligned}$$

Lemma 3.2 *Let $\alpha \in (1,2]$ and $\beta > 0$ be arbitrary. Then the following statements hold*:

(i) *For all $\mu > 0$, we have*

$$\begin{aligned} A_{\alpha,\beta}\bigl(-\mu t^{\alpha}\bigr) = {} & \int_0^{\infty} Q_1(r,t)\,dr + \frac{2t^{1-\beta}}{\alpha\mu^{1-\frac{1}{\alpha}}} \exp\biggl(t\mu^{\frac{1}{\alpha}} \cos\biggl(\frac{\pi}{\alpha}\biggr)\biggr) \\ & \times \cos\biggl[t\mu^{\frac{1}{\alpha}} \sin\biggl(\frac{\pi}{\alpha}\biggr) - \frac{\pi}{\alpha}(\beta-1)\biggr], \end{aligned}$$

where

$$\begin{aligned} Q_1(r,t) &= \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} \exp\bigl(-r^{\frac{1}{\alpha}}\bigr) \frac{r\sin(\pi(1-\beta)) + \mu t^{\alpha}\sin(\pi(1-\beta+\alpha))}{r^2 + 2r\mu t^{\alpha}\cos(\pi\alpha) + \mu^2 t^{2\alpha}} \\ &\overset{w=-\mu t^{\alpha}<0}{=} Q(r,w). \end{aligned}$$

(ii) *For all* $\mu > 0$, *we have*

$$A_{\alpha,\beta}\left(\mu t^{\alpha}\right) = \frac{1}{\alpha}\mu^{\frac{1-\beta}{\alpha}} t^{1-\beta} \exp\left(\mu^{\frac{1}{\alpha}} t\right) + \int_0^{\infty} Q_2(r,t)\,dr$$
$$+ \frac{2t^{1-\beta}}{\alpha\mu^{1-\frac{1}{\alpha}}} \exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right],$$

where

$$Q_2(r,t) = \frac{1}{\pi\alpha} r^{\frac{1-\beta}{\alpha}} \exp\left(-r^{\frac{1}{\alpha}}\right) \frac{r\sin(\pi(1-\beta)) - \mu t^{\alpha}\sin(\pi(1-\beta+\alpha))}{r^2 - 2r\mu t^{\alpha}\cos(\pi\alpha) + \mu^2 t^{2\alpha}}$$
$$\overset{w=\mu t^{\alpha}>0}{=} Q(r,w).$$

Proof (i) By applying Proposition 2.1 of [19] for $\alpha \in (1,2)$, $\gamma > 0$, we can deduce

$$t^{\alpha-1}A_{\alpha,\alpha}\left(-\gamma^{\alpha}t^{\alpha}\right) = \frac{\sin(\pi\alpha)}{\pi}\int_0^{\infty}\exp(-rt)\frac{r^{\alpha}}{r^{2\alpha} + 2r^{\alpha}\gamma^{\alpha}\cos(\pi\alpha) + \gamma^{2\alpha}}\,dr$$
$$- \frac{2}{\alpha\gamma^{\alpha-1}}\exp\left(t\gamma\cos\left(\frac{\pi}{\alpha}\right)\right)\cos\left[t\gamma\sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right]. \tag{8}$$

Letting $\mu = \gamma^{\alpha}$, (8) can be simplified as

$$\begin{aligned} A_{\alpha,\alpha}(t,\mu) &= t^{\alpha-1}A_{\alpha,\alpha}\left(-\mu t^{\alpha}\right) \\ &= \frac{\sin(\pi\alpha)}{\pi}\int_0^{\infty}\exp(-rt)\frac{r^{\alpha}}{r^{2\alpha} + 2r^{\alpha}\mu\cos(\pi\alpha) + \mu^2}\,dr \\ &\quad - \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right] \\ &= \int_0^{\infty}\exp(-rt)Q_{\alpha,\alpha}(r,\mu)\,dr - \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right) \\ &\quad \times \cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\alpha-1)\right]. \end{aligned} \tag{9}$$

On the other hand, from Corollary 1 of [20], for all $\alpha \in (1,2]$, $\beta > 0$, and $t > 0$, we get

$$\begin{aligned} A_{\alpha,\beta}(t) &= A_{\alpha,\beta}(t,1) \\ &= \int_0^{\infty}\exp(-rt)Q_{\alpha,\beta}(r)\,dr + \frac{2}{\alpha}\exp\left(t\cos\left(\frac{\pi}{\alpha}\right)\right) \\ &\quad \times \cos\left[t\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right], \end{aligned} \tag{10}$$

where $Q_{\alpha,\beta}(r,1) = Q_{\alpha,\beta}(r)$.

By comparing formulas (9) and (10) we can derive a general formula of $A_{\alpha,\beta}(t,\mu)$ as follows:

$$\begin{aligned} A_{\alpha,\beta}(t,\mu) &= \int_0^{\infty}\exp(-rt)Q_{\alpha,\beta}(r,\mu)\,dr + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right) \\ &\quad \times \cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right]. \end{aligned}$$

Let $r = x^\alpha t^\alpha$. The last equality can be simplified as

$$A_{\alpha,\beta}\big(-\mu t^\alpha\big) = \int_0^\infty Q_1(r,t)\,dr + \frac{2t^{1-\beta}}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right) \times \cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right],$$

which proves part (i).

(ii) Due to Lemma 3.1(ii) via the previous result, changing $-\mu t^\alpha$ to μt^α, we easily obtain

$$A_{\alpha,\beta}\big(\mu t^\alpha\big) = \frac{1}{\alpha}\mu^{\frac{1-\beta}{\alpha}}t^{1-\beta}\exp\big(\mu^{\frac{1}{\alpha}}t\big) + \int_0^\infty Q_2(r,t)\,dr + \frac{2t^{1-\beta}}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right],$$

where

$$Q_2(r,t) = \frac{1}{\pi\alpha}r^{\frac{1-\beta}{\alpha}}\exp\big(-r^{\frac{1}{\alpha}}\big)\frac{r\sin(\pi(1-\beta)) - \mu t^\alpha\sin(\pi(1-\beta+\alpha))}{r^2 - 2r\mu t^\alpha\cos(\pi\alpha) + \mu^2t^{2\alpha}}.$$

The proof is finished. □

Lemma 3.3 *Let $\mu > 0$ be arbitrary. For any $\alpha \in (1,2]$ and $\beta > 0$, we define*

$$P(\alpha,\beta,\mu) = \max\big\{P_1(\alpha,\beta,\mu), P_2(\alpha,\beta,\mu)\big\},$$

where

$$P_1(\alpha,\beta,\mu) = \frac{|\sin(\pi\beta)|\int_0^\infty r^{\frac{1-\beta+\alpha}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\mu^2},$$

$$P_2(\alpha,\beta,\mu) = \frac{|\sin(\pi(\beta-\alpha))|\int_0^\infty r^{\frac{1-\beta}{\alpha}}\exp(-r^{\frac{1}{\alpha}})\,dr}{\sin^2(\pi\alpha)\pi\alpha\mu}.$$

(i) *For all $t > 0$, we have*

$$\left|t^{\beta-1}A_{\alpha,\beta}\big(\mu t^\alpha\big) - \frac{1}{\alpha}\mu^{\frac{1-\beta}{\alpha}}\exp\big(\mu^{\frac{1}{\alpha}}t\big)\right| \le P(\alpha,\beta,\mu)\left(\frac{1}{t^{2\alpha-\beta+1}} + \frac{1}{t^{\alpha-\beta+1}}\right) + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right).$$

In particular,

$$\left|A_\alpha\big(\mu t^\alpha\big) - \frac{1}{\alpha}\exp\big(\mu^{\frac{1}{\alpha}}t\big)\right| \le P(\alpha,1,\mu)\left(\frac{1}{t^{2\alpha}} + \frac{1}{t^\alpha}\right) + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right).$$

(ii) *For all $t > 0$, we have*

$$\left|t^{\beta-1}A_{\alpha,\beta}\big(-\mu t^\alpha\big)\right| \le P(\alpha,\beta,\mu)\left(\frac{1}{t^{2\alpha-\beta+1}} + \frac{1}{t^{\alpha-\beta+1}}\right) + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right).$$

In particular,

$$\left|A_\alpha\left(-\mu t^\alpha\right)\right| \le P(\alpha,1,\mu)\left(\frac{1}{t^{2\alpha}}+\frac{1}{t^\alpha}\right)+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right).$$

Proof (i) From Lemma 3.2 we deduce

$$\begin{aligned}
&\left|t^{\beta-1}A_{\alpha,\beta}\left(\mu t^\alpha\right)-\frac{1}{\alpha}\mu^{\frac{1-\beta}{\alpha}}\exp\left(\mu^{\frac{1}{\alpha}}t\right)\right| \\
&\quad=\left|t^{\beta-1}\int_0^\infty Q_2(r,t)\,dr+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\right. \\
&\qquad\left.\times\cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)-\frac{\pi}{\alpha}(\beta-1)\right]\right| \\
&\quad\le P(\alpha,\beta,\mu)\left(\frac{1}{t^{2\alpha-\beta+1}}+\frac{1}{t^{\alpha-\beta+1}}\right)+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right),
\end{aligned}$$

where we apply the same computation as in Lemma 2.5 of [16] to get the estimate of $|t^{\beta-1}\int_0^\infty Q_2(r,t)\,dr|$.

In particular,

$$\left|A_\alpha\left(\mu t^\alpha\right)-\frac{1}{\alpha}\exp\left(\mu^{\frac{1}{\alpha}}t\right)\right|\le\frac{P(\alpha,1,\mu)}{t^\alpha}+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right),$$

which proves part (i).

(ii) According to Lemma 3.2, we obtain

$$\begin{aligned}
&\left|t^{\beta-1}A_{\alpha,\beta}\left(\mu t^\alpha\right)\right| \\
&\quad=\left|t^{\beta-1}\int_0^\infty Q_1(r,t)\,dr+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\right. \\
&\qquad\left.\times\cos\left[t\mu^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)-\frac{\pi}{\alpha}(\beta-1)\right]\right| \\
&\quad\le P(\alpha,\beta,\mu)\left(\frac{1}{t^{2\alpha-\beta+1}}+\frac{1}{t^{\alpha-\beta+1}}\right)+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right),
\end{aligned}$$

where we applied the same computation as in Lemma 2.5 of [16] to get the estimate of $|t^{\beta-1}\int_0^\infty Q_1(r,t)\,dr|$.

In particular,

$$\left|A_\alpha\left(\mu t^\alpha\right)\right|\le\frac{P(\alpha,1,\mu)}{t^\alpha}+\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right).$$

The proof of the lemma is finished. □

We further state the continuity, monotonicity, and nonnegativity of Mittag-Leffler functions $A_{\alpha,\beta}$, $\alpha\in(1,2]$, $\beta>0$.

Lemma 3.4 *Let $\alpha\in(1,2]$ and $\beta>0$ be arbitrary. Then the functions $A_\alpha(\cdot)$, $A_{\alpha,\alpha}(\cdot)$, and $A_{\alpha,\beta}(\cdot)$ are nonnegative and have the following properties*:

(i) *For all* $\mu > 0$ *and* $t, \hat{t}_1, \hat{t}_2 \in I$ *such that* $0 < \hat{t}_2 \le \hat{t}_1$,

$$A_\alpha(\hat{t}_2^\alpha \mu) \le A_\alpha(\hat{t}_1^\alpha \mu), \qquad A_{\alpha,\beta}(\hat{t}_2^\alpha \mu) \le A_{\alpha,\beta}(\hat{t}_1^\alpha \mu),$$
$$A_\alpha(-t^\alpha \mu) \le 1, \qquad A_{\alpha,\alpha}(-t^\alpha \mu) \le \frac{1}{\Gamma(\alpha)}, \qquad A_{\alpha,\beta}(-t^\alpha \mu) \le \frac{1}{\Gamma(\beta)}.$$

(ii) *For all* $\mu > 0$ *and* $\hat{t}_1, \hat{t}_2 \in I := [0,1]$,

$$A_\alpha(\hat{t}_1^\alpha \mu) \to A_\alpha(\hat{t}_2^\alpha \mu) \quad \text{as } \hat{t}_1 \to \hat{t}_2,$$
$$A_\alpha(-\hat{t}_1^\alpha \mu) \to A_\alpha(-\hat{t}_2^\alpha \mu) \quad \text{as } \hat{t}_1 \to \hat{t}_2,$$
$$A_{\alpha,\beta}(\hat{t}_1^\alpha \mu) \to A_{\alpha,\beta}(\hat{t}_2^\alpha \mu) \quad \text{as } \hat{t}_1 \to \hat{t}_2,$$
$$A_{\alpha,\beta}(-\hat{t}_1^\alpha \mu) \to A_{\alpha,\beta}(-\hat{t}_2^\alpha \mu) \quad \text{as } \hat{t}_1 \to \hat{t}_2.$$

Proof (i) Suppose that $A_\alpha(w)$ and $A_{\alpha,\beta}(w)$ are increasing functions for $w > 0$. It is easy to get the first conclusion of (i). On the other hand, from Lemma 2.7 of [13] we know $A_\alpha(-t^\alpha\mu) \le 1$ and $A_{\alpha,\alpha}(-t^\alpha\mu) \le \frac{1}{\Gamma(\alpha)}$. As for $A_{\alpha,\beta}(-t^\alpha\mu)$, we have

$$\begin{aligned}
A_{\alpha,\beta}(-t^\alpha \mu) &= \sum_{l=0}^{\infty} \frac{(-t^\alpha\mu)^l}{\Gamma(\alpha l + \beta)} \\
&= \sum_{l=0}^{\infty} \frac{(-t^\alpha\mu)^l}{\Gamma(\alpha l + \alpha)\Gamma(\beta - \alpha)} B(\alpha l + \alpha, \beta - \alpha) \\
&= \sum_{l=0}^{\infty} \frac{(-t^\alpha\mu)^l}{\Gamma(\alpha l + \alpha)\Gamma(\beta - \alpha)} \int_0^1 v^{\alpha l + \alpha - 1}(1-v)^{\beta-\alpha-1}\,dv \\
&= \int_0^1 \sum_{l=0}^{\infty} \frac{(-t^\alpha\mu)^l v^{\alpha l}}{\Gamma(\alpha l + \alpha)\Gamma(\beta - \alpha)} v^{\alpha-1}(1-v)^{\beta-\alpha-1}\,dv \\
&= \int_0^1 \sum_{l=0}^{\infty} \frac{(-(tv)^\alpha\mu)^l}{\Gamma(\alpha l + \alpha)\Gamma(\beta - \alpha)} v^{\alpha-1}(1-v)^{\beta-\alpha-1}\,dv \\
&= \frac{A_{\alpha,\alpha}(-(tv)^\alpha\mu)}{\Gamma(\beta-\alpha)} \int_0^1 v^{\alpha-1}(1-v)^{\beta-\alpha-1}\,dv \\
&= \frac{A_{\alpha,\alpha}(-(tv)^\alpha\mu)}{\Gamma(\beta-\alpha)} B(\alpha, \beta-\alpha) \\
&\le \frac{B(\alpha,\beta-\alpha)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} = \frac{1}{\Gamma(\beta)}.
\end{aligned}$$

(ii) In this part, we verify the second conclusion of the lemma. Without loss of generality, we only prove the third result of (ii). The remaining three results of (ii) can be proved by a similar method.

By Lemma 3.2(ii), for all $\mu > 0$, we have

$$\begin{aligned}
A_{\alpha,\beta}(\mu t^\alpha) &= \frac{1}{\alpha}\mu^{\frac{1-\beta}{\alpha}} t^{1-\beta} \exp(\mu^{\frac{1}{\alpha}} t) + \int_0^\infty Q_2(r,t)\,dr \\
&\quad + \frac{2t^{1-\beta}}{\alpha\mu^{1-\frac{1}{\alpha}}} \exp\left(t\mu^{\frac{1}{\alpha}} \cos\left(\frac{\pi}{\alpha}\right)\right) \cos\left[t\mu^{\frac{1}{\alpha}} \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta-1)\right].
\end{aligned}$$

Define

$$V(t) = r^2 - 2rt^\alpha \mu \cos(\pi\alpha) + \mu^2 t^{2\alpha}$$

and

$$W(t) = t^{1-\beta} \exp\left(t\mu^{\frac{1}{\alpha}} \cos\left(\frac{\pi}{\alpha}\right)\right) \cos\left[t\mu^{\frac{1}{\alpha}} \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}(\beta - 1)\right].$$

In fact,

$$r^2 - 2rt^\alpha \mu \cos(\pi\alpha) + \mu^2 t^{2\alpha} - \mu^2 t^{2\alpha} \sin^2(\pi\alpha) = \left(r - t^\alpha \mu \cos(\pi\alpha)\right)^2 \geq 0,$$

so that

$$V(t) = r^2 - 2rt^\alpha \mu \cos(\pi\alpha) + \mu^2 t^{2\alpha} \geq \mu^2 t^{2\alpha} \sin^2(\pi\alpha).$$

Assume that $\hat{t}_1 > \hat{t}_2 > 0$ and $\mu > 0$. According to Lagrange's mean value theorem, we obtain

$$\begin{aligned}
&\left|A_{\alpha,\beta}\left(\hat{t}_1^\alpha \mu\right) - A_{\alpha,\beta}\left(\hat{t}_2^\alpha \mu\right)\right| \\
&\quad \leq \frac{\mu^{\frac{1-\beta}{\alpha}}}{\alpha}\left|\hat{t}_1^{1-\beta} \exp\left(\hat{t}_1 \mu^{\frac{1}{\alpha}}\right) - \hat{t}_2^{1-\beta} \exp\left(\hat{t}_2 \mu^{\frac{1}{\alpha}}\right)\right| \\
&\qquad + \left|\frac{1}{\pi\alpha} \int_0^\infty r^{\frac{1-\beta}{\alpha}} \exp\left(-r^{\frac{1}{\alpha}}\right)\left(\frac{r\sin(\pi(1-\beta)) - \hat{t}_1^\alpha \mu \sin(\pi(1-\beta+\alpha))}{r^2 - 2r\hat{t}_1^\alpha \mu \cos(\pi\alpha) + \hat{t}_1^{2\alpha}\mu^2}\right.\right. \\
&\qquad \left.\left. - \frac{r\sin(\pi(1-\beta)) - \hat{t}_2^\alpha \mu \sin(\pi(1-\beta+\alpha))}{r^2 - 2r\hat{t}_2^\alpha \mu \cos(\pi\alpha) + \hat{t}_2^{2\alpha}\mu^2}\right) dr\right| + \left|\frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\left[W(\hat{t}_1) - W(\hat{t}_2)\right]\right| \\
&\quad \leq \frac{\mu^{\frac{1-\beta}{\alpha}}}{\alpha}\left|\left[(1-\beta)\eta^{-\beta} \exp\left(\eta\mu^{\frac{1}{\alpha}}\right) + \eta^{1-\beta}\mu^{\frac{1}{\alpha}} \exp\left(\eta\mu^{\frac{1}{\alpha}}\right)\right](\hat{t}_1 - \hat{t}_2)\right| \\
&\qquad + \left|\frac{1}{\pi\alpha} \int_0^\infty r^{\frac{1-\beta}{\alpha}} \exp\left(-r^{\frac{1}{\alpha}}\right)\right. \\
&\qquad \left. \times \frac{[(V(t_2') - V(t_1'))r\sin(\pi(1-\beta))] + [(V(\hat{t}_1)\hat{t}_2^\alpha - V(\hat{t}_2)\hat{t}_1^\alpha)\mu \sin(\pi(1-\beta+\alpha))]}{V(\hat{t}_1)V(\hat{t}_2)} dr\right| \\
&\qquad + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\left|\left[(1-\beta)\rho^{-\beta} + 2\rho^{1-\beta}\mu^{\frac{1}{\alpha}}\right](\hat{t}_1 - \hat{t}_2)\right| \\
&\quad \leq \frac{\mu^{\frac{1-\beta}{\alpha}}}{\alpha}\left|(1-\beta)\eta^{-\beta} \exp\left(\eta\mu^{\frac{1}{\alpha}}\right) + \eta^{1-\beta}\mu^{\frac{1}{\alpha}} \exp\left(\eta\mu^{\frac{1}{\alpha}}\right)\right||\hat{t}_1 - \hat{t}_2| \\
&\qquad + \frac{1}{\pi\alpha \sin^4(\pi\alpha)\mu^4 \hat{t}_1^{2\alpha}\hat{t}_2^{2\alpha}} \\
&\qquad \times \left|\int_0^\infty r^{\frac{1-\beta}{\alpha}} \exp\left(-r^{\frac{1}{\alpha}}\right)\left[r\left(V(\hat{t}_2) - V(\hat{t}_1)\right) + \mu\left(V(\hat{t}_1)\hat{t}_1^\alpha - V(\hat{t}_2)\hat{t}_2^\alpha\right)\right] dr\right| \\
&\qquad + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}}\left|(1-\beta)\rho^{-\beta} + 2\rho^{1-\beta}\mu^{\frac{1}{\alpha}}\right|\left|(\hat{t}_1 - \hat{t}_2)\right| \\
&\quad \leq O\left(|\hat{t}_1 - \hat{t}_2|\right)
\end{aligned}$$

$$+\frac{1}{\pi\alpha\sin^4(\pi\alpha)\mu^4\hat{t}_1^{2\alpha}\hat{t}_2^{2\alpha}}\left|\int_0^\infty r^{\frac{1-\beta}{\alpha}}\exp\bigl(-r^{\frac{1}{\alpha}}\bigr)\bigl[r\bigl(2\alpha\mu^{2\alpha-1}\mu^2-2r\mu\alpha\mu^{\alpha-1}\cos(\pi\alpha)\bigr)\right.$$

$$\left.+\mu\bigl(\alpha\xi^{\alpha-1}r^2-4\alpha r\xi^{2\alpha-1}\mu\cos(\pi\alpha)+3\alpha\xi^{3\alpha-1}\mu^2\bigr)\bigr]\,dr\right||\hat{t}_1-\hat{t}_2|$$

$$\le O\bigl(|\hat{t}_1-\hat{t}_2|\bigr)+\frac{1}{\pi\alpha\sin^4(\pi\alpha)\mu^4t_1'^{2\alpha}t_2'^{2\alpha}}\bigl(2\mu^2\alpha^2\mu^{2\alpha-1}\Gamma(2\alpha-\beta+1)$$

$$+2\mu\alpha^2\mu^{\alpha-1}\cos(\pi\alpha)\Gamma(3\alpha-\beta+1)+\mu\alpha^2\xi^{\alpha-1}\Gamma(3\alpha-\beta+1)$$

$$+4\mu^2\alpha^2\xi^{2\alpha-1}\cos(\pi\alpha)\Gamma(2\alpha-\beta+1)+3\mu^3\alpha^2\xi^{3\alpha-1}\Gamma(\alpha-\beta+1)\bigr)|\hat{t}_1-\hat{t}_2|$$

$$:=O\bigl(|\hat{t}_1-\hat{t}_2|\bigr)$$

as $\hat{t}_1$ tends to $\hat{t}_2$. where $\eta,\mu,\xi,\rho\in(\hat{t}_2,\hat{t}_1)$. The proof of this part is completed. □

4 Existence results

4.1 Existence results for $\mu>0$

In this section, we mainly prove the existence and uniqueness of solutions for equation (2). For convenience of the following presentation, set

$$k=1-\sum_{i=1}^m a_iA_{\alpha,1}\bigl(-\mu t_i^\alpha\bigr)-\mu\sum_{i=1}^m a_it_i^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_i^\alpha\bigr),$$

$$p=b\sum_{i=1}^m a_it_iA_{\alpha,2}\bigl(-\mu t_i^\alpha\bigr)+(b\mu+d)\sum_{i=1}^m a_it_i^{\alpha+1}A_{\alpha,\alpha+2}\bigl(-\mu t_i^\alpha\bigr)+c\sum_{i=1}^m a_it_i^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_i^\alpha\bigr),$$

$$N=\frac{|p|}{|k|}\left(1+\frac{\mu}{\Gamma(\alpha+1)}\right)+\frac{|b|}{\Gamma(2)}+\frac{|c|}{\Gamma(\alpha+1)}+\frac{|b\mu+d|}{\Gamma(\alpha+2)}.$$

According to (3) and (4), we get a general expression of the solution for ${}^{c}D_t^\beta({}^{c}D_t^\alpha+\mu)x(t)=f(t,x(t))$, $t\in I:=[0,1]$, $1<\alpha,\beta<2$, as follows:

$$\begin{aligned}x(t)={}&x(0)A_\alpha\bigl(-\mu t^\alpha\bigr)+x'(0)tA_{\alpha,2}\bigl(-\mu t^\alpha\bigr)\\&+\bigl[x(0)\mu+\bigl[{}^{c}D_t^\alpha x(t)\bigr]_{t=0}\bigr]t^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr)\\&+\bigl[x'(0)\mu+\bigl[{}^{c}D_t^\alpha x(t)\bigr]'_{t=0}\bigr]t^{\alpha+1}A_{\alpha,\alpha+2}\bigl(-\mu t^\alpha\bigr)\\&+\int_0^t(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)f\bigl(\tau,x(\tau)\bigr)\,d\tau.\end{aligned}\tag{11}$$

In (2), since $a=x(0)=\sum_{i=1}^m a_ix(t_i)$, we obtain

$$\begin{aligned}x(0)={}&\sum_{i=1}^m a_iaA_\alpha\bigl(-\mu t_i^\alpha\bigr)+\sum_{i=1}^m a_ibt_iA_{\alpha,2}\bigl(-\mu t_i^\alpha\bigr)+\sum_{i=1}^m a_i(a\mu+c)t_i^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_i^\alpha\bigr)\\&+\sum_{i=1}^m a_i(b\mu+d)t_i^{\alpha+1}A_{\alpha,\alpha+2}\bigl(-\mu t_i^\alpha\bigr)\\&+\sum_{i=1}^m a_i\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)f\bigl(\tau,x(\tau)\bigr)\,d\tau\\={}&a.\end{aligned}$$

Furthermore, we derive:

$$a = \frac{p + \sum_{i=1}^{m} a_i \int_0^{t_i} (t_i - \tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}(-\mu(t_i - \tau)^{\alpha}) f(\tau, x(\tau))\, d\tau}{k}. \tag{12}$$

On the other hand, in (2), we have

$$x'(0) = b, \qquad \left[{}^{c}D_t^{\alpha} x(t)\right]_{t=0} = c, \qquad \left[{}^{c}D_t^{\alpha} x(t)\right]'_{t=0} = d. \tag{13}$$

Thus, substituting (12) and (13) into (11), we get the following expression for the solution of (2):

$$\begin{aligned} x(t) = {} & \frac{1}{k}\left(A_{\alpha,1}\left(-\mu t^{\alpha}\right) + \mu t^{\alpha} A_{\alpha,\alpha+1}\left(-\mu t^{\alpha}\right)\right)\left(p + \sum_{i=1}^{m} a_i \int_0^{t_i} (t_i - \tau)^{\alpha+\beta-1}\right. \\ & \left. \times A_{\alpha,\alpha+\beta}\left(-\mu(t_i - \tau)^{\alpha}\right) f\left(\tau, x(\tau)\right) d\tau\right) \\ & + btA_{\alpha,2}\left(-\mu t^{\alpha}\right) + ct^{\alpha} A_{\alpha,\alpha+1}\left(-\mu t^{\alpha}\right) + (b\mu + d) t^{\alpha+1} A_{\alpha,\alpha+2}\left(-\mu t^{\alpha}\right) \\ & + \int_0^t (t - \tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\left(-\mu(t - \tau)^{\alpha}\right) f\left(\tau, x(\tau)\right) d\tau. \end{aligned}$$

Next, we introduce some conclusions about the existence and uniqueness of the solution for equation (2). Before that, we make the following assumptions:

(A_1) $f : I \times R \to R$ is continuous.

(A_2) There exists a positive constant L such that

$$\left|f(t,x) - f(t,\tilde{x})\right| \le L|x - \tilde{x}| \quad \text{for all } t \in I \text{ and } x, \tilde{x} \in R.$$

(A_3) Set $\beta > \alpha$. Suppose that $0 < L\sigma\omega < 1$, where

$$\omega = P(\alpha, \alpha + \beta, \mu)\left(\frac{1}{\beta - \alpha} + \frac{1}{\beta}\right) + \frac{2}{\alpha\mu\cos(\frac{\pi}{\alpha})}\left(\exp\left(\mu^{\frac{1}{\alpha}} \cos\left(\frac{\pi}{\alpha}\right)\right) - 1\right) > 0$$

and

$$\sigma = \frac{mA(\Gamma(\alpha + 1) + \mu)}{|k|\Gamma(\alpha + 1)} + 1, \qquad A = \max\left\{|a_i|, i = 1, \ldots, m\right\},$$

where $P(\alpha, \alpha + \beta, \mu)$ is defined in Lemma 3.3.

Let

$$B_r = \left\{x \in C(I, R) : \|x\|_{\infty} \le r\right\},$$

where

$$r \ge \frac{N + M\sigma\omega}{1 - L\sigma\omega},$$

and denote $M = \max\{|f(t, 0)| : t \in I\}$.

Theorem 4.1 *Assuming that conditions* (A_1)-(A_3) *are satisfied, equation* (2) *has a unique solution on* I.

Proof Define the operator $F: B_r \to C(I,R)$ by

$$\begin{aligned}(Fx)(t) ={}& \frac{1}{k}\bigl(A_{\alpha,1}\bigl(-\mu t^\alpha\bigr) + \mu t^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr)\bigr)\Biggl(p + \sum_{i=1}^m a_i \int_0^{t_i} (t_i-\tau)^{\alpha+\beta-1} \\ & \times A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau\Biggr) \\ & + btA_{\alpha,2}\bigl(-\mu t^\alpha\bigr) + ct^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr) + (b\mu+d)t^{\alpha+1}A_{\alpha,\alpha+2}\bigl(-\mu t^\alpha\bigr) \\ & + \int_0^t (t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr) f\bigl(\tau,x(\tau)\bigr)\,d\tau. \end{aligned} \tag{14}$$

According to (A_1), we know that F is well defined on $C(I,R)$. Next, we divide the proof into two steps.

Step 1. In this step, we show that $F(B_r) \subset B_r$:

$$\begin{aligned} &\bigl|(Fx)(t)\bigr| \\ &\quad\le \frac{|p|}{|k|}\biggl(1+\frac{\mu}{\Gamma(\alpha+1)}\biggr) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} + \frac{\Gamma(\alpha+1)+\mu}{|k|\Gamma(\alpha+1)} \\ &\qquad \times \sum_{i=1}^m |a_i| \int_0^{t_i} \bigl|(t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) \\ &\qquad \times \bigl[\bigl(f\bigl(\tau,x(\tau)\bigr) - f(\tau,0)\bigr) + f(\tau,0)\bigr]\bigr|\,d\tau \\ &\qquad + \int_0^t \bigl|(t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\bigl[\bigl(f\bigl(\tau,x(\tau)\bigr) - f(\tau,0)\bigr) + f(\tau,0)\bigr]\bigr|\,d\tau \\ &\quad\le \frac{|p|}{|k|}\biggl(1+\frac{\mu}{\Gamma(\alpha+1)}\biggr) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} + \Biggl[\frac{\Gamma(\alpha+1)+\mu}{|k|\Gamma(\alpha+1)} \\ &\qquad \times \sum_{i=1}^m |a_i| \int_0^{t_i} \Biggl[P(\alpha,\alpha+\beta,\mu)\biggl(\frac{1}{(t_i-\tau)^{\alpha-\beta+1}} + \frac{1}{(t_i-\tau)^{1-\beta}}\biggr) \\ &\qquad + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}} \exp\biggl((t_i-\tau)\mu^{\frac{1}{\alpha}}\cos\biggl(\frac{\pi}{\alpha}\biggr)\biggr)\Biggr]\,d\tau + \int_0^t \biggl(P(\alpha,\alpha+\beta,\mu) \\ &\qquad \times \biggl(\frac{1}{(t-\tau)^{\alpha-\beta+1}} + \frac{1}{(t-\tau)^{1-\beta}}\biggr) + \frac{2}{\alpha\mu^{1-\frac{1}{\alpha}}} \exp\biggl((t-\tau)\mu^{\frac{1}{\alpha}}\cos\biggl(\frac{\pi}{\alpha}\biggr)\biggr)\biggr)\,d\tau\Biggr] \\ &\qquad \times \bigl(L\|x\|_\infty + M\bigr) \\ &\quad\le \frac{|p|}{|k|}\biggl(1+\frac{\mu}{\Gamma(\alpha+1)}\biggr) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} + \frac{\Gamma(\alpha+1)+\mu}{|k|\Gamma(\alpha+1)}\bigl(L\|x\|_\infty + M\bigr) \\ &\qquad \times \sum_{i=1}^m A\biggl[P(\alpha,\alpha+\beta,\mu)\biggl(\frac{t_i^{\beta-\alpha}}{\beta-\alpha} + \frac{t_i^\beta}{\beta}\biggr) + \frac{2}{\alpha\mu\cos(\frac{\pi}{\alpha})}\biggl(\exp\biggl(t_i\mu^{\frac{1}{\alpha}}\cos\biggl(\frac{\pi}{\alpha}\biggr)\biggr) - 1\biggr)\biggr] \\ &\qquad + \bigl(L\|x\|_\infty + M\bigr)\biggl[P(\alpha,\alpha+\beta,\mu)\biggl(\frac{t^{\beta-\alpha}}{\beta-\alpha} + \frac{t^\beta}{\beta}\biggr) \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{\alpha\mu\cos(\frac{\pi}{\alpha})}\left(\exp\left(t\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)-1\right)\Bigg]\\
&\leq N+\left(\frac{mA(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}+1\right)\left(L\|x\|_\infty+M\right)\left[P(\alpha,\alpha+\beta,\mu)\left(\frac{1}{\beta-\alpha}+\frac{1}{\beta}\right)\right.\\
&\left.+\frac{2}{\alpha\mu\cos(\frac{\pi}{\alpha})}\left(\exp\left(\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)-1\right)\right]\\
&\leq N+\left(\frac{mA(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}+1\right)(Lr+M)\omega\\
&\leq N+\sigma(Lr+M)\omega\\
&\leq r.
\end{aligned}
$$

Step 2. In this step, we prove that F is a contraction mapping for $x,y\in B_r$. For each $t\in I$, by Lemma 3.3(ii) and Lemma 3.4(ii) we get

$$
\begin{aligned}
&\bigl|(Fx)(t)-(Fy)(t)\bigr|\\
&=\frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}\left(\sum_{i=1}^m|a_i|\int_0^{t_i}\bigl|(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)\right.\\
&\left.\times\bigl(f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr)\bigr|\,d\tau\right)\\
&+\int_0^t\bigl|(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\bigr|\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|\,d\tau\\
&\leq L\frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}\left(\sum_{i=1}^m\int_0^{t_i}|a_i|\bigl|(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)\bigr|\,d\tau\right)d\tau\,\|x-y\|_\infty\\
&+\int_0^t\bigl|(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\bigr|\,d\tau\,\|x-y\|_\infty\\
&\leq L\left(\frac{(mA\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}+1\right)\left[P(\alpha,\alpha+\beta,\mu)\left(\frac{1}{\beta-\alpha}+\frac{1}{\beta}\right)\right.\\
&\left.+\frac{2}{\alpha\mu\cos(\frac{\pi}{\alpha})}\left(\exp\left(\mu^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)-1\right)\right]\|x-y\|_\infty\\
&\leq L\sigma\omega\|x-y\|_\infty,
\end{aligned}
$$

which shows that F is a contraction mapping. By applying contraction mapping principles and (A_3) we obtain the conclusion of the theorem. The proof is completed. □

Next, we will use the Krasnoselskii fixed point theorem to derive the existence result for equation (2). Before the derivation, we give a new assumption:

(A_4) There exist $0<q<1$ and a real function $m(\cdot)\in L^{\frac{1}{q}}(I,R_+)$ such that $|f(t,x)|\leq m(t)$ for all $(t,x)\in I\times R$.

Let $B_{\bar r}=\{x\in C(I,R):\|x\|_\infty\leq\bar r\}$, where

$$
\bar r\geq N+\frac{1-q}{\Gamma(\alpha+\beta)(\alpha+\beta-q)}\sigma\,\|m\|_{L^{\frac{1}{q}}}.
$$

Theorem 4.2 *Assume that* (A_1) *and* (A_4) *hold. Then equation* (2) *has at least one solution on* I.

Proof Define two operators G and H on $B_{\bar{r}}$ as follows:

$$\begin{aligned}(Gx)(t) &= \int_0^t (t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr) f\bigl(\tau,x(\tau)\bigr)\,d\tau \\ &\quad + \left(\frac{A_{\alpha,1}(-\mu t^\alpha) + \mu t^\alpha A_{\alpha,\alpha+1}(-\mu t^\alpha)}{k}\right) \\ &\quad \times \left(\sum_{i=1}^m a_i \int_0^{t_i} (t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) f\bigl(\tau,x(\tau)\bigr)\,d\tau\right), \\ (Hy)(t) &= \frac{p}{k}\bigl(A_{\alpha,1}\bigl(-\mu t^\alpha\bigr) + \mu t^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr)\bigr) + btA_{\alpha,2}\bigl(-\mu t^\alpha\bigr) + ct^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr) \\ &\quad + (b\mu + d)t^{\alpha+1} A_{\alpha,\alpha+2}\bigl(-\mu t^\alpha\bigr).\end{aligned}$$

For any $x, y \in B_{\bar{r}}$ and $t \in I$, we have

$$\begin{aligned}&\bigl|(Gx)(t) + (Hy)(t)\bigr| \\ &\quad\le \frac{|p|}{|k|}\left(1 + \frac{\mu}{\Gamma(\alpha+1)}\right) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} + \frac{\Gamma(\alpha+1)+\mu}{|k|\Gamma(\alpha+1)} \\ &\qquad \times \sum_{i=1}^m |a_i| \int_0^{t_i} \bigl|(t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) m(\tau)\bigr|\,d\tau \\ &\qquad + \int_0^t \bigl|(t-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr) m(\tau)\bigr|\,d\tau \\ &\quad\le \frac{|p|}{|k|}\left(1 + \frac{\mu}{\Gamma(\alpha+1)}\right) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} \\ &\qquad + \frac{1}{\Gamma(\alpha+\beta)} \cdot \frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)} \sum_{i=1}^m |a_i| \int_0^{t_i} \bigl|(t_i-\tau)^{\alpha+\beta-1} m(\tau)\bigr|\,d\tau \\ &\qquad + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t \bigl|(t-\tau)^{\alpha+\beta-1} m(\tau)\bigr|\,d\tau \\ &\quad\le \frac{|p|}{|k|}\left(1 + \frac{\mu}{\Gamma(\alpha+1)}\right) + \frac{|b|}{\Gamma(2)} + \frac{|c|}{\Gamma(\alpha+1)} + \frac{|b\mu+d|}{\Gamma(\alpha+2)} \\ &\qquad + \frac{1}{\Gamma(\alpha+\beta)} \cdot \frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)} \sum_{i=1}^m |a_i| \left(\int_0^{t_i} \bigl|\bigl((t_i-\tau)^{\alpha+\beta-1}\bigr)^{\frac{1}{1-q}}\bigr|\,d\tau\right)^{1-q} \\ &\qquad \times \left(\int_0^{t_i} \bigl|\bigl(m(\tau)\bigr)\bigr|^{\frac{1}{q}}\,d\tau\right)^q \\ &\qquad + \frac{1}{\Gamma(\alpha+\beta)} \left(\int_0^t \bigl|\bigl((t-\tau)^{\alpha+\beta-1}\bigr)^{\frac{1}{1-q}}\bigr|\,d\tau\right)^{1-q} \left(\int_0^t \bigl|\bigl(m(\tau)\bigr)\bigr|^{\frac{1}{q}}\,d\tau\right)^q \\ &\quad\le N + \frac{1}{\Gamma(\alpha+\beta)}\left[\frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)} \sum_{i=1}^m \frac{A(1-q)}{\alpha+\beta-q} + \frac{1-q}{\alpha+\beta-q}\right] \|m\|_{L^{\frac{1}{q}}} \\ &\quad\le N + \frac{1-q}{\Gamma(\alpha+\beta)(\alpha+\beta-q)}\left[\frac{mA(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)} + 1\right] \|m\|_{L^{\frac{1}{q}}}\end{aligned}$$

$$
\begin{aligned}
&\le N + \frac{1-q}{\Gamma(\alpha+\beta)(\alpha+\beta-q)}\sigma\|m\|_{L^{\frac{1}{q}}} \\
&\le \bar{r}.
\end{aligned}
$$

The inequality obtained shows that $(G+H)(B_{\bar{r}}) \subset B_{\bar{r}}$.

It is easy to prove that H is a contraction mapping with zero Lipschitz constant. On the other hand, by the continuity of f we know the operator G is continuous. At the same time, we have

$$
\|Gx\|_\infty \le \frac{1-q}{\Gamma(\alpha+\beta)(\alpha+\beta-q)}\sigma\|m\|_{L^{\frac{1}{q}}} \le \bar{r}.
$$

So the operator G is uniformly bounded on $B_{\bar{r}}$. Next, we prove that G is a compact operator.

Define $f_{\max} = \sup\{|f(t,x)| : t \in I, x \in B_{\bar{r}}\}$. For any $t_1, t_2 \in I$ such that $t_1 < t_2$, by using Lemma 3.4(ii) we get

$$
\begin{aligned}
&\bigl|(Gx)(t_2) - (Gx)(t_1)\bigr| \\
&= \Bigg| \int_0^{t_2} (t_2-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \\
&\quad - \int_0^{t_1} (t_1-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_1-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \\
&\quad + \frac{1}{k}\bigl[A_{\alpha,1}\bigl(-\mu t_1^\alpha\bigr) - A_{\alpha,1}\bigl(-\mu t_2^\alpha\bigr)\bigr] + \bigl[\mu t_1^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_1^\alpha\bigr) - \mu t_2^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_2^\alpha\bigr)\bigr] \\
&\quad \times \left(\sum_{i=1}^m a_i \int_0^{t_i} (t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \right) \Bigg| \\
&\le \Bigg| \int_0^{t_1} (t_2-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) - (t_1-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_1-\tau)^\alpha\bigr) \\
&\quad \times f\bigl(\tau, x(\tau)\bigr)\,d\tau + \int_{t_1}^{t_2} (t_2-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \Bigg| \\
&\quad + \frac{1}{|k|}\bigl|\bigl(A_{\alpha,1}\bigl(-\mu t_1^\alpha\bigr) - A_{\alpha,1}\bigl(-\mu t_2^\alpha\bigr)\bigr) + \bigl(\mu t_1^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_1^\alpha\bigr) - \mu t_2^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_2^\alpha\bigr)\bigr)\bigr| \\
&\quad \times \left| \sum_{i=1}^m a_i \int_0^{t_i} (t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \right| \\
&\le \Bigg| \int_0^{t_1} \bigl((t_2-\tau)^{\alpha+\beta-1} - (t_1-\tau)^{\alpha+\beta-1}\bigr) A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \\
&\quad + \int_0^{t_1} (t_1-\tau)^{\alpha+\beta-1} \bigl(A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) - A_{\alpha,\alpha+\beta}\bigl(-\mu(t_1-\tau)^\alpha\bigr)\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \\
&\quad + \int_{t_1}^{t_2} (t_2-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_2-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \Bigg| \\
&\quad + \frac{1}{|k|}\bigl|\bigl(A_{\alpha,1}\bigl(-\mu t_1^\alpha\bigr) - A_{\alpha,1}\bigl(-\mu t_2^\alpha\bigr)\bigr) + \bigl(\mu t_1^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_1^\alpha\bigr) - \mu t_2^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t_2^\alpha\bigr)\bigr)\bigr| \\
&\quad \times \left| \sum_{i=1}^m a_i \int_0^{t_i} (t_i-\tau)^{\alpha+\beta-1} A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr) f\bigl(\tau, x(\tau)\bigr)\,d\tau \right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \left|\frac{f_{\max}}{\Gamma(\alpha+\beta)}\int_0^{t_1}\big((t_2-\tau)^{\alpha+\beta-1}-(t_1-\tau)^{\alpha+\beta-1}\big)\,d\tau \right.\\
&\quad \left. + \frac{f_{\max}}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha+\beta-1}d\tau\right|\\
&\quad + f_{\max}\int_0^{t_1}\big|(t_1-\tau)^{\alpha+\beta-1}\big(A_{\alpha,\alpha+\beta}\big(-\mu(t_2-\tau)^{\alpha}\big)-A_{\alpha,\alpha+\beta}\big(-\mu(t_1-\tau)^{\alpha}\big)\big)\big|\,d\tau\\
&\quad + \frac{f_{\max}}{|k|\Gamma(\alpha+\beta)}\big|\big(A_{\alpha,1}\big(-\mu t_1^{\alpha}\big)-A_{\alpha,1}\big(-\mu t_2^{\alpha}\big)\big)\\
&\quad + \big(\mu t_1^{\alpha}A_{\alpha,\alpha+1}\big(-\mu t_1^{\alpha}\big)-\mu t_2^{\alpha}A_{\alpha,\alpha+1}\big(-\mu t_2^{\alpha}\big)\big)\big|\left|\sum_{i=1}^{m}a_i\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}\,d\tau\right|\\
&\leq \frac{f_{\max}}{(\alpha+\beta)\Gamma(\alpha+\beta)}\big(2(t_2-t_1)^{\alpha+\beta}+\big(t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\big)\big)+\frac{f_{\max}}{\alpha+\beta}O\big(|t_2-t_1|\big)\\
&\quad + \frac{Amf_{\max}}{(\alpha+\beta)|k|\Gamma(\alpha+\beta)}\left((1+\mu)O\big(|t_2-t_1|\big)+\frac{\mu}{\Gamma(\alpha+1)}\big(t_2^{\alpha}-t_1^{\alpha}\big)\right),
\end{aligned}$$

which tends to zero as t_2 tends to t_1.

This result shows that G is equicontinuous, and thus G is relatively compact. Therefore, G is compact. Consequently, we get that $G+H$ is a condensing map on $B_{\tilde{r}}$. By using the Krasnoselskii fixed point theorem, problem (2) has at least one solution. The proof of this part is complete. □

Next, we will apply the Krasnoselskii-Zabreiko fixed point theorem to derive an existence result. We introduce two new assumptions:

(A_5) The function $f(t,0)\neq 0$ for some $t\in I$, and

$$\lim_{\|x\|_{\infty}\to\infty}\frac{f(t,x)}{x}=\phi(t).$$

(A_6) $\phi_{\sup} := \sup_{t\in I}|\phi(t)| < \frac{(\alpha+\beta)\Gamma(\alpha+\beta)}{\sigma}$.

Theorem 4.3 *Assume that* (A_1), (A_5), *and* (A_6) *are satisfied. Then equation* (2) *has at least one solution on* I.

Proof Let $B_{\tilde{r}}=\{x\in C(I,R):\|x\|_{\infty}\leq\tilde{r}\}$, where $\tilde{r}\geq N+\sigma f_{\max}\omega$. We set $f(t,x(t))=\phi(t)x(t)$; thus, problem (2) can be regarded as a linear problem. Define the bounded linear operator $L:B_{\tilde{r}}\to B_{\tilde{r}}$ by

$$\begin{aligned}
(Lx)(t) &= btA_{\alpha,2}\big(-\mu t^{\alpha}\big)+ct^{\alpha}A_{\alpha,\alpha+1}\big(-\mu t^{\alpha}\big)+(b\mu+d)t^{\alpha+1}A_{\alpha,\alpha+2}\big(-\mu t^{\alpha}\big)\\
&\quad + \frac{1}{k}\big(A_{\alpha,1}\big(-\mu t^{\alpha}\big)+\mu t^{\alpha}A_{\alpha,\alpha+1}\big(-\mu t^{\alpha}\big)\big)\left(p+\sum_{i=1}^{m}a_i\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}\right.\\
&\quad \left.\times A_{\alpha,\alpha+\beta}\big(-\mu(t_i-\tau)^{\alpha}\big)\phi(\tau)x(\tau)\,d\tau\right)\\
&\quad + \int_0^{t}(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\big(-\mu(t-\tau)^{\alpha}\big)\phi(\tau)x(\tau)\,d\tau.
\end{aligned}$$

Now, we claim that

$$\begin{aligned}\sup_{t\in I}\bigl|Lx(t)\bigr| &\le \frac{|p|}{|k|}\biggl(1+\frac{\mu}{\Gamma(\alpha+1)}\biggr)+\frac{|b|}{\Gamma(2)}+\frac{|c|}{\Gamma(\alpha+1)}+\frac{|b\mu+d|}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+\beta)}\\ &\quad\times\phi_{\sup}\biggl(\frac{mA(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}+1\biggr)\int_0^1(1-\tau)^{\alpha+\beta-1}\,d\tau\,\|x\|_\infty\\ &\le N+\frac{\phi_{\sup}\sigma}{(\alpha+\beta)\Gamma(\alpha+\beta)}\|x\|_\infty\\ &< N+\|x\|_\infty. \end{aligned}\tag{15}$$

If not, we can derive that

$$\phi_{\sup}\sigma=\lim_{\|x\|_\infty\to\infty}\frac{N+\frac{\sigma\phi_{\sup}}{(\alpha+\beta)\Gamma(\alpha+\beta)}\|x\|_\infty}{\|x\|_\infty}\ge 1,$$

which contradicts with (A_6). Consequently, we deduce that 1 is not an eigenvalue of the operator L due to (15).

It is obvious that F is well defined due to (A_1). Next we will show that $\frac{\|Fx-Lx\|_\infty}{\|x\|_\infty}$ vanishes as $\|x\|_\infty\to\infty$, where F is defined in equation (14). For $x\in B_{\tilde{r}}$, we have

$$\begin{aligned}&\bigl|(Fx)(t)-(Lx)(t)\bigr|\\ &=\frac{1}{|k|}\bigl[A_{\alpha,1}\bigl(-\mu t^\alpha\bigr)+\mu t^\alpha A_{\alpha,\alpha+1}\bigl(-\mu t^\alpha\bigr)\bigr]\Biggl|\sum_{i=1}^m a_i\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)\\ &\quad\times\bigl(f\bigl(\tau,x(\tau)\bigr)-\phi(\tau)x(\tau)\bigr)\,d\tau\\ &\quad+\int_0^t(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\bigl(f\bigl(\tau,x(\tau)\bigr)-\phi(\tau)x(\tau)\bigr)\,d\tau\Biggr|\\ &\le\Biggl[\frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}\sum_{i=1}^m|a_i|\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)\biggl|\frac{f(\tau,x(\tau))}{x(\tau)}-\phi(\tau)\biggr|\,d\tau\\ &\quad+\int_0^t(t-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\biggl|\frac{f(\tau,x(\tau))}{x(\tau)}-\phi(\tau)\biggr|\,d\tau\Biggr]\|x\|_\infty.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\frac{\|Fx-Lx\|_\infty}{\|x\|_\infty}\\ &\le\frac{(\Gamma(\alpha+1)+\mu)}{|k|\Gamma(\alpha+1)}\sum_{i=1}^m|a_i|\int_0^{t_i}(t_i-\tau)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t_i-\tau)^\alpha\bigr)\biggl|\frac{f(\tau,x(\tau))}{x(\tau)}-\phi(\tau)\biggr|\,d\tau\\ &\quad+\int_0^t(t-s)^{\alpha+\beta-1}A_{\alpha,\alpha+\beta}\bigl(-\mu(t-\tau)^\alpha\bigr)\biggl|\frac{f(\tau,x(\tau))}{x(\tau)}-\phi(\tau)\biggr|\,d\tau,\end{aligned}$$

which implies that

$$\lim_{\|x\|_\infty\to\infty}\frac{\|Fx-Lx\|_\infty}{\|x\|_\infty}=0$$

due to (A_5).

The proof is completed. □

4.2 Existence results for $\mu < 0$

In this section, we give three existence results for problem (2) with $\mu < 0$ similarly as in Section 3. Here, we need a new assumption:

(A_7) Let $0 < L\delta\varrho < 1$ and $\beta > \alpha$, where

$$\begin{aligned}\varrho &= P(\alpha,\alpha+\beta,-\mu)\left(\frac{1}{\beta-\alpha}+\frac{1}{\beta}\right)+\frac{2}{\alpha(-\mu)\cos(\frac{\pi}{\alpha})}\left(\exp\left((-\mu)^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)-1\right)\\&\quad+\frac{1}{\alpha}(-\mu)^{-\frac{\beta}{\alpha}}\left(e^{(-\mu)^{\frac{1}{\alpha}}}-1\right)>0,\\ \delta &= \frac{Am(A_{\alpha,1}(-\mu)+|\mu|A_{\alpha,\alpha+1}(-\mu))}{|k|}+1.\end{aligned}$$

For convenience of the following presentation, let

$$\bar{N} = |b|A_{\alpha,2}(-\mu)+|c|A_{\alpha,\alpha+1}(-\mu)+|d+b\mu|A_{\alpha,\alpha+2}(-\mu)+\left|\frac{p}{k}\right|\left(A_{\alpha,1}(-\mu)+|\mu|A_{\alpha,\alpha+1}(-\mu)\right).$$

Theorem 4.4 *Assume that* (A_1), (A_2), *and* (A_7) *hold. Then equation* (2) *has a unique solution on* I.

Proof Now we define $B_{r'} = \{x \in C(I,R) : \|x\|_\infty \leq r'\}$, where

$$r' \geq \frac{\bar{N}+M\delta\varrho}{1-L\delta\varrho}. \tag{16}$$

Like in Theorem 4.1, we consider $F : B_{r'} \to C(I,R)$ again. Then, we go on to prove that $F(B_{r'}) \subset B_{r'}$ for $\mu < 0$. For all $x \in B_{r'}$, applying (A_2) via Lemma 3.3(i) and Lemma 3.4(i), we have

$$\begin{aligned}&\left|(Fx)(t)\right|\\&\quad\leq \bar{N}+\frac{A}{|k|}\left(A_{\alpha,1}(-\mu)+|\mu|A_{\alpha,\alpha+1}(-\mu)\right)\\&\qquad\times\sum_{i=1}^{m}\int_0^{t_i}\left|P(\alpha,\alpha+\beta,-\mu)\left(\frac{1}{(t_i-\tau)^{\alpha-\beta+1}}+\frac{1}{(t_i-\tau)^{1-\beta}}\right)\right.\\&\qquad+\left.\frac{2}{\alpha(-\mu)^{1-\frac{1}{\alpha}}}\exp\left((t_i-\tau)(-\mu)^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\right|\\&\qquad+\left|\frac{1}{\alpha}(-\mu)^{\frac{1-\alpha-\beta}{\alpha}}\exp\left((-\mu)^{\frac{1}{\alpha}}(t_i-\tau)\right)\right|d\tau\left(L\|x\|_\infty+M\right)\\&\qquad+\int_0^{t}\left|P(\alpha,\alpha+\beta,-\mu)\left(\frac{1}{(t-\tau)^{\alpha-\beta+1}}+\frac{1}{(t-\tau)^{1-\beta}}\right)\right.\\&\qquad+\left.\frac{2}{\alpha(-\mu)^{1-\frac{1}{\alpha}}}\exp\left((t-\tau)(-\mu)^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right)\right|\\&\qquad+\left|\frac{1}{\alpha}(-\mu)^{\frac{1-\alpha-\beta}{\alpha}}\exp\left((-\mu)^{\frac{1}{\alpha}}(t-\tau)\right)\right|d\tau\left(L\|x\|_\infty+M\right)\end{aligned}$$

$$\begin{aligned}
&\leq \bar{N} + \left(\frac{Am(A_{\alpha,1}(-\mu) + |\mu|A_{\alpha,\alpha+1}(-\mu))}{|k|} + 1\right)\bigl(Lr' + M\bigr)\varrho \\
&\leq \bar{N} + \delta\bigl(Lr' + M\bigr)\varrho \\
&\leq r',
\end{aligned}$$

which implies that $F(B_{r'}) \subset B_{r'}$. For $x, y \in B_{r'}$ and $t \in I$, according to Lemma 3.3(i) and Lemma 3.4(i), we obtain

$$\begin{aligned}
&\bigl|(Fx)(t) - (Fy)(t)\bigr| \\
&\quad\leq L\left(\frac{Am(A_{\alpha,1}(-\mu) + |\mu|A_{\alpha,\alpha+1}(-\mu))}{|k|} + 1\right)\left[P(\alpha,\alpha+\beta,-\mu)\left(\frac{1}{\beta-\alpha} + \frac{1}{\beta}\right)\right. \\
&\qquad \left. + \frac{2}{\alpha(-\mu)\cos(\frac{\pi}{\alpha})}\left(\exp\left((-\mu)^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)\right) - 1\right) + \frac{1}{\alpha}(-\mu)^{-\frac{\beta}{\alpha}}\bigl(e^{(-\mu)^{\frac{1}{\alpha}}} - 1\bigr)\right]\|x-y\|_\infty \\
&\quad\leq L\left(\frac{Am(A_{\alpha,1}(-\mu) + |\mu|A_{\alpha,\alpha+1}(-\mu))}{|k|} + 1\right)\varrho\|x-y\|_\infty \\
&\quad\leq L\delta\varrho\|x-y\|_\infty,
\end{aligned}$$

which shows that

$$\|Fx - Fy\|_\infty \leq L\delta\varrho\|x-y\|_\infty.$$

By (A_7) and the contraction mapping principle we complete the proof of this theorem. □

Next, we are ready to give another existence result.

Theorem 4.5 *Assume that* (A_1) *and* (A_4) *are satisfied. Then equation* (2) *has at least one solution on* I.

Proof Let $B_{\bar{r}'} = \{x \in C(I,R) : \|x\|_\infty \leq \bar{r}'\}$, where

$$\bar{r}' \geq \bar{N} + \frac{(1-q)A_{\alpha,\alpha+\beta}(-\mu)}{\alpha+\beta-q}\delta.$$

We consider the operators G and H in Theorem 4.2 again. Applying the same method as in Step 1 of Theorem 4.2, we deduce that $(G+H)(B_{\bar{r}'}) \subset B_{\bar{r}'}$ for some positive number $\bar{r}'$. At the same time, we get that H is a contraction mapping due to (A_4).

To prove the compactness of the operator G, we only need to verify that operator G is equicontinuous. For any $t_1, t_2 \in I$ such that $t_1 < t_2$, we have

$$\begin{aligned}
&\bigl|(Gx)(t_2) - (Gx)(t_1)\bigr| \\
&\quad\leq \frac{f_{\max}A_{\alpha,\alpha+\beta}(-\mu)}{\alpha+\beta}\bigl[2(t_2-t_1)^{\alpha+\beta} + \bigl(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}\bigr)\bigr] + \frac{f_{\max}}{\alpha+\beta}O\bigl(|t_2-t_1|\bigr) \\
&\qquad + \frac{Amf_{\max}A_{\alpha,\alpha+\beta}(-\mu)}{|k|(\alpha+\beta)}\bigl((1+|\mu|)O\bigl(|t_2-t_1|\bigr) + |\mu|A_{\alpha,\alpha+1}(-\mu)\bigl(t_2^\alpha - t_1^\alpha\bigr)\bigr),
\end{aligned}$$

which tends to zero as $t_2 \to t_1$, where we used Lemmas 3.3(i) and 3.4(i). So the operator G is equicontinuous.

The remaining proof is the same as that of Theorem 4.2. □

Now we give the last assumption:

(A_8) $\phi_{\sup} := \sup_{t\in I} |\phi(t)| < \frac{\alpha+\beta}{A_{\alpha,\alpha+\beta}(-\mu)\delta}$, where ϕ is defined in (A_5).

Theorem 4.6 *Assume that* (A_1), (A_5), *and* (A_8) *are satisfied. Then equation* (2) *has at least one solution on* I.

Proof Choose $r \geq \bar{N} + \delta f_{\max}\varrho$. Then, similarly as in Theorem 4.3, we get the existence result for equation (2). □

5 Examples

In this section, we give two examples to illustrate our main results.

Example 5.1 Let $\alpha = \frac{4}{3}$, $\beta = \frac{3}{2}$, $\mu = 1$, $m = 1$, $a_1 = 4$, $x(t_1) = \frac{1}{2}$. We consider

$$\begin{cases} {}^cD_t^{\frac{3}{2}}({}^cD_t^{\frac{4}{3}} + 1)x(t) = \frac{\sin t}{(t+4)^2}\frac{x(t)}{1+x(t)}, & t \in I := [0,1], x \in R, 1 < \alpha, \beta < 2, \\ x(0) = a_1 x(t_1) = 2, \quad x'(0) = b, & [{}^cD_t^{\alpha}x(t)]_{t=0} = c, \quad [{}^cD_t^{\alpha}x(t)]'_{t=0} = d, \end{cases} \tag{17}$$

where b, c, d are some constants.

Case 1. Define $f(t,x(t)) = \frac{\sin t}{(t+4)^2}\frac{x(t)}{1+x(t)}$, $t \in [0,1]$. For $x, y \in [0,\infty)$, we have

$$\left|f(t,x(t)) - f(t,y(t))\right| \leq \left|\frac{\sin t}{(t+4)^2}\right|\left|\frac{x-y}{(1+x)(1+y)}\right| \leq \frac{1}{16}|x-y|.$$

Thus, we take $L = \frac{1}{16}$. Then, $|k| = 2.9999$ and $P(\frac{4}{3}, \frac{3}{2} + \frac{4}{3}, 1) = 0.3802$. Further, $\omega = P(\frac{4}{3}, \frac{3}{2} + \frac{4}{3}, 1)(\frac{1}{\frac{3}{2}-\frac{4}{3}} + \frac{1}{\frac{3}{2}}) + \frac{3(e^{\cos\frac{3\pi}{4}})}{2\cos(\frac{3\pi}{4}-1)} = 3.9689$, $\sigma = \frac{4(\Gamma(\frac{4}{3}+1)+1)}{|k|\Gamma(\frac{4}{3}+1)} + 1 = 3.4533$.

Now $L\omega\sigma = \frac{1}{16} \times 3.9689 \times 3.4533 = 0.8566 < 1$. Then ($A_1$)-($A_3$) hold. By Theorem 4.1, equation (17) has a unique solution.

Case 2. Let $m(t) = \frac{\sin t}{(t+4)^2}$ and $0 < q < 1$. It is obvious that $|f(t,x(t))| \leq m(t) \in L^{\frac{1}{q}}(I,R)$. Thus, ($A_1$) and ($A_4$) are satisfied. By Theorem 4.2, equation (17) has at least one solution.

Example 5.2 Let $\alpha = \frac{6}{5}$, $\beta = \frac{4}{3}$, $\mu = -1$, $m = 1$, $a_1 = 4$, $x(t_1) = \frac{1}{2}$. We consider

$$\begin{cases} {}^cD_t^{\frac{4}{3}}({}^cD_t^{\frac{6}{5}} - 1)x(t) = \frac{|x(t)|}{(2t+10)^2}, & t \in I := [0,1], x \in R, 1 < \alpha, \beta < 2, \\ x(0) = a_1 x(t_1) = 2, \quad x'(0) = b, & [{}^cD_t^{\alpha}x(t)]_{t=0} = c, \quad [{}^cD_t^{\alpha}x(t)]'_{t=0} = d, \end{cases} \tag{18}$$

where b, c, d are some constants.

Case 1. Define $f(t,x(t)) = \frac{|x(t)|}{(2t+10)^2}$, $t \in [0,1]$. For $x, y \in R$, obviously, $|f(t,x(t)) - f(t,y(t))| \leq \frac{1}{100}|x-y|$. So take $L = \frac{1}{100}$. Then, we obtain $|k| = 2.9998$ and $P(\frac{6}{5}, \frac{6}{5} + \frac{4}{3}, 1) = 1.0117$.

Further, $\varrho = P(\frac{6}{5}, \frac{6}{5} + \frac{4}{3}, 1)(\frac{1}{\frac{4}{3}-\frac{6}{5}} + \frac{1}{\frac{4}{3}}) + \frac{5(e^{\cos\frac{5\pi}{6}} - 1)}{3\cos(\frac{5\pi}{6})} + \frac{5}{6}(e-1) = 3.5587$, $\delta = \frac{4(A_{\frac{6}{5},1}(1) + A_{\frac{6}{5},\frac{6}{5}+1}(1))}{|k|} + 1 = 5.8830$. Now $L\delta\varrho = \frac{1}{100} \times 5.8830 \times 3.5587 = 0.2094 < 1$. Then ($A_1$)-($A_3$) hold. By Theorem 4.4, equation (18) has a unique solution.

Case 2. Define $f(t,x(t)) = \frac{|x(t)|}{(2t+10)^2}$ for $t \in I$. Then, (A_1) is satisfied, and $\lim_{\|x\|_\infty\to\infty} \frac{f(t,x)}{x} = \frac{1}{(2t+10)^2} := \phi(t)$. Set $\phi_{\sup} = \frac{1}{100}$. Further, we have $\delta = 5.8830$, so $\phi_{\sup} = 0.01 < \frac{\frac{6}{5}+\frac{4}{3}}{A_{\frac{6}{5},\frac{6}{5}+\frac{4}{3}}(1)\varrho} = 0.4227$.

Now (A_5) and (A_8) hold. By Theorem 4.6, equation (18) has a unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration of all the authors. ZG, XY, and JRW proved the theorems, interpreted the results, and wrote the article. All authors defined the research theme and read and approved the manuscript.

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