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Long-time behavior of a semilinear wave equation with memory

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Abstract

In this paper we study the long-time dynamics of the semilinear viscoelastic equation

$$u_{tt} - \Delta u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + f(u) = h,$$

defined in a bounded domain of $\mathbb{R}^3$ with Dirichlet boundary condition. The functions $f = f(u)$ and $h = h(x)$ represent forcing terms and the kernel function $\mu \geq 0$ is assumed to decay exponentially. Then, by exploring only the dissipation given by the memory term, we establish the existence of a global attractor to the corresponding dynamical system.

MSC: 35L71; 35B41; 74D99

Keywords: wave equation; global attractor; memory; viscoelasticity

1 Introduction

This is paper is concerned with the long-time behavior of a class of wave equations with memory of the form

$$u_{tt} - \Delta u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + f(u) = h \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{1.2}$$

with initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x), \quad \forall x \in \Omega, \\ &u(x,-s) = \varphi(x,s), \quad \forall (x,s) \in \Omega \times \mathbb{R}^+, \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ, and φ is a prescribed past history.

This problem is related to a model of extensional vibrations of thin rods

$$u_{tt} - \Delta u_{tt} - \Delta u = 0,$$

described in Love [1], Chapter 20, which is a conservative system. Here, we have added a nonlinear forcing $f(u)$ and a dissipation of memory type.

We observe that such a system was extensively studied in the more general form

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + f(u) = 0, \tag{1.4}$$

with $0 \leq \rho < 2$. Most of results are concerned with the exponential stability of the system under additional damping $-\Delta u_t$ or u_t. We refer the reader to, *e.g.*, [2–5]. The existence of global attractors to (1.4) was first proved in Araújo *et al.* [6], with the assumption $\rho > 1$ and with the additional damping $-\Delta u_t$. The assumption $\rho > 1$ was technical and related to the uniqueness of the problem. Later, it was shown in [7] that the strong damping $-\Delta u_t$ could be replaced by the weak damping u_t, but yet with $\rho > 1$. On the other hand, in [8], the existence of a global attractor for the problem with $\rho = \mu = 0$ was studied with a strong damping.

More recently, it was proved by Conti *et al.* [9] that existence and uniqueness for the mixed problem (1.4) holds for $\rho \geq 0$ and without additional damping terms, that is, keeping only the dissipation given by the memory. This means that the restriction $\rho > 1$ can be dropped.

Motivated by results in [6] and [9], we propose to study the existence of global attractors of (1.4) with $\rho = 0$ and exploring only the dissipation given by the memory term. That is, we consider the problem (1.1)-(1.3). Then our result extends or complements the ones in [6–9]. See Theorem 3.1.

Of course, if the rotational inertia Δu_{tt} is dropped, then equation (1.1) becomes the well-known viscoelastic wave equation of memory type. On this matter, we refer the reader to some relevant results in [10–14], among others.

2 History setting

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ is the standard inner product and norm on $L^2(\Omega)$. It is well known that the operator A with domain $D(A)$ defined by

$$A = -\Delta, \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

is self-adjoint and strictly positive. See, *e.g.*, [15]. We adopt the notation

$$H_0 = L^2(\Omega), \qquad H_1 = H_0^1(\Omega) \quad \text{and} \quad H_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

Next we establish the history setting of the problem (1.1)-(1.3) in order to deal with the non-autonomous character of the memory term in (1.1). We follow the arguments of [9, 12, 14], based on [16]. Let $\mu : \mathbb{R}^+ \to [0,\infty)$ be a summable function. We denote by $\mathcal{M}$ the L^2-weighted space defined by

$$L^2_\mu(\mathbb{R}^+; H_1) = \{\eta : \mathbb{R}^+ \to H_1 : \|\eta\|_{\mathcal{M}} < \infty\},$$

where $\|\cdot\|_{\mathcal{M}} = (\cdot,\cdot)_{\mathcal{M}}^{\frac{1}{2}}$ and $(\eta,\xi)_{\mathcal{M}} = \int_0^\infty \mu(\tau)(\eta(\tau),\xi(\tau))_1\,d\tau$. Similarly we define the space $\mathcal{M}_0$ as

$$L^2_\mu(\mathbb{R}^+; H_0) = \{\eta : \mathbb{R}^+ \to H_0 : \|\eta\|_{\mathcal{M}_0} < \infty\},$$

where $\|\cdot\|_{\mathcal{M}_0} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{M}_0}$ and $(\eta,\xi)_{\mathcal{M}_0} = \int_0^\infty \mu(\tau)(\eta(\tau),\xi(\tau))_0\,d\tau$. From classical theory, the spaces $\mathcal{M}$ and $\mathcal{M}_0$ are separable Hilbert spaces.

Let T be the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, that is,

$$T\eta = -\eta',$$

for all $\eta \in D(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M}, \eta(0) = 0\}$, where $\eta'(t) = \frac{\partial\eta}{\partial t}$ in the sense of distributions and $\eta(0) = \lim_{s\to 0}\eta(s)$. It is well known that

$$(T\eta,\eta)_{\mathcal{M}} \leq 0, \quad \forall \eta \in D(T).$$

We also introduce the Hilbert space

$$\mathcal{H} = H_1 \times H_1 \times \mathcal{M},$$

endowed with the norms $\|\cdot\|_{\mathcal{H}} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{H}}$ where

$$\big((u_1,v_1,\eta_1),(u_2,v_2,\eta_2)\big)_{\mathcal{H}} = (u_1,u_2)_1 + (v_1,v_2)_1 + (\eta_1,\eta_2)_{\mathcal{M}}.$$

Then, as in [12, 14], we define

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad s \in \mathbb{R}^+.$$

Using this new variable η we can reformulate the system (1.1)-(1.3) to become

$$\begin{cases} u_{tt} + Au_{tt} + Au - \int_0^\infty \mu(s)A\eta(s)\,ds + f(u) = h, \\ \eta_t = T\eta + u_t, \end{cases} \tag{2.1}$$

with initial conditions

$$u(0) = u_0, \qquad u_t(0) = v_0, \qquad \eta^0(s) = \eta_0, \tag{2.2}$$

where $\eta_0(s) = u_0 - \varphi(s)$ for all $s \in \mathbb{R}^+$.

The system (2.1)-(2.2) is a particular case of the system considered in [9]. There, the authors established the well-posedness for a class of problems with $|u_t|^\rho u_{tt}$ instead of u_{tt} as in (2.1). They proved, among other results, that the system (2.1)-(2.2) with initial data $z = (u_0,v_0,\eta_0) \in \mathcal{H}$ admits a unique weak solution

$$(u,\eta) \in W^{2,\infty}(0,\tau;H_1) \times C\big([0,\tau];\mathcal{M}\big),$$

satisfying the identity

$$(u_{tt},\phi) + (u_{tt},\phi)_1 + (u,\phi)_1 + \int_0^\infty \mu(s)\big(\eta(s),\phi\big)_1\,ds + \big(f(u),\phi\big) = (h,\phi), \tag{2.3}$$

for every $\phi \in H_1$ and for a.e. $t > 0$. Here, η is a mild solution to the non-homogeneous linear equation in the Hilbert space $\mathcal{M}$,

$$\frac{d}{dt}\eta = T\eta + u_t,$$

where τ is a positive real number arbitrarily fixed. In addition, it was shown in Estimate (4.4) of [9],

$$\|u_{tt}\|_1 \le C, \quad \text{a.e. } t \in [0, \tau], \tag{2.4}$$

where $C > 0$ depends only on the initial data.

Now, due to the continuous dependence on initial data, the weak solution (u, η) of the system (2.1)-(2.2), with initial data $(u(0), u_t(0), \eta^0) = z$, can be rewritten in the form

$$S(t)z = \bigl(u(t), u_t(t), \eta^t\bigr), \tag{2.5}$$

generating a C_0-semigroup $S(t)$ on $\mathcal{H}$.

We end this section by recalling that a global attractor for a C_0-semigroup $S(t)$ on $\mathcal{H}$ is a compact subset $\mathbf{A} \subset \mathcal{H}$ which is strictly invariant, that is, $S(t)\mathbf{A} = \mathbf{A}$, $\forall t \ge 0$, and uniformly attracting, that is,

$$\operatorname{dist}_{\mathcal{H}}\bigl(S(t)B, \mathbf{A}\bigr) = \sup_{x \in S(t)B} \inf_{y \in \mathbf{A}} \|x - y\|_{\mathcal{H}} \to 0 \quad \text{as } t \to \infty,$$

for any bounded set $B \subset \mathcal{H}$.

3 Global attractors

In this section we establish our main result. The assumptions we make in this paper are as follows.

(H_1) Assume $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and satisfying the following conditions:
  - (i) $\mu(s) \ge 0$ for all $s \in \mathbb{R}^+$;
  - (ii) there exists a positive constant k_1 such that $\mu'(s) \le -k_1\mu(s)$ for all $s \in \mathbb{R}^+$.

(H_2) The nonlinearity $f \in C^1(\mathbb{R})$ and verifies the following conditions:
  - (i) $|f(r) - f(s)| \le C(1 + |r|^p + |s|^p)|r - s|$ for all $r, s \in \mathbb{R}$, where $0 \le p < 4$;
  - (ii) there exists a positive constant ρ such that $f(s)s - \hat{f}(s) \ge -\rho$, where $\hat{f}(s) = \int_0^s f(\tau)\,d\tau$.

(H_3) The forcing h belongs to the dual space of H_1.

To simplify the notation we write $\|\mu\|_{L^1(\mathbb{R}^+)} = k_0$ in our estimates. We also observe that the energy associated with the problem (2.1)-(2.2) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u_t\|_1^2 + \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 \\ & + \int_\Omega \hat{f}(u)\,dx - \int_\Omega hu\,dx, \quad t \ge 0. \end{aligned}$$

Our main result is the following.

Theorem 3.1 *Suppose that the conditions* (H_1)-(H_3) *are verified. Then the dynamical system* $(S(t), \mathcal{H})$ *generated by the problem* (2.1)-(2.2) *has a global attractor.*

The proof of this theorem will be completed at the end of this section.

3.1 Abstract theory

Let us present a small collection of well-known results of from the theory of attractors. This can be found in, *e.g.*, [17–21]. A dynamical system $(\mathcal{H}, S(t))$ is called dissipative if the semigroup $S(t)$ has an absorbing set, that is, a bounded set $B_0 \subset \mathcal{H}$ such that

$$S(t)B \subset B_0, \quad \forall t \geq t_B,$$

for all bounded set $B \subset \mathcal{H}$. A semigroup $S(t)$ is asymptotically smooth in $\mathcal{H}$ if for any bounded positive invariant set $B \subset \mathcal{H}$, that is, $S(t)B \subseteq B$ for all $t \geq 0$, there exists a compact set $K \subset \overline{B}$, such that

$$\operatorname{dist}_{\mathcal{H}}\big(S(t)B, K\big) \to 0 \quad \text{as } t \to \infty.$$

Then a classical result asserts that a dissipative C_0-semigroup $S(t)$ defined on $\mathcal{H}$ has a compact global attractor in $\mathcal{H}$ if and only if it is asymptotically smooth in $\mathcal{H}$.

Now, it is well known from Proposition 2.10 in [18] that a $S(t)$ is asymptotically smooth in $\mathcal{H}$ if for any positively invariant set $B \subset \mathcal{H}$, and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that

$$\big\| S(T)x - S(T)y \big\|_{\mathcal{H}} \leq \varepsilon + \phi_T(x, y), \quad \forall x, y \in B, \tag{3.1}$$

where $\phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z_n, z_m) = 0, \tag{3.2}$$

for any sequence (z_n) in B.

3.2 Dissipativeness

In this section we shall construct a bounded absorbing set to our system $(\mathcal{H}, S(t))$ where $S(t)$ is the solution operator defined in (2.5). Let (u, u_t, η) be a weak solution of the system (2.1)-(2.2). Since H_1 is dense in H_2 and $u \in L^\infty(0, \tau; H_1)$ we can assume that u is more regular and obtain

$$\|u\|_{L^{p+2}}^{p+2} \leq K_\Omega \|u\|_1^{p+2} \|u\|,$$

where K_Ω is an embedding constant. To simplify the notation denote all the embedding constants by C_Ω. Replacing ϕ by u_t in (2.3) and adding the identity $(\eta_t, \eta) = (T\eta, \eta)_{\mathcal{M}} + (u_t, \eta)_{\mathcal{M}}$ we get

$$\frac{d}{dt} E(t) = (T\eta, \eta)_{\mathcal{M}} = \frac{1}{2} \int_0^\infty \mu'(s) \big\| \eta(s) \big\|_1^2 \, ds \leq 0.$$

We are going to apply the perturbed energy method. We consider the maps

$$\Psi(t) = \big(u_t(t), u(t)\big) + \big(u_t(t), u(t)\big)_1, \qquad \Phi(t) = \big(u_t(t), \eta^t\big)_{\mathcal{M}} + \big(u_t(t), \eta^t\big)_{\mathcal{M}_0} \tag{3.3}$$

and

$$F_\varepsilon(t) = \varepsilon^{-1}E(t) + \varepsilon\Psi(t) - \Phi(t), \tag{3.4}$$

where $0 < \varepsilon \leq 1$. Then we have the following result.

Lemma 3.2 *Let* Ψ, Φ *and* F_ε *be the maps defined in* (3.3) *and* (3.4). *Then*

(a) *there exist constants* $\beta = \beta(k_0, \lambda_1) \geq 1$ *and* $C_{\rho,\lambda_1,h,\Omega} \geq 0$ *such that*

$$\left|\varepsilon\Psi(t) - \Phi(t)\right| \leq \beta\left(E(t) + C_{\rho,\lambda_1,h,\Omega}\right);$$

(b) *for every* $0 < \varepsilon < \beta^{-1}$, *the positive constants* $\beta_1 = \varepsilon^{-1} - \beta$ *and* $\beta_2 = \varepsilon^{-1} + \beta$ *satisfy the inequality*

$$\beta_1 E(t) - \frac{1}{2}C_{\rho,\lambda_1,h,\Omega} \leq F_\varepsilon(t) \leq \beta_2 E(t) + \frac{1}{2}C_{\rho,\lambda_1,h,\Omega}.$$

Proof By using Sobolev embeddings and condition (H_2) we have

$$\begin{aligned}
\left|-\int_\Omega \left(\hat{f}(u) - hu\right) dx\right| &= \left|-\int_\Omega \left(\hat{f}(u) - f(u)u + f(u)u - hu\right) dx\right| \\
&\leq \left|\int_\Omega \left(\hat{f}(u) - f(u)u\right) dx\right| + \left|\int_\Omega \left(f(u)u - hu\right) dx\right| \\
&\leq \rho|\Omega| + \int_\Omega \left|\left(f(u)u - hu\right)\right| dx \\
&\leq \rho|\Omega| + \int_\Omega \left(1 + |u|^q\right)|u|^2\, dx + 6\lambda_1^{-1}\|h\|^2 + \frac{1}{12}\|u\|_1^2 \\
&\leq \rho|\Omega| + \|u\|^2 + \|u\|_{p+2}^{p+2} + 6\lambda_1^{-1}\|h\|^2 + \frac{1}{12}\|u\|_1^2 \\
&\leq \rho|\Omega| + K_\Omega\|u\|_1 + K_\Omega\|u\|_1 + 6\lambda_1^{-1}\|h\|^2 + \frac{1}{12}\|u\|_1^2 \\
&\leq \rho|\Omega| + 12(K_\Omega)^2 + \frac{1}{12}\|u\|_1^2 + \frac{1}{12}\|u\|_1^2 \\
&\quad + 6\lambda_1^{-1}\|h\|^2 + \frac{1}{12}\|u\|_1^2,
\end{aligned} \tag{3.5}$$

where $|\Omega|$ is the measure of Ω. Therefore, there is a positive constant $C_{\rho,\lambda_1,h,\Omega}$ such that

$$\frac{1}{4}\|u\|_1^2 + \int_\Omega \left(\hat{f}(u) - hu\right) dx + C_{\rho,\lambda_1,h,\Omega} \geq 0. \tag{3.6}$$

Applying the inequality (3.6) we obtain

$$\begin{aligned}
\left|\varepsilon\Psi(t) - \Phi(t)\right| &\leq \varepsilon\left|(u_t, u) + (u_t, u)_1\right| + \left|(u_t, \eta)_{\mathcal{M}} + (u_t, \eta)_{\mathcal{M}_0}\right| \\
&\leq \varepsilon\left(2\lambda_1^{-1}\|u_t\|^2 + \frac{1}{8}\|u\|_1^2\right) + \varepsilon\left(2\|u_t\|_1^2 + \frac{1}{8}\|u\|_1^2\right) \\
&\quad + \left(k_0\|u_t\|_1^2 + \frac{1}{4}\|\eta\|_{\mathcal{M}}^2\right) + \left(k_0\|u_t\|^2 + \frac{1}{4}\|\eta\|_{\mathcal{M}}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \big(k_0 + 2\lambda_1^{-1}\varepsilon\big)\|u_t\|^2 + (k_0 + 2\varepsilon)\|u_t\|_1^2 + \frac{\varepsilon}{4}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 \\
&\le \big(k_0 + 2\lambda_1^{-1}\varepsilon\big)\|u_t\|^2 + (k_0 + 2\varepsilon)\|u_t\|_1^2 + \frac{\varepsilon}{4}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 \\
&\quad + \left(\frac{1}{4}\|u\|_1^2 + \int_\Omega \big(\hat{f}(u) - hu\big)\,dx + C_{\rho,\lambda_1,h,\Omega}\right) \\
&\le \beta\big(E(t) + C_{\rho,\lambda_1,h,\Omega}\big),
\end{aligned}
$$

where $\beta = \sup_{0<\varepsilon\le 1}\{1, \varepsilon, 2(k_0 + 2\varepsilon), 2(k_0 + 2\lambda_1^{-1}\varepsilon)\}$, which ends the proof of (a). Now, let $0 < \varepsilon < \frac{1}{\beta}$. By item (a) we obtain

$$
\begin{aligned}
\big|F_\varepsilon(t) - \varepsilon^{-1}E(t)\big| &= \big|\varepsilon\Psi(t) - \Phi(t)\big| \\
&\le \beta\big(E(t) + C_{\rho,\lambda_1,h,\Omega}\big),
\end{aligned}
$$

which provides

$$
\left(\frac{1}{\varepsilon} - \beta\right)E(t) - \frac{1}{2}C_{\rho,\lambda_1,h,\Omega} \le F_\varepsilon(t) \le \left(\frac{1}{\varepsilon} + \beta\right)E(t) + \frac{1}{2}C_{\rho,\lambda_1,h,\Omega}.
$$

This proves (b). □

Lemma 3.3 *There exists $\varepsilon_1 > 0$ such that*

$$
F_\varepsilon'(t) \le -\varepsilon E(t) + C_\varepsilon, \quad \forall t \ge 0, \forall \varepsilon \in (0, \varepsilon_1).
$$

Proof By (3.3) we get

$$
\begin{aligned}
\Psi'(t) &= (u_{tt}, u) + (u_t, u_t) + (u_{tt}, u)_1 + (u_t, u_t)_1 \\
&= \|u_t\|^2 + \|u_t\|_1^2 - \|u\|_1^2 - (u, \eta)_{\mathcal{M}} - \big(f(u), u\big) + (h, u) \\
&\le \|u_t\|^2 + \|u_t\|_1^2 - \|u\|_1^2 - (u, \eta)_{\mathcal{M}} + \rho|\Omega| + \left(-\int_\Omega \hat{f}(u)\,dx + (h, u)\right) \\
&= \|u_t\|^2 + \|u_t\|_1^2 - \|u\|_1^2 - (u, \eta)_{\mathcal{M}} + \rho|\Omega| - E(t) + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u_t\|_1^2 \\
&\quad + \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 \\
&\le -E(t) + \frac{3}{2}\|u_t\|^2 + \frac{3}{2}\|u_t\|_1^2 - \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 - (u, \eta)_{\mathcal{M}} + \rho|\Omega| \\
&\le -E(t) + \big(1 + \lambda_1^{-1}\big)\|u_t\|_1^2 - \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2 + \frac{1}{4}\|u\|_1^2 + k_0\|\eta\|_{\mathcal{M}}^2 + \rho|\Omega| \\
&\le -E(t) + \big(1 + \lambda_1^{-1}\big)\|u_t\|_1^2 - \frac{1}{4}\|u\|_1^2 + \left(\frac{1}{2} + k_0\right)\|\eta\|_{\mathcal{M}}^2 + \rho|\Omega|
\end{aligned}
$$

and

$$
\begin{aligned}
-\Phi'(t) &= -(u_{tt}, \eta)_{\mathcal{M}} - (u_t, \eta_t)_{\mathcal{M}} - (u_{tt}, \eta)_{\mathcal{M}_0} - (u_t, \eta_t)_{\mathcal{M}_0} \\
&= -(u_{tt}, \eta)_{\mathcal{M}} - (u_{tt}, \eta)_{\mathcal{M}_0} - \|u_t\|_{\mathcal{M}}^2 - (T\eta, u_t)_{\mathcal{M}} - \|u_t\|_{\mathcal{M}_0}^2 + (T\eta, u_t)_{\mathcal{M}_0} \\
&= -\|u_t\|_{\mathcal{M}}^2 - \|u_t\|_{\mathcal{M}_0}^2 - (T\eta, u_t)_{\mathcal{M}} - (T\eta, u_t)_{\mathcal{M}_0} + (u, \eta)_{\mathcal{M}} + \big(f(u), \eta\big)_{\mathcal{M}_0}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (h,\eta)_{\mathcal{M}_0} - \int_0^\infty \int_0^\infty \mu(s)\mu(\tau)\big(\eta(\tau),\eta(s)\big)_1\,d\tau\,ds \\
&= -\|u_t\|_{\mathcal{M}}^2 - \|u_t\|_{\mathcal{M}_0}^2 + \int_0^\infty \mu'(s)(\eta,u_t)_1\,ds + \int_0^\infty \mu'(s)(\eta,u_t)\,ds + \big(f(u),\eta\big)_{\mathcal{M}_0} \\
&\quad - (h,\eta)_{\mathcal{M}_0} - \int_0^\infty \int_0^\infty \mu(s)\mu(\tau)\big(\eta(\tau),\eta(s)\big)_1\,d\tau\,ds \\
&\le -\|u_t\|_{\mathcal{M}}^2 - \|u_t\|_{\mathcal{M}_0}^2 - k_1(\eta,u_t)_{\mathcal{M}} - k_1(\eta,u_t)_{\mathcal{M}_0} + \big(f(u),\eta\big)_{\mathcal{M}_0} - (h,\eta)_{\mathcal{M}_0} \\
&\quad - \int_0^\infty \int_0^\infty \mu(s)\mu(\tau)\big(\eta(\tau),\eta(s)\big)_1\,d\tau\,ds \\
&\le -\frac{1}{2}\|u_t\|_{\mathcal{M}}^2 - \frac{1}{2}\|u_t\|_{\mathcal{M}_0}^2 + k_1^2\|\eta\|_{\mathcal{M}}^2 + k_1^2\|\eta\|_{\mathcal{M}_0}^2 + \big(f(u),\eta\big)_{\mathcal{M}_0} - (h,\eta)_{\mathcal{M}_0} \\
&\quad - \int_0^\infty \int_0^\infty \mu(s)\mu(\tau)\big(\eta(\tau),\eta(s)\big)_1\,d\tau\,ds \\
&\le -\frac{k_0}{2}\big(1+\lambda_1^{-1}\big)\|u_t\|_1 + k_1^2\big(1+\lambda_1^{-1}\big)\|\eta\|_{\mathcal{M}}^2 + \frac{\delta}{4}\|u\|_1^2 + C_\delta\|\eta\|_{\mathcal{M}}^2 \\
&\quad + \frac{\lambda_1^{-1}k_0}{2}\|h\|^2 + \frac{1}{2}\|\eta\|_{\mathcal{M}}^2,
\end{aligned}
$$

where $0 < \delta \le 1$ and C_δ is a positive constant that verifies the inequality

$$
\begin{aligned}
\big|\big(f(u),\eta\big)_{\mathcal{M}_0}\big| &\le \int_0^\infty \mu(s)\int_\Omega \big(1+|u|^p\big)|u||\eta|\,dx\,ds \\
&\le \int_0^\infty \mu(s)\int_\Omega |u||\eta|\,dx\,ds + \int_0^\infty \mu(s)\int_\Omega |u|^{p+1}|\eta|\,dx\,ds \\
&\le \lambda_1^{-1}\int_0^\infty \mu(s)\|u\|_1\|\eta\|_1\,ds + \int_0^\infty \mu(s)\|u\|_{p+2}^{p+1}\|\eta\|_{p+2}\,ds \\
&\le \lambda_1^{-1}\int_0^\infty \mu(s)\|u\|_1\|\eta\|_1\,ds + K_\Omega\int_0^\infty \mu(s)\|u\|_1\|\eta\|_1\,ds \\
&\le K_\Omega\|u\|_1\int_0^\infty \mu(s)\|\eta\|_1\,ds \\
&\le \frac{\delta}{4}\|u\|_1^2 + C_\delta\|\eta\|_{\mathcal{M}}^2.
\end{aligned}
$$

Now, for every $0 < \varepsilon < \frac{k_0}{2}$, the above inequalities provide

$$
\begin{aligned}
F_\varepsilon'(t) - \varepsilon^{-1}E'(t) &= \varepsilon\Psi'(t) - \Phi'(t) \\
&\le \big(1+\lambda_1^{-1}\big)\left(\varepsilon - \frac{k_0}{2}\right)\|u_t\|_1 - \varepsilon E(t) + \frac{1}{4}(\delta-\varepsilon)\|u\|_1^2 + \tilde{C}_{\varepsilon,\delta} \\
&\le -\varepsilon E(t) + \frac{1}{4}(\delta-\varepsilon)\|u\|_1^2 + \tilde{C}_{\varepsilon,\delta},
\end{aligned}
$$

where $\tilde{C}_{\varepsilon,\delta}$ is a positive constant. As $E'(t) \le 0$ we can choose $\delta \le \varepsilon$ in the previous inequality to obtain

$$
F_\varepsilon'(t) \le -\varepsilon E(t) + C_\varepsilon,
$$

which ends the proof of the lemma. □

Lemma 3.4 (Absorbing set) *Let $S(t)$ be the C_0-semigroup defined in* (2.5). *Then* $(\mathcal{H}, S(t))$ *is a dissipative dynamical system.*

Proof We shall prove that $S(t)$ has a bounded absorbing set. Let $\varepsilon_0 = \min\{\frac{1}{2\beta}, \varepsilon_1\}$. By item (b) of Lemma 3.2 we have

$$\beta_1 E(t) - \frac{1}{2} C_{\rho,\lambda_1,h,\Omega} \le F_\varepsilon(t) \le \beta_2 E(t) + \frac{1}{2} C_{\rho,\lambda_1,h,\Omega}, \tag{3.7}$$

where $\beta_1 = \varepsilon^{-1} - \beta$ and $\beta_2 = \varepsilon^{-1} + \beta$. Multiplying the inequality (3.7) for $\frac{\varepsilon}{\beta_2}$ we get

$$\frac{\varepsilon}{\beta_2} F_\varepsilon(t) \le \varepsilon E(t) + \mathbf{a}_\varepsilon, \tag{3.8}$$

where $\mathbf{a}_\varepsilon = \varepsilon \beta_2^{-1} C_{\rho,\lambda_1,h,\Omega}$. Now by Lemma 3.3 we have

$$F_\varepsilon'(t) \le -\varepsilon E(t) + C_\varepsilon. \tag{3.9}$$

Adding the inequalities (3.8) and (3.9) we obtain

$$F_\varepsilon'(t) + \frac{\varepsilon}{\beta_2} F_\varepsilon(t) \le \mathbf{b}_\varepsilon, \tag{3.10}$$

where $\mathbf{b}_\varepsilon = \mathbf{a}_\varepsilon + C_\varepsilon$. By (3.10) we conclude that

$$F_\varepsilon(t) \le \bigl(F_\varepsilon(0) - \varepsilon^{-1}\mathbf{b}_\varepsilon \beta_2\bigr) e^{-\frac{\varepsilon}{\beta_2} t} + \varepsilon^{-1} \mathbf{b}_\varepsilon \beta_2, \quad \text{for all } t \ge 0. \tag{3.11}$$

But by inequality (3.8) we have $F_\varepsilon(0) \le \beta_2 E(0) + \varepsilon^{-1}\mathbf{a}_\varepsilon \beta_2$. Therefore, by inequality (3.11) we get

$$F_\varepsilon(t) \le \bigl(\beta_2 E(0) - \varepsilon^{-1} C_\varepsilon \beta_2\bigr) e^{-\frac{\varepsilon}{\beta_2} t} + \varepsilon^{-1} \mathbf{b}_\varepsilon \beta_2, \quad \text{for all } t \ge 0. \tag{3.12}$$

Combining (3.7) and (3.12) we obtain

$$\begin{aligned} E(t) &\le \frac{\beta_2}{\beta_1}\bigl(E(0) - \varepsilon^{-1} C_\varepsilon\bigr) e^{-\frac{\varepsilon}{\beta_2} t} + \frac{\beta_2}{\beta_1} C_\varepsilon + \frac{3}{2\beta_1} C_{\rho,\lambda_1,h,\Omega} \\ &\le \frac{\beta_2}{\beta_1} E(0) e^{-\frac{\varepsilon}{\beta_2} t} + \frac{\beta_2}{\beta_1} C_\varepsilon + \frac{3}{2\beta_1} C_{\rho,\lambda_1,h,\Omega}. \end{aligned} \tag{3.13}$$

By (3.6) we have

$$E(t) \ge \sigma \bigl\| \bigl(u(t), u_t(t), \eta^t\bigr) \bigr\|_{\mathcal{H}}^2 - C_{\rho,\lambda_1,h,\Omega}, \quad \text{for all } t \ge 0, \tag{3.14}$$

where $\sigma = \min\{1, \frac{\lambda_1}{2}\}$. Combining (3.13) and (3.14) we get

$$\bigl\| \bigl(u(t), u_t(t), \eta^t\bigr) \bigr\|_{\mathcal{H}}^2 \le \frac{\beta_2}{\beta_1 \sigma} E(0) e^{-\frac{\varepsilon}{\beta_2} t} + \frac{\beta_2}{\beta_1 \sigma} C_\varepsilon + \left(\frac{3}{2\beta_1 \sigma} + \frac{1}{\sigma}\right) C_{\rho,\lambda_1,h,\Omega}. \tag{3.15}$$

By inequality (3.15) the ball $B(0,R) \subset \mathcal{H}$, where

$$R > \sqrt{\frac{2\beta_2}{\beta_1 \sigma} C_\varepsilon + \left(\frac{3}{\beta_1 \sigma} + \frac{2}{\sigma}\right) C_{\rho,\lambda_1,h,\Omega}},$$

is an absorbing set of the semigroup $S(t)$. □

3.3 Compactness

In this section we shall prove that the system $(\mathcal{H}, S(t))$ is asymptotically smooth.

Lemma 3.5 (Stabilization inequality) *Let $B \subset \mathcal{H}$ be a bounded invariant set and $z = (u_0, v_0, \eta_0)$, $\tilde{z} = (\tilde{u}_0, \tilde{v}_0, \tilde{\eta}_0)$ two initial data in B. Then there exists $\nu > 0$ such that*

$$\|S(t)z - S(t)\tilde{z}\|_{\mathcal{H}}^2 \le C_B e^{-\nu t} + C_B \int_0^t \left(\|w(s)\|_{L^{p+2}}^2 + \|w_t(s)\|_{L^{p+2}}^2 \right) ds, \tag{3.16}$$

where (u, η), $(\tilde{u}, \tilde{\eta})$ are the corresponding weak solutions of (2.1)-(2.2), *$w = u - \tilde{u}$, and C_B is a positive constant depending on B but not on t.*

Proof Let us also write $\xi = \eta - \tilde{\eta}$. Then w is a weak solution of the system

$$\begin{cases} w_{tt} - \Delta w_{tt} - \Delta w - \int_0^\infty \mu(s)\Delta\xi(s)\,ds = f(u) - f(\tilde{u}), \\ \xi_t = T\xi + w_t, \end{cases} \tag{3.17}$$

with Dirichlet boundary condition and initial data

$$w(0) = u_0 - \tilde{u}_0, \qquad w_t(0) = v_0 - \tilde{v}_0, \qquad \xi^0 = \eta_0 = \tilde{\eta}_0.$$

We define the energy functional

$$G(t) = \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\|w_t(t)\|_1^2 + \frac{1}{2}\|w(t)\|_1^2 + \frac{1}{2}\|\xi^t\|_{\mathcal{M}}^2.$$

In the following, C_0 will denote several positive constants dependent on B but not on t.

Claim 1 *There exists a constant $C_0 > 0$ such that*

$$G'(t) \le \frac{1}{2}\int_0^\infty \mu'(s)\|\xi^t(s)\|_1^2\,ds + C_0\left(\|w(t)\|_{L^{p+2}}^2 + \|w_t(t)\|_{L^{p+2}}^2\right). \tag{3.18}$$

To prove the claim, we multiply the first equation in (3.17) by w_t and integrate over Ω. Then we obtain

$$G'(t) = \frac{1}{2}\int_0^\infty \mu'(s)\|\xi(s)\|_1^2\,ds - \int_\Omega \left(f(u) - f(\tilde{u})\right)w_t\,dx.$$

Using (H_2) we have

$$\begin{aligned} \left|\int_\Omega \left(f(u) - f(\tilde{u})\right)w_t\,dx\right| &\le C_f\left(1 + \|u\|_{L^{p+2}}^p + \|\tilde{u}\|_{L^{p+2}}^p\right)\|w\|_{L^{p+2}}\|w_t\|_{L^{p+2}} \\ &\le C_0\left(\|w(t)\|_{L^{p+2}}^2 + \|w_t(t)\|_{L^{p+2}}^2\right), \end{aligned}$$

since B is bounded and invariant. Then we see that (3.18) holds.

Now, let us define the perturbed functional

$$J(t) = NG(t) + \varepsilon\tilde{\Psi}(t) + \tilde{\Phi}(t),$$

where

$$\tilde{\Psi}(t) = \big(w(t), w_t(t)\big) + \big(w(t), w_t(t)\big)_1, \qquad \tilde{\Phi}(t) = \big(w(t), \xi^t\big)_{\mathcal{M}} + \big(w(t), \xi^t\big)_{\mathcal{M}_0},$$

and $N \geq 1$, $0 < \varepsilon < 1$ are constants to be determined. Then the following claims can be proved with similar arguments to the above one and to the proof of the absorbing set.

Claim 2 *There exist constants* $\beta_1, \beta_2, C_\beta > 0$ *such that, if* $N > C_\beta$,

$$\beta_1 G(t) \leq J(t) \leq \beta_2 G(t), \quad t \geq 0. \tag{3.19}$$

Claim 3 *There exists a constant* $C_1 > 0$ *such that*

$$\begin{aligned}\tilde{\Psi}'(t) \leq{} & -G(t) - \frac{1}{4}\|w(t)\|_1^2 + \frac{3}{2}\|w_t(t)\|^2 + \frac{3}{2}\|w_t(t)\|_1^2 \\ & - C_1 \int_0^\infty \mu'(s)\|\xi^t(s)\|_1^2\,ds + C_0\|w(t)\|_{L^{p+2}}^2.\end{aligned} \tag{3.20}$$

Claim 4 *Given* $\delta > 0$ *there exists a constant* $C_\delta > 0$ *such that*

$$\tilde{\Phi}'(t) \leq \delta\|w(t)\|_1^2 - \frac{k_0}{2}\|w_t(t)\|_1^2 - C_\delta \int_0^t \mu'(s)\|\xi^t(s)\|_1^2\,ds. \tag{3.21}$$

Now, taking $\varepsilon > 0$ sufficiently small and $N > 0$ sufficiently large, we obtain from (3.18), (3.20), and (3.21),

$$J'(t) \leq -\varepsilon G(t) + C_0\big(\|w(t)\|_{L^{p+2}}^2 + \|w_t(t)\|_{L^{p+2}}^2\big), \quad t \geq 0.$$

Combining this with (3.19) we have, as in the proof of the absorbing set,

$$G(t) \leq \frac{\beta_2}{\beta_1} G(0) e^{-\frac{\varepsilon}{\beta_2}t} + C_0 \int_0^t e^{-\frac{\varepsilon}{\beta_2}(t-s)}\big(\|w(s)\|_{L^{p+2}}^2 + \|w_t(s)\|_{L^{p+2}}^2\big)\,ds, \quad \forall t \geq 0.$$

This implies (3.16) by taking $\nu = \varepsilon/\beta_2$ and in view of the definition of $G(t)$. □

Lemma 3.6 (Asymptotic smoothness) *Let* $S(t)$ *be the* C_0*-semigroup defined in* (2.5). *Then the system* $(\mathcal{H}, S(t))$ *is asymptotically smooth.*

Proof We apply the compactness criterion presented in Proposition 2.10 of [18]. As recalled in Section 3.1, we must check conditions (3.1) and (3.2).

Given a forward invariant set $B \subset \mathcal{H}$ and $\varepsilon > 0$, we can take $T > 0$ such that

$$\sqrt{2C_B} e^{-\frac{\nu}{2}T} < \varepsilon.$$

Then from (3.16), using notation

$$S(t)z^n = \big(u^n(t), u_t^n(t), \eta_n^t\big),$$

we obtain for any $z^1, z^2 \in B$,

$$\|S(t)z^1 - S(t)z^2\|_{\mathcal{H}} \leq \varepsilon + \left(2C_B \int_0^T \|u^1 - u^2\|_{L^{p+2}}^2 + \|u_t^1 - u_t^2\|_{L^{p+2}}^2\, ds\right)^{\frac{1}{2}}$$

with $0 < t < T$. Then defining

$$\phi_T(z^1, z^2) = \sqrt{2C_B}\left(\int_0^T \|u^1(s) - u^2(s)\|_{L^{p+2}}^2 + \|u_t^1(s) - u_t^2(s)\|_{L^{p+2}}^2\, ds\right)^{\frac{1}{2}},$$

we see that condition (3.1) holds.

It remains to show that (3.2) also holds. Given any sequence $(z^n) \subset B$, from the positive invariance of B we see that $S(t)z^n = (u^n(t), u_t^n(t), \eta_n^t)$ is uniformly bounded in $\mathcal{H}$. Then we conclude that

$$\begin{aligned} &u^n \quad \text{is bounded in } L^\infty(0,T,H_1),\\ &u_t^n \quad \text{is bounded in } L^\infty\big(0,T,L^2(\Omega)\big) \cap L^\infty(0,T,H_1), \end{aligned}$$

and from (2.4),

$$u_{tt}^n \quad \text{is bounded in } L^2\big(0,T,L^2(\Omega)\big).$$

Then from Simon's theorem [22] we have

$$u^n, u_t^n \quad \text{converge strongly in } C\big([0,T], L^{p+2}(\Omega)\big),$$

since H_1 is compactly embedded in $L^{p+2}(\Omega)$. Therefore there is a subsequence such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|u^{n_k}(s) - u^{n_l}(s)\|_{L^{p+2}}^2 + \|u_t^{n_k}(s) - u_t^{n_l}(s)\|_{L^{p+2}}^2\, ds = 0.$$

This shows that (3.2) also holds. □

Proof of Theorem 3.1 Since we have proved that $(\mathcal{H}, S(t))$ is dissipative and asymptotically smooth, the existence of a global attractor follows from a classical result, as noticed in Section 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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