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Positive fixed points for convex and decreasing operators in probabilistic Banach spaces with an application to a two-point boundary value problem

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Abstract
We establish a fixed point theorem for decreasing and convex operators in a probabilistic Banach space partially ordered by a normal cone. We give an application to the study of the existence and uniqueness of positive solutions to a two-point boundary value problem.

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1 Introduction and preliminaries

The concept of probabilistic Banach spaces was introduced by Serstnev [1] by adapting the idea of Menger [2] to linear spaces. Fixed point theory in such spaces was studied and developed by many authors (see [3–8] and the references mentioned therein).

This paper deals with the existence and uniqueness of fixed points for a certain class of convex and decreasing operators defined in a probabilistic Banach space partially ordered by a cone.

For the sake of convenience, we first give some definitions and known results from the existing literature. For more details, we refer to [4, 9].

Definition 1.1 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a distribution function if it satisfies the following conditions:

(i) f is non-decreasing;
(ii) f is left-continuous;
(iii) $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote by D the set of all distribution functions.

Definition 1.2 A triangular norm, briefly a T-norm, is a mapping $T : [0,1] \times [0,1] \to [0,1]$ that is continuous and such that for every $a, b, c, d \in [0,1]$,

(i) $T(a,1) = a$;
(ii) $T(a,b) = T(b,a)$;

(iii) $c \geq a, d \geq b \Longrightarrow T(c,d) \geq T(a,b)$;
(iv) $T(T(a,b),c) = T(a,T(b,c))$.

As standard examples, $T_m(a,b) = \min\{a,b\}$ and $T_p(a,b) = ab$ on $[0,1]$ are T-norms.

Definition 1.3 Let X be a real vector space, T be a T-norm and $N : X \to D$ be a given mapping. We say that N is a probabilistic norm on X if the following conditions hold:
(i) $N_x(0) = 0$ for every $x \in X$;
(ii) $N_x(t) = 1$ for all $t > 0$ iff $x = 0$;
(iii) $N_{\alpha x}(t) = N_x(\frac{t}{|\alpha|})$ for all $x \in X$ and $\alpha \in \mathbb{R}\setminus\{0\}$;
(iv) $N_{x+y}(s+t) \geq T(N_x(s), N_y(t))$ for all $x, y \in X$ and $s, t \geq 0$.
In this case, the triplet (X, N, T) is said to be a probabilistic normed space.

In the above definition, for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value N_x at $t \in \mathbb{R}$.

Example 1.4 Let $(X, \|\cdot\|)$ be a normed linear space. For all $x \in X$, define the mapping

$$N_x(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{t}{t+\|x\|} & \text{if } t > 0. \end{cases}$$

Then (X, N, T_p) and (X, N, T_m) are probabilistic normed spaces.

Example 1.5 Let $(X, \|\cdot\|)$ be a normed linear space. For all $x \in X$, define the mapping

$$N_x(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ e^{\frac{-\|x\|}{t}} & \text{if } t > 0. \end{cases}$$

Then (X, N, T_p) is a probabilistic normed space.

Now, let us recall some topological properties of probabilistic normed spaces.

Definition 1.6 Let (X, N, T) be a probabilistic normed space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$N_{x_n - x}(\varepsilon) > 1 - \lambda$$

for every $n \geq N$.

Definition 1.7 Let (X, N, T) be a probabilistic normed space. A sequence $\{x_n\}$ in X is said to be Cauchy if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$N_{x_n - x_m}(\varepsilon) > 1 - \lambda$$

for every $n, m \geq N$.

Definition 1.8 Let (X,N,T) be a probabilistic normed space. It is said to be a Banach probabilistic normed space (or complete) if every Cauchy sequence in X is convergent to a point in X.

Definition 1.9 Let (X,N,T) be a probabilistic normed space. A subset A of X is said to be closed if every convergent sequence in A converges to an element of A.

Definition 1.10 Let (X,N,T) be a probabilistic Banach space. A nonempty subset $P\subseteq X$ is a cone if it satisfies the following conditions:

(i) P is closed and convex;
(ii) if $p\in P$, $tp\in P$ for every $t\geq 0$;
(iii) if both p and $-p$ are in P, then $p=0$.

Let $\preceq$ be the partial order on X induced by the cone P in X. That is,

$$p,q\in X,\quad p\preceq q \quad\Longleftrightarrow\quad p-q\in P.$$

Thus X becomes a partially ordered probabilistic Banach space. If $x,y\in X$, the notation $x\prec y$ means that $x\preceq y$ and $x\neq y$.

Definition 1.11 Let (X,N,T) be a probabilistic Banach space. A cone P in X is said to be normal if there is some constant $K>0$ (normal constant) such that

$$x,y\in X,\quad 0\preceq x\preceq y \quad\Longrightarrow\quad N_x(t)\geq N_y\left(\frac{t}{K}\right),\quad t\in\mathbb{R}.$$

Definition 1.12 Let (X,N,T) be a probabilistic Banach space and $P\subseteq X$ be a cone in X. Let C be a convex subset in X. An operator $A:C\to X$ is called a convex operator if

$$A\bigl(tx+(1-t)y\bigr)\preceq tAx+(1-t)Ay$$

for all $x,y\in C$, $x\preceq y$ and $t\in[0,1]$.

Definition 1.13 Let (X,N,T) be a probabilistic Banach space and $P\subseteq X$ be a cone in X. An operator $A:C\to X$ is said to be a decreasing operator if

$$x,y\in C,\quad x\preceq y \quad\Longrightarrow\quad Ax\succeq Ay.$$

The paper is organized as follows. In Section 2, we study the existence and uniqueness of positive fixed points for a certain class of decreasing and convex operators $A:P\to P$. In Section 3, we study the existence and uniqueness of positive solutions to the nonlinear functional equation $x=x_0+Bx$, where $x_0\in P$ and $B:P\to P$ is a given operator satisfying certain conditions. Section 4 contains a Banach version of our man result established in Section 2. Finally, in Section 5, we present an application of our main result to the study of the existence and uniqueness of positive solutions to a nonlinear differential equation of second order with two-point boundary value problem.

2 Main result and proof

Before stating our main result, we need some lemmas.

Lemma 2.1 *Let (X, N, T) be a probabilistic Banach space and $\{x_n\}$ be a sequence in X that converges to some $x \in X$. Then any subsequence of $\{x_n\}$ converges to x.*

Proof Let $\{x_{\varphi(n)}\}$ be a subsequence of $\{x_n\}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a mapping satisfying

$$\varphi(n+1) > \varphi(n)$$

for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $\lambda > 0$. Since $\{x_n\}$ converges to $x \in X$, there is some positive integer N such that

$$N_{x_n - x}(\varepsilon) > 1 - \lambda$$

for every $n \geq N$. On the other hand,

$$n \geq N \quad \Longrightarrow \quad \varphi(n) \geq \varphi(N) \geq N.$$

Then

$$N_{x_{\varphi(n)} - x}(\varepsilon) > 1 - \lambda$$

for every $n \geq N$. This proves that $\{x_{\varphi(n)}\}$ converges to x. □

Lemma 2.2 (see [9]) *Let (X, N, T) be a probabilistic Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $\{\alpha_n\}$ be a real sequence. The following properties hold*:

(i) *if $\{x_n\}$ converges to $x \in X$ and $\{y_n\}$ converges to $y \in X$, then $\{x_n + y_n\}$ converges to $x + y$;*

(ii) *if $\{\alpha_n\}$ converges to some $\alpha \in \mathbb{R}$ and $\{x_n\}$ converges to some $x \in X$, then $\{\alpha_n x_n\}$ converges to αx.*

Lemma 2.3 *Let (X, N, T) be a probabilistic Banach space and $P \subseteq X$ be a normal cone in X with normal constant $K > 0$. Let $\{u_n\}$ be a sequence in X such that*

$$0 \preceq u_m - u_n \preceq \xi_n a$$

for every $m, n \geq N$, where $a \in X$ and $\{\xi_n\}$ is a real sequence such that $\xi_n \to 0$ as $n \to \infty$. Then $\{u_n\}$ is a Cauchy sequence in the probabilistic Banach space (X, N, T).

Proof Let $\varepsilon > 0$ and $\lambda > 0$. Without restriction of the generality, we may assume that $\xi_n \neq 0$ for every $n \geq N$. Since P is a normal cone, we have

$$N_{u_m - u_n}(\varepsilon) \geq N_{\xi_n a}\left(\frac{\varepsilon}{K}\right) = N_a\left(\frac{\varepsilon}{K|\xi_n|}\right) \tag{2.1}$$

for every $m, n \geq N$. On the other hand, since $N_a \in D$, we have

$$\sup_{t \in \mathbb{R}} N_a(t) = 1.$$

Then there is some $t^* \in \mathbb{R}$ such that

$$N_a\left(t^*\right) > 1 - \lambda.$$

Since $\xi_n \to 0$ as $n \to \infty$, there is some positive integer N' such that

$$\frac{\varepsilon}{K|\xi_n|} > t^*$$

for every $n \geq N'$. Since N_a is non-decreasing, we get

$$N_a\left(\frac{\varepsilon}{K|\xi_n|}\right) \geq N_a\left(t^*\right) > 1 - \lambda \tag{2.2}$$

for every $n \geq N'$. Finally, using (2.1) and (2.2), we obtain

$$N_{u_m - u_n}(\varepsilon) > 1 - \lambda$$

for every $n, m \geq \max\{N, N'\}$. This proves that $\{u_n\}$ is a Cauchy sequence in the probabilistic Banach space (X, N, T). □

Lemma 2.4 *Let (X, N, T) be a probabilistic Banach space and $P \subseteq X$ be a normal cone in X with normal constant $K > 0$. Let us consider two sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$0 \preceq u_n \preceq v_n$$

for every $n \geq N$. Then

$$\{v_n\} \text{ converges to } 0 \quad \Longrightarrow \quad \{u_n\} \text{ converges to } 0.$$

Proof Let $\varepsilon > 0$ and $\lambda > 0$. Since $\{v_n\}$ converges to 0, there exists some $N' \in \mathbb{N}$ such that

$$N_{v_n}\left(\frac{\varepsilon}{K}\right) > 1 - \lambda$$

for any $n \geq N'$. On the other hand, since P is normal, we have

$$N_{u_n}(\varepsilon) \geq N_{v_n}\left(\frac{\varepsilon}{K}\right)$$

for any $n \geq N$. Thus we proved that

$$N_{u_n}(\varepsilon) > 1 - \lambda$$

for any $n \geq \max\{N, N'\}$, which implies that $\{u_n\}$ converges to 0. □

The following result is an immediate consequence of Lemma 2.2 and Lemma 2.4.

Lemma 2.5 *Let (X, N, T) be a probabilistic Banach space and $P \subseteq X$ be a normal cone in X with normal constant $K > 0$. Let us consider three sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in X*

such that

$$u_n \preceq v_n \preceq w_n$$

for every $n \geq N$. *Then*

$$\{u_n\}, \{w_n\} \text{ converge to } \ell \in X \quad \Longrightarrow \quad \{v_n\} \text{ converges to } \ell.$$

Now, we are ready to state and prove our main result.

Let (X, N, T) be a probabilistic Banach space and $P \subseteq X$ be a normal cone in X with normal constant $K > 0$. We denote by $\mathcal{A}$ the set of operators $A : P \to P$ satisfying the following conditions:

($\mathcal{A}1$) $0 \prec A0$;

($\mathcal{A}2$) A is a convex and decreasing operator;

($\mathcal{A}3$) there exist $\gamma \in (0,1)$ and $m_0, n_0 \in \mathbb{N}$ with $n_0 > m_0$ such that

$$A^{2m_0+2}0 - A^{2m_0}0 \succeq \gamma \bigl(A^{2m_0+3}0 - A^{2m_0}0\bigr) \tag{2.3}$$

and

$$A^{2n_0}0 \succeq \frac{1}{2}\bigl(A^{2m_0+1}0 + A^{2m_0}0\bigr). \tag{2.4}$$

Theorem 2.6 *Let* $A \in \mathcal{A}$. *Then*

(i) *A has a unique fixed point* $x^* \in P$;

(ii) *for any initial value* $x_0 \in P$, *the Picard sequence* $\{x_n\}$ *in* X *defined by*

$$x_n = Ax_{n-1}, \quad n \geq 1$$

converges to x^*;

(iii) *we have the estimates*

$$N_{x_{2(m_0+n)}-x^*}(t) \geq T\left(N_{A0}\left(\frac{(n-2-n_0+m_0)t}{K^2}\right), N_{A0}\left(\frac{(n-2-n_0+m_0)t}{K^2}\right)\right) \tag{2.5}$$

for every $n > n_0 + 2 - m_0$, $t \in \mathbb{R}$, *and*

$$N_{x_{2(m_0+n)+1}-x^*}(t) \geq T\left(N_{A0}\left(\frac{(n-1-n_0+m_0)t}{K^2}\right), N_{A0}\left(\frac{(n-1-n_0+m_0)t}{K^2}\right)\right) \tag{2.6}$$

for every $n > n_0 + 1 - m_0$, $t \in \mathbb{R}$.

Proof Let us consider the sequence $\{u_n\}$ in P defined by

$$u_n = A^n 0, \quad n \in \mathbb{N}.$$

By (2.3) and (2.4), we have

$$u_{2(m_0+1)} \succeq \gamma u_{2m_0+3} + (1-\gamma)u_{2m_0} \tag{2.7}$$

and

$$2u_{2n_0} - u_{2m_0} \succeq u_{2m_0+1}. \tag{2.8}$$

Since A is a decreasing operator, we have

$$\begin{aligned} 0 = u_0 &\preceq u_2 \preceq \cdots \preceq u_{2m_0} \preceq \cdots \preceq u_{2(m_0+n)} \preceq \cdots \\ &\preceq u_{2(m_0+n)+1} \preceq \cdots \preceq u_{2m_0+3} \preceq \cdots \preceq u_3 \preceq u_1 = A0. \end{aligned} \tag{2.9}$$

Using (2.7) and (2.9), we obtain

$$\begin{aligned} 0 &\preceq \gamma(u_{2(m_0+n)+1} - u_{2m_0}) \preceq \cdots \preceq \gamma(u_{2m_0+3} - u_{2m_0}) \\ &\preceq u_{2(m_0+1)} - u_{2m_0} \preceq \cdots \preceq u_{2(m_0+n)} - u_{2m_0}. \end{aligned}$$

For every $n \in \mathbb{N}$, we define the set

$$\mathcal{S}_n = \{s \in (0,1] : u_{2(m_0+n)} \succeq s u_{2(m_0+n)+1} + (1-s)u_{2m_0}\}.$$

Clearly $\mathcal{S}_n \neq \emptyset$ since $\gamma \in \mathcal{S}_n$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$s_n = \sup_{\mathbb{R}} \mathcal{S}_n \quad \text{and} \quad e_n = 1 - s_n.$$

Let $n \in \mathbb{N}$ be fixed. By the definition of s_n, there exists a sequence $\{a_p\} \subset \mathcal{S}_n$ such that $a_p \to s_n$ as $p \to \infty$. Thus we have

$$u_{2(m_0+n)} \succeq a_p u_{2(m_0+n)+1} + (1-a_p)u_{2m_0}$$

for every $p \in \mathbb{N}$. This means that

$$u_{2(m_0+n)} - a_p u_{2(m_0+n)+1} - (1-a_p)u_{2m_0} \in P$$

for every $p \in \mathbb{N}$. Since P is closed, letting $p \to \infty$, using Lemma 2.2 and (2.9), we obtain

$$\begin{aligned} &u_{2(m_0+n)} \succeq s_n u_{2(m_0+n)+1} + e_n u_{2m_0}, \\ &\quad 0 < \gamma \le s_1 \le s_2 \le \cdots \le s_n \le \cdots \le 1, 0 \le e_n \le 1-\gamma. \end{aligned} \tag{2.10}$$

Now, we shall prove that $e_n \to 0$ as $n \to \infty$. Using inequalities (2.8), (2.9), (2.10) and the fact that A is a decreasing convex operator, for every $n \ge n_0 - m_0 - 1$, we have

$$\begin{aligned} u_{2(m_0+n)+1} = Au_{2(m_0+n)} &\preceq A(s_n u_{2(m_0+n)+1} + e_n u_{2m_0}) \\ &\preceq s_n u_{2(m_0+n+1)} + e_n u_{2m_0+1} \end{aligned}$$

$$\preceq s_n u_{2(m_0+n+1)} + e_n(2u_{2n_0} - u_{2m_0})$$
$$\preceq (1+e_n)u_{2(m_0+n+1)} - e_n u_{2m_0}.$$

Thus we proved that

$$u_{2(m_0+n+1)} \succeq \frac{1}{1+e_n}u_{2(m_0+n)+1} + \frac{e_n}{1+e_n}u_{2m_0}$$

for every $n \geq n_0 - m_0 - 1$. Using that A is convex and decreasing and inequality (2.8), we obtain

$$\begin{aligned} u_{2(m_0+n+1)+1} = Au_{2(m_0+n+1)} &\preceq \frac{1}{1+e_n}u_{2(m_0+n+1)} + \frac{e_n}{1+e_n}u_{2m_0+1} \\ &\preceq \frac{1}{1+e_n}u_{2(m_0+n+1)} + \frac{e_n}{1+e_n}(2u_{2n_0} - u_{2m_0}) \\ &\preceq \frac{1+2e_n}{1+e_n}u_{2(m_0+n+1)} - \frac{e_n}{1+e_n}u_{2m_0}, \end{aligned}$$

which implies that

$$u_{2(m_0+n+1)} \succeq \frac{1+e_n}{1+2e_n}u_{2(m_0+n+1)+1} + \frac{e_n}{1+2e_n}u_{2m_0}$$

for every $n \geq n_0 - m_0 - 1$. This implies that

$$\frac{1+e_n}{1+2e_n} \in \mathcal{S}_{n+1}$$

for every $n \geq n_0 - m_0 - 1$. So, for every $n \geq n_0 - m_0 - 1$, we have

$$1 - e_{n+1} = s_{n+1} \geq \frac{1+e_n}{1+2e_n},$$

that is,

$$e_{n+1} \leq \frac{e_n}{1+2e_n}.$$

Now, let us consider the function $f : [0,1] \to [0,1]$ defined by

$$f(r) = \frac{r}{1+2r}, \quad r \in [0,1].$$

Since

$$e_{n+1} \leq f(e_n)$$

for every $n \geq n_0 - m_0 - 1$ and f is a non-decreasing function, we obtain

$$\begin{aligned} 0 \leq e_{n+1} &\leq f^{n-n_0+m_0+2}(e_{n_0-m_0-1}) \\ &\leq \frac{e_{n_0-m_0-1}}{1+2(n-n_0+m_0)e_{n_0-m_0-1}}. \end{aligned}$$

Thus we have

$$0 \le e_{n+1} \le \frac{1}{2(n - n_0 + m_0)} \tag{2.11}$$

for every $n \ge n_0 - m_0 + 1$. Letting $n \to \infty$ in (2.11), we get

$$\lim_{n\to\infty} e_n = 0,$$

that is,

$$\lim_{n\to\infty} s_n = 1.$$

Using (2.10), we deduce that for all $n, p \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\preceq u_{2(m_0+n+p)} - u_{2(m_0+n)} \preceq u_{2(m_0+n)+1} - u_{2(m_0+n)} \\ &\preceq e_n(u_{2(m_0+n)+1} - u_{2m_0}) \preceq e_n(u_1 - u_{2m_0}) \preceq e_n u_1. \end{aligned} \tag{2.12}$$

Since P is a normal cone and $e_n \to 0$ as $n \to \infty$, using Lemma 2.3, we obtain that $\{u_{2(m_0+n)}\}$ is a Cauchy sequence. Since P is closed and (X, N, T) is a complete probabilistic normed space, there is $x^* \in P$ such that $\{u_{2n}\}$ converges to x^*. Using (2.12), Lemma 2.2 and Lemma 2.4, we deduce that

$$\{u_{2n}\} \text{ and } \{u_{2n+1}\} \text{ converge to } x^*. \tag{2.13}$$

On the other hand, using (2.9), we have

$$u_{2(m_0+n)} \preceq u_{2(m_0+n+p)}$$

for every $n, p \in \mathbb{N}$. This implies that

$$u_{2(m_0+n+p)} - u_{2(m_0+n)} \in P$$

for every $n, p \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and letting $p \to \infty$, from (2.13) and since P is closed, we get

$$x^* - u_{2(m_0+n)} \in P,$$

that is,

$$u_{2(m_0+n)} \preceq x^*.$$

Similarly, we can observe that

$$x^* \preceq u_{2(m_0+n)+1}.$$

Thus we proved that

$$u_{2(m_0+n)} \preceq x^* \preceq u_{2(m_0+n)+1} \tag{2.14}$$

for all $n \in \mathbb{N}$. Using the fact that A is a decreasing operator, (2.14) yields

$$u_{2(m_0+n+1)} \preceq Ax^* \preceq u_{2(m_0+n)+1}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality and using (2.13), we obtain

$$x^* \preceq Ax^* \preceq x^*,$$

that is, $x^* = Ax^*$. Thus we proved that $x^* \in P$ is a fixed point of the operator A.

Now, let $x_0 \in P$ be an arbitrary point. Let us consider the Picard sequence $\{x_n\}$ in P defined by

$$x_n = A^n x_0 \quad \text{for every } n \in \mathbb{N}. \tag{2.15}$$

Since A is a decreasing operator, using that $u_0 \preceq x_0$ and $u_0 \preceq x_1$, by induction we obtain

$$u_{2(m_0+n)} \preceq x_{2(m_0+n)} \preceq u_{2(m_0+n)-1}, \tag{2.16}$$

$$u_{2(m_0+n)} \preceq x_{2(m_0+n)+1} \preceq u_{2(m_0+n)+1} \tag{2.17}$$

for every $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.16) and (2.17), using (2.13) and Lemma 2.5, we obtain

$$\{x_{2n}\} \text{ and } \{x_{2n+1}\} \text{ converge to } x^*,$$

which implies that $\{x_n\}$ converges to x^*.

Let us prove that x^* is the unique fixed point of the operator A. Suppose that $y^* \in P$ is another fixed point of A. Let $x_0 = y^*$ and consider the Picard sequence $\{x_n\}$ in P defined by (2.15). Then we have

$$x_n = A^n x_0 = y^* \quad \text{for every } n \in \mathbb{N}.$$

Letting $n \to \infty$ in (2.16) and (2.17), using (2.13), we obtain $x^* = y^*$. Thus we proved that x^* is the unique fixed point of A.

Let us prove estimate (2.5). Let $t \in \mathbb{R}$, we have

$$\begin{aligned} N_{x_{2(m_0+n)}-x^*}(t) &= N_{(x_{2(m_0+n)}-u_{2(m_0+n)})+(u_{2(m_0+n)}-x^*)}(t) \\ &\geq T\left(N_{x_{2(m_0+n)}-u_{2(m_0+n)}}\left(\frac{t}{2}\right), N_{u_{2(m_0+n)}-x^*}\left(\frac{t}{2}\right)\right). \end{aligned} \tag{2.18}$$

By (2.16), we have

$$0 \preceq x_{2(m_0+n)} - u_{2(m_0+n)} \preceq u_{2(m_0+n)-1} - u_{2(m_0+n)}.$$

Since P is a normal cone, we get

$$N_{x_{2(m_0+n)}-u_{2(m_0+n)}}\left(\frac{t}{2}\right) \geq N_{u_{2(m_0+n)-1}-u_{2(m_0+n)}}\left(\frac{t}{2K}\right). \tag{2.19}$$

Using (2.14) and (2.9), we have

$$0 \preceq x^* - u_{2(m_0+n)} \preceq u_{2(m_0+n)-1} - u_{2(m_0+n)}.$$

Since P is a normal cone, we get

$$N_{x^*-u_{2(m_0+n)}}\left(\frac{t}{2}\right) \geq N_{u_{2(m_0+n)-1}-u_{2(m_0+n)}}\left(\frac{t}{2K}\right). \tag{2.20}$$

It follows from (2.18), (2.19) and (2.20) that

$$N_{x_{2(m_0+n)}-x^*}(t) \geq T\left(N_{u_{2(m_0+n)-1}-u_{2(m_0+n)}}\left(\frac{t}{2K}\right), N_{u_{2(m_0+n)-1}-u_{2(m_0+n)}}\left(\frac{t}{2K}\right)\right). \tag{2.21}$$

On the other hand, by (2.9) and (2.10), we have

$$u_{2(m_0+n)} \succeq u_{2(m_0+n-1)} \succeq s_{n-1}u_{2(m_0+n)-1} + e_{n-1}u_{2m_0}.$$

Using (2.9), (2.12) and the above inequality, we obtain

$$0 \preceq u_{2(m_0+n)-1} - u_{2(m_0+n)} \preceq e_{n-1}A0.$$

Using (2.11) and the above inequality, we obtain

$$0 \preceq u_{2(m_0+n)-1} - u_{2(m_0+n)} \preceq \frac{1}{2(n-2-n_0+m_0)}A0$$

for every $n > n_0 + 2 - m_0$. Since P is a normal cone, we have

$$N_{u_{2(m_0+n)-1}-u_{2(m_0+n)}}\left(\frac{t}{2K}\right) \geq N_{A0}\left(\frac{(n-2-n_0+m_0)t}{K^2}\right) \tag{2.22}$$

for every $n > n_0 + 2 - m_0$. Now, (2.5) follows immediately from (2.21) and (2.22). The proof of estimate (2.6) follows using similar arguments as above. This ends the proof. □

3 Positive solutions for the nonlinear functional equation: $x = x_0 + Bx$

In this section, from our main theorem (Theorem 2.6), we deduce an existence and uniqueness result for the nonlinear operator equation on ordered probabilistic Banach spaces

$$x = x_0 + Bx, \tag{3.1}$$

where $x_0 \in P$ and $B : P \to P$ is a given operator satisfying certain conditions. There have appeared a series of research results concerning this kind of nonlinear operator Eq. (3.1) because of the crucial role played by nonlinear equations in applied science as well as in mathematics (see [10–13]).

Let (X, N, T) be a probabilistic Banach space and $P \subseteq X$ be a normal cone in X with normal constant $K > 0$. We denote by $\mathcal{B}$ the set of operators $B : P \to P$ satisfying the following conditions:

($\mathcal{B}1$) $B0 = 0$;
($\mathcal{B}2$) B is a convex and decreasing operator.

We have the following result.

Theorem 3.1 *Let $B \in \mathcal{B}$ and $x_0 \in P$ such that $0 \prec x_0$. Then the operator Eq.* (3.1) *has a unique solution $x^* \in P$.*

Proof Let us define the operator $A : P \to P$ by

$$Ax = x_0 + Bx, \quad x \in P.$$

Obviously, $x \in P$ is a solution to Eq. (3.1) if and only if x is a fixed point of A. We have just to prove that the operator A satisfies the required conditions of Theorem 2.6, that is, A belongs to the class of operators $\mathcal{A}$.

• Condition ($\mathcal{A}1$). Since $x_0 \succ 0$, using ($\mathcal{B}1$), we have

$$A0 = x_0 + B0 = x_0 \succ 0.$$

Thus condition ($\mathcal{A}1$) is satisfied.

• Condition ($\mathcal{A}2$). Let $x, y \in P$ such that $x \preceq y$. Using the fact that B is a decreasing operator, we obtain

$$Bx \succeq By,$$

which implies that

$$x_0 + Bx \succeq x_0 + By,$$

that is,

$$Ax \succeq Ay.$$

Thus A is a decreasing operator.

Now, let $t \in [0,1]$, $x, y \in P$ such that $x \preceq y$. Since B is a convex operator, we have

$$B\big(tx + (1-t)y\big) \preceq tBx + (1-t)By,$$

which implies that

$$x_0 + B\big(tx + (1-t)y\big) \preceq x_0 + tBx + (1-t)By = t(x_0 + Bx) + (1-t)(x_0 + By).$$

Thus we have

$$A\big(tx + (1-t)y\big) \preceq tAx + (1-t)Ay.$$

Then A is a convex operator. Condition ($\mathcal{A}2$) is then satisfied.

• Condition ($\mathcal{A}3$). Let $n_0 = 1$ and $m_0 = 0$. Since $B0 = 0$, we have

$$A^{2n_0}0 = A^2 0 = A(x_0 + B0) = Ax_0 = x_0 + Bx_0.$$

On the other hand,

$$\frac{1}{2}\left(A^{2m_0+1}0 + A^{2m_0}0\right) = \frac{1}{2}A0 = \frac{1}{2}x_0.$$

Clearly, we have

$$A^{2n_0}0 \succeq \frac{1}{2}\left(A^{2m_0+1}0 + A^{2m_0}0\right).$$

Now, we have

$$A^{2m_0+2}0 - A^{2m_0}0 = A^2 0 = x_0 + Bx_0.$$

Moreover, for any $\gamma \in (0,1)$, we have

$$\gamma\left(A^{2m_0+3}0 - A^{2m_0}0\right) = \gamma A^3 0 = \gamma x_0 + \gamma B(x_0 + Bx_0).$$

Thus we have

$$\left(A^{2m_0+2}0 - A^{2m_0}0\right) - \gamma\left(A^{2m_0+3}0 - A^{2m_0}0\right) = (1-\gamma)x_0 + Bx_0 - \gamma B(x_0 + Bx_0).$$

Since $\gamma \in (0,1)$ and B is a decreasing operator, we have

$$(1-\gamma)x_0 \succeq 0$$

and

$$Bx_0 - \gamma B(x_0 + Bx_0) \succeq Bx_0 - B(x_0 + Bx_0) \succeq 0.$$

Then we get

$$A^{2m_0+2}0 - A^{2m_0}0 \succeq \gamma\left(A^{2m_0+3}0 - A^{2m_0}0\right)$$

for every $\gamma \in (0,1)$.

Hence $A \in \mathcal{A}$ and the result follows from Theorem 2.6. □

4 The case of Banach spaces

Let $(X, \|\cdot\|)$ be a Banach space and $P \subset X$ be a cone in X. Let us suppose that P is a normal cone with normal constant $K > 0$, that is,

$$x, y \in X, \quad 0 \preceq x \preceq y \quad \Longrightarrow \quad \|x\| \leq K\|y\|.$$

Let $A : P \to P$ be an operator satisfying the following conditions:

(i) $0 \prec A0$;

(ii) A is a convex and decreasing operator;

(iii) there exist $\gamma \in (0,1)$ and $m_0, n_0 \in \mathbb{N}$ with $n_0 > m_0$ such that

$$A^{2m_0+2}0 - A^{2m_0}0 \succeq \gamma\big(A^{2m_0+3}0 - A^{2m_0}0\big)$$

and

$$A^{2n_0}0 \succeq \frac{1}{2}\big(A^{2m_0+1}0 + A^{2m_0}0\big).$$

Let us consider the probabilistic Banach space (X, N, T_m), where

$$N_x(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{t}{t+\|x\|} & \text{if } t > 0 \end{cases}$$

for every $x \in X$.

Lemma 4.1 *P is a normal cone in the probabilistic Banach space (X, N, T_m) with normal constant K.*

Proof At first, let us prove that P is also a cone in the probabilistic Banach space (X, N, T_m). Indeed, we have just to prove that P is also closed in (X, N, T_m). Let $\{x_n\}$ be a sequence in P such that $\{x_n\}$ converges to some $x \in X$ in (X, N, T_m). Let $\varepsilon > 0$, by the definition of the convergence in a probabilistic normed space, there exists some $N \in \mathbb{N}$ such that

$$N_{x_n - x}(\varepsilon) > \frac{1}{2}$$

for every $n \geq N$. Then we have

$$\frac{\varepsilon}{\varepsilon + \|x_n - x\|} > \frac{1}{2}$$

for every $n \geq N$, which is equivalent to

$$\|x_n - x\| < \varepsilon$$

for every $n \geq N$. Thus $\{x_n\}$ converges to x with respect to $\|\cdot\|$. Since P is closed with respect to the topology of the norm $\|\cdot\|$, we have $x \in P$. Then we have proved that P is also closed in the probabilistic Banach space (X, N, T_m).

Now, let us prove that P is normal in (X, N, T_m). Let $x, y \in X$ such that

$$0 \preceq x \preceq y.$$

Since P is a normal cone in the Banach space $(X, \|\cdot\|)$ with normal constant K, we have

$$\|x\| \leq K\|y\|.$$

Then, for every $t > 0$, we have

$$N_x(t) = \frac{t}{t + \|x\|} \geq \frac{t}{t + K\|y\|} = \frac{\frac{t}{K}}{\frac{t}{K} + \|y\|} = N_y\left(\frac{t}{K}\right).$$

If $t \leq 0$, obviously we have

$$0 = N_x(t) = N_y\left(\frac{t}{K}\right).$$

Thus, for every $t \in \mathbb{R}$, we have

$$N_x(t) \geq N_y\left(\frac{t}{K}\right).$$

This proves that P is a normal cone in (X, N, T_m) with normal constant K. □

Now, using Theorem 2.6 and Lemma 4.1, we obtain the following fixed point result in Banach spaces ([14], Theorem 2.1).

Corollary 4.2 *Suppose that conditions* (i)-(iii) *are satisfied. Then*

(I) *A has a unique fixed point $x^* \in P$;*

(II) *for any initial value $x_0 \in P$, the Picard sequence $\{x_n\}$ in X defined by*

$$x_n = Ax_{n-1}, \quad n \geq 1$$

converges to x^*;

(III) *we have the estimates*

$$\left\|x_{2(m_0+n)} - x^*\right\| \leq \frac{K^2\|A0\|}{n - 2 - n_0 + m_0}$$

for every $n > n_0 + 2 - m_0$, and

$$\left\|x_{2(m_0+n)+1} - x^*\right\| \leq \frac{K^2\|A0\|}{n - 1 - n_0 + m_0}$$

for every $n > n_0 + 1 - m_0$.

5 An application to a two-point boundary value problem

In this section, we present an application of Theorem 2.6 to a two-point boundary value problem.

Let $a : [0,1] \to \mathbb{R}$ be a given function satisfying the following conditions:

(a_1) a is a continuous function;

(a_2) $a(x(1-x)) = a(x)$ for every $x \in [0,1]$;

(a_3) $0 < m \leq a(x) \leq M$ for every $x \in [0,1]$.

Clearly, the set of functions $a : [0,1] \to \mathbb{R}$ satisfying the above conditions is not empty.

Example 5.1 Let $a : [0,1] \to \mathbb{R}$ be a positive constant function. Then a satisfies conditions (a_1)-(a_4).

Example 5.2 Let $a : [0,1] \to \mathbb{R}$ be the function defined by

$$a(x) = \alpha + \beta x(1-x), \quad x \in [0,1],$$

where $\alpha > 0$ and $\beta \geq 0$ are constants. Then a satisfies conditions (a_1)-(a_4) with

$$m = \alpha \quad \text{and} \quad M = \alpha + \frac{\beta}{4}.$$

Let $f : [0,\infty) \to \mathbb{R}$ be a function satisfying the following conditions:

(f_1) f is a continuous function, $f \geq 0$;
(f_2) f is a decreasing and convex function;
(f_3) $f(0) = 1$, $0 < f(\frac{M}{8}) < 1$;
(f_4) $f(\gamma x) \geq \frac{1}{2}$ for every $x \in [0, \frac{M}{8}]$, where

$$\gamma = 1 - \frac{1}{6} f\left(\frac{M}{8}\right)\left(1 - f\left(\frac{m}{2}\right)\right).$$

The set of functions $f : [0,\infty) \to \mathbb{R}$ satisfying the above conditions is not empty.

Example 5.3 Let $m = M = 1$, that is,

$$a(x) = 1, \quad x \in [0,1].$$

Let $f : [0,\infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{2}{2 + 17x}, \quad x \geq 0.$$

Then f satisfies conditions (f_1)-(f_4) with $\gamma = \frac{1{,}943}{2{,}079}$.

Now, let us consider the following two-point boundary value problem:

$$\begin{cases} -u''(x) = f(a(x)u(x)) & \text{if } 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{5.1}$$

Let (X, N, T_m) be the probabilistic Banach space, where $X = C([0,1])$ is the set of real continuous functions in $[0,1]$ and $N : X \to D$ is given by

$$N_u(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{t}{t + \max_{0 \leq x \leq 1} |u(x)|} & \text{if } t > 0, \end{cases} \quad u \in C\big([0,1]\big).$$

Let

$$P = \big\{u \in C\big([0,1]\big) : u(x) \geq 0 \text{ for all } x \in [0,1]\big\}.$$

Then P is a normal cone in the probabilistic Banach space (X, N, T_m). The partial order $\preceq$ induced by the cone P in the set X is defined by

$$u, v \in C([0,1]), \quad u \preceq v \quad \Longleftrightarrow \quad u(x) \leq v(x) \quad \text{for all } x \in [0,1].$$

We have the following result.

Theorem 5.4 *The boundary value problem* (5.1) *has a unique positive solution* $u^* \in P$.

Proof The Green function associated to (5.1) is given by

$$G(x,y) = \begin{cases} y(1-x) & \text{if } 0 \leq y \leq x \leq 1, \\ x(1-y) & \text{if } 0 \leq x \leq y \leq 1. \end{cases}$$

Then problem (5.1) is equivalent to the integral equation

$$u(x) = \int_0^1 G(x,y) f\big(a(y)u(y)\big)\, dy, \quad x \in [0,1].$$

Let us consider the nonlinear operator $A : P \to P$ defined by

$$(Au)(x) = \int_0^1 G(x,y) f\big(a(y)u(y)\big)\, dy, \quad x \in [0,1].$$

We have to prove that A has a unique fixed point in P. Theorem 2.6 will be used for the proof.

Clearly, the operator A is convex and decreasing with respect to the partial order $\preceq$.

For every $x \in [0,1]$, we have

$$(A0)(x) = \int_0^1 G(x,y) f(0)\, dy = \int_0^1 G(x,y)\, dy = \frac{x(1-x)}{2} \geq 0.$$

Then we have

$$0 \prec A0.$$

Moreover, we have

$$0 \leq a(x)(A0)(x) \leq \frac{M}{8}, \quad x \in [0,1],$$

which implies that

$$f\big(a(x)(A0)(x)\big) \geq f\left(\frac{M}{8}\right), \quad x \in [0,1].$$

The above inequality yields

$$\big(A^2 0\big)(x) = \int_0^1 G(x,y) f\big(a(y)(A0)(y)\big)\, dy \geq f\left(\frac{M}{8}\right)(A0)(x), \quad x \in [0,1]. \tag{5.2}$$

On the other hand, for every $x \in [0,1]$, we have

$$\begin{aligned}(A^2 0)(x) &= \int_0^1 G(x,y) f\big(a(y)(A0)(y)\big)\,dy \\ &= (1-x)\int_0^x y f\left(a(y)\frac{y(1-y)}{2}\right)dy + x\int_x^1 (1-y) f\left(a(y)\frac{y(1-y)}{2}\right)dy.\end{aligned}$$

Using the fact that f is a decreasing and convex function, we obtain

$$\begin{aligned}&(1-x)\int_0^x y f\left(a(y)\frac{y(1-y)}{2}\right)dy \\ &\le (1-x)\int_0^x y(1-y) f\left(\frac{a(y)y}{2}\right)dy + (1-x)\int_0^x y^2\,dy \\ &\le (1-x)\int_0^x y^2(1-y) f\left(\frac{a(y)}{2}\right)dy + (1-x)\int_0^x y(1-y)^2\,dy + (1-x)\int_0^x y^2\,dy \\ &\le (1-x) f\left(\frac{m}{2}\right)\int_0^x y^2(1-y)\,dy + (1-x)\int_0^x y(1-y)^2\,dy + (1-x)\int_0^x y^2\,dy \\ &= \frac{1}{6}\left[f\left(\frac{m}{2}\right)\big(4x^2-3x^3\big) + 6x - 4x^2 + 3x^3\right](A0)(x).\end{aligned}$$

Similarly, using condition (a_2), we have

$$\begin{aligned}&x\int_x^1 (1-y) f\left(a(y)\frac{y(1-y)}{2}\right)dy \\ &\le \frac{1}{6}\left[f\left(\frac{m}{2}\right)\big(1+x-5x^2+3x^3\big) + 5 - 7x + 5x^2 - 3x^3\right](A0)(x).\end{aligned}$$

Thus we have

$$(A^2 0)(x) \le \frac{1}{6}\left[5 + f\left(\frac{m}{2}\right)\right](A0)(x) \tag{5.3}$$

for every $x \in [0,1]$. Moreover, using (5.2), we get

$$f\big(a(x)(A^2 0)(x)\big) \le f\left(a(x) f\left(\frac{M}{8}\right)(A0)(x)\right) \tag{5.4}$$

for every $x \in [0,1]$. Now, (5.3) and (5.4) yield

$$\begin{aligned}(A^3 0)(x) &= \int_0^1 G(x,y) f\big(a(y)(A^2 0)(y)\big)\,dy \\ &\le \int_0^1 G(x,y) f\left(a(y) f\left(\frac{M}{8}\right)(A0)(y)\right)dy \\ &\le f\left(\frac{M}{8}\right)\int_0^1 G(x,y) f\big(a(y)(A0)(y)\big)\,dy + \left[1 - f\left(\frac{M}{8}\right)\right](A0)(x) \\ &= f\left(\frac{M}{8}\right)(A^2 0)(x) + \left[1 - f\left(\frac{M}{8}\right)\right](A0)(x) \\ &\le \gamma (A0)(x)\end{aligned}$$

for every $x \in [0,1]$. Thus we have

$$f\bigl(a(x)\bigl(A^3 0\bigr)(x)\bigr) \geq f\bigl(\gamma a(x)(A0)(x)\bigr), \quad x \in [0,1].$$

Using condition (f_4), we obtain

$$\begin{aligned}\bigl(A^4 0\bigr)(x) &= \int_0^1 G(x,y) f\bigl(a(y)\bigl(A^3 0\bigr)(y)\bigr)\,dy \\ &\geq \int_0^1 G(x,y) f\bigl(\gamma a(y)(A0)(y)\bigr)\,dy \\ &\geq \frac{1}{2}(A0)(x)\end{aligned}$$

for every $x \in [0,1]$. Finally, using (5.2), for all $x \in [0,1]$, we have

$$\bigl(A^2 0\bigr)(x) \geq \epsilon (A0)(x) \geq \epsilon \bigl(A^3 0\bigr)(x),$$

where $\epsilon = f(\frac{M}{8}) \in (0,1)$.

Now, the desired result follows from Theorem 2.6 with $m_0 = 0$ and $n_0 = 2$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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