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Classical Topology and Quantum States*

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Abstract

Any two infinite-dimensional (separable) Hilbert spaces are unitarily isomorphic. The sets of all their self-adjoint operators are also therefore unitarily equivalent. Thus if all self-adjoint operators can be observed, and if there is no further major axiom in quantum physics than those formulated for example in Dirac's 'Quantum Mechanics', then a quantum physicist would not be able to tell a torus from a hole in the ground. We argue that there are indeed such axioms involving observables with smooth time evolution: they contain commutative subalgebras from which the spatial slice of spacetime with its topology (and with further refinements of the axiom, its C^K- and $C^\infty-$ structures) can be reconstructed using Gel'fand - Naimark theory and its extensions. Classical topology is an attribute of only certain quantum observables for these axioms, the spatial slice emergent from quantum physics getting progressively less differentiable with increasingly higher excitations of energy and eventually altogether ceasing to exist. After formulating these axioms, we apply them to show the possibility of topology change and to discuss quantized fuzzy topologies. Fundamental issues concerning the role of time in quantum physics are also addressed.

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1 Introduction

Conventional expositions of classical physics assume that the concept of the spatial slice Q and its topological and differential geometric attributes are somehow known , and formulate dynamics of particles or fields using Q and further metaprinciples like locality and causality. The space Q thus becomes an irreducible background , immune to analysis, for a classical physicist, even though it is an indispensable ingredient in the formulation of physical theory.

Quantum physics is a better approximation to reality than is classical physics. Still, models of quantum physics are seldom autonomous, but are rather emergent from a classical substructure. Thus we generally formulate a quantum model by canonical or path integral quantisation of a classical Lagrangian based on the space Q . We thus see that Q and its properties are tamely accepted , and they are not subjected to physical or mathematical analysis, in such conservative quantum physics too.

Classical topology is in this manner incorporated in conventional quantum physics by formulating it using smooth functions on Q. There is reason to be uneasy with this method of encoding classical data in quantum physics. In quantum theory, the fundamental physical structure is the algebra of observables, and it would be greatly more satisfactory if we can learn if and how operator algebras describe classical topology and its differential attributes.

This note will report on certain ongoing research with several colleagues concerning this question which is fundamentally an enquiry into the nature of space and time in quantum physics. Some of our ideas have already been published elsewhere [1, 2, 3, 4]. Our work touches both on issues of relevance to quantum gravity such as the meaning of "quantized topology" and the possibility of topology change, and on topics of significance for foundations of quantum physics. I think that we have progressively approached a measure of precision in the formulation of relatively inarticulated questions, but our responses are still tentative and lacking in physical and mathematical completeness and rigor.

2 The Problem as a Parable

We restate the problem to be addressed here.It is best introduced as a little story about a quantum baby. The story will set the framework for the rest of the talk. Its proper enjoyment calls for a willing suspension of disbelief for the moment.

All babies are naturally quantum, so my adjective for the baby can be objected to as redundant and provocative, but it invites attention to a nature of infants of central interest to us, so let us leave it there.

Parable of the Quantum Baby

Entertain the conjecture of a time, long long ago, when there lived a quantum baby of cheerful semblance and sweet majesty. It was brought up by its doting parents on a nourishing diet of self-adjoint operators on a Hilbert space. All it could experience as it grew up were their mean values in quantum states. It did not have a clue when it was little that there is our classical world with its topology, dimension and metric. It could not then tell a torus from a hole in the ground.

Yet the baby learned all that as it grew up.

And the wise philosopher is struck with wonder: How did the baby manage this amazing task?

For the problem is this: Even in a quantum theory emergent from a smooth classical configuration space Q, there is no need for a wave function ψ, or a probability density $\psi^*\psi$, to be continuous on Q. It is enough that the integral $\int \omega\psi^*\psi$ over Q for an appropriate volume form ω is finite. Probability interpretation requires no more.

But if the baby can observe all self-adjoint operators with equal ease, and thereby prepare all sorts of discontinuous quantum states, how then does it ever learn of Q, its topology and its differential attributes ? The problem is even worse: We shall see below that any two (separable)Hilbert spaces are isometric so that there is only one abstract Hilbert space.

This then is our central question. All that follows is charged with its emotional content, and comes from trying to find its answer.

3 Another Statement

We can explain the baby problem in yet another way.

In quantum physics,observables come from bounded operators on a (separable [5]) Hilbert space $\mathcal{H}$. [We will deal only with separable Hilbert spaces.] The latter is generally infinite-dimensional.

But all infinite-dimensional Hilbert spaces are isomorphic, in fact unitary so. If $|n>^{(i)}$ $(n \in \mathbb{N})$ gives the orthonormal basis for the Hilbert space $\mathcal{H}^{(i)}(i = 1, 2)$, we can achieve this equivalence by setting $|n>^{(2)} = V|n>^{(1)}$. That being so, any operator $A^{(1)}$ on $\mathcal{H}^{(1)}$ has a corresponding operator $A^{(2)} = VA^{(1)}V^{-1}$ on $\mathcal{H}^{2)}$.

How then does a quantum baby tell a torus from a hole in the ground?

Without further structure in quantum physics besides those to be found in standard text books, this task is in fact entirely beyond the baby.

In conventional quantum physics of particles say,we generally start from smooth functions (or smooth sections of hermitean vector bundles) on Q and complete them into a Hilbert space $\mathcal{H}$ using a suitable scalar product. In this way, we somehow incorporate knowledge about Q right at the start.

But this approach requires realizing $\mathcal{H}$ in a particular way, as square integrable functions (or sections of hermitean vector bundles)on Q. The presentation of $\mathcal{H}$ in this manner is reminiscent of the presentation of a manifold in a preferred manner, as for instance using a particular coordinate chart.

Can we give a reconstruction of Q in an intrinsic way? What new structures are needed for this purpose?

In the scheme we develop as a response to these questions, Q emerges with its $C^{\infty}-$ structure only from certain observables, *topology and differential features being attributes of particular classes of observables and not universal properties of all observables.* Thus Q emerges as a manifold only if the high energy components in the observables are suppressed. When higher and higher energies are excited, it gets more and more rough and eventually altogether ceases to exist as a topological space modelled on a manifold. Here by becoming more rough we mean that C^{∞} becomes C^{K} and correspondingly the $C^{\infty}-$ manifold Q becomes a $C^{K}-$ space Q^{K}.

The epistemological problems we raise here are not uniquely quantal.They are encountered in classical physics too , but we will not discuss them here.

4 What is Our Quantum System?

The system we consider is generic. If K is the configuration space of a generic system,such as that of a single particle or a quantum field,its algebra of observables normally contains the algebra $C^\infty(Q)$ of smooth functions on the spatial slice Q.For a charged field, for example,suitably smeared charge, energy and momentum densities can generate this algebra. That is (provisionally) enough for our central goal of recovering Q from quantum observables.

5 Time is Special

We have to assume that time evolution is given as a unitary operator $U(t)$ which is continuous in t. Our analysis needs this input. Time therefore persists as an *a priori* irreducible notion even in our quantum approach. It would be very desirable to overcome this limitation. (See [6] in this connection.)

There is more to be said on time, its role in measurement theory and as the mediation between quantum and classical physics. There are brief remarks on these matters below.

It is true that in so far as our main text is concerned, $U(t)$ or the Hamiltonian can be substituted by spatial translations, momenta or other favorite observables. But we think that time evolution is something special, being of universal and central interest to science. It is for this reason that we have singled out $U(t)$.

6 The Gel'fand-Naimark Theory

The principal mathematical tool of our analysis involves this remarkable theory [7] and, to some extent its developments in Noncommutative Geometry [8, 9, 10, 11, 12]. We shall now give a crude and short sketch of this theory.

A C^*-algebra $\mathcal{A}$ with elements c has the following properties: a) It is an algebra over $\mathbb{C}$. b) It is closed under an antinvolution $*$:

$$* : c_j \in \mathcal{A} \Rightarrow c_j^* \in \mathcal{A},\ c_j^{**} = c_j,\ (c_1 c_2)^* = c_2^* c_1^*,\ (\xi c_j)^* = \xi^* c_j^*, \tag{6.1}$$

where ξ is a complex number and ξ^* is its complex conjugate. c) It has a norm $||.||$ with the properties $||c^*|| = ||c||,\ ||c^*c|| = ||c||^2$ for $c \in \mathcal{A}$. d) It is complete under this norm.

A $*-$ representation ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a representation of $\mathcal{A}$ by a C^*-algebra of bounded operators on a Hilbert space [13] with the following features: i) The $*$ and norm for $\rho(\mathcal{A})$ are the operator adjoint $\dagger$ and operator norm (also denoted by $||\cdot||$). ii) $\rho(c^*) = \rho(c)^\dagger$.

ρ is said to be a $*$-homomorphism because of ii). We can also in a similar manner speak of $*$-isomorphisms.

We will generally encounter $\mathcal{A}$ concretely as an algebra of operators. In any case, we will usually omit the symbol ρ.

Note that a $*$-algebra (even if it is not C^*) is by definition closed under an antinvolution $*$.

Let $\mathcal{C}$ denote a commutative C^*-algebra. Let $\{x\}$ denote its space of inequivalent irreducible $*$-representations (IRR's) or its spectrum. [So $a \in \mathcal{C} \Rightarrow x(a) \in \mathbb{C}$.] The Gel'fand-Naimark theory then makes the following striking assertions: α) There is a natural topology on $\{x\}$ making it into a Hausdorff topological space [14] Q^0. [We will denote the IRR's prior to introducing topology by $\{x\}$ and after doing so by $Q^{\cdot}$ with suitable superscripts.Q is the same as Q^∞ below.] β) Let $\mathcal{C}^0(Q)$ be the C^*-algebra of $\mathbb{C}$-valued continuous functions on Q. Its $*$ is complex conjugation and its norm $||\cdot||$ is the supremum norm, $||\phi|| = \sup_{x\in Q^0} |\phi(x)|$. Then $\mathcal{C}^0(Q)$ is $*$-isomorphic to $\mathcal{C}$.

We can thus identify $\mathcal{C}^0(Q)$ with $\mathcal{C}$, as we will often do.

The above results can be understood as follows. By "duality", the collection of $x(a)$'s for all x defines a function a_c on $\{x\}$ by $a_c(x) := x(a)$. a_c is said to be the Gel'fand transform of a.

$\{x\}$ is as yet just a collection of points with no topology. How can we give it a natural topology? We want a_c to be C^0 in this topology. Now the set of zeros of a continuous function is closed. So let us identify the set of zeros C_a of each a_c with a closed set:

$$C_a = \{x : x(a) \equiv a_c(x) = 0\}. \tag{6.2}$$

The topology we seek is given by these closed sets. The Gel'fand-Naimark theorem then asserts α) and β) for this topology, the isomorphism $\mathcal{C} \to \mathcal{C}^0(Q)$ being $a \to a_c$.

A Hausdorff topological space can therefore be equally well described by a commutative C^* -algebra $\mathcal{C}$, presented for example using generators. That would be an intrinsic coordinate-free description of the space and an alternative to using coordinate charts.

A C^K - structure can now be specified by identifying an appropriate subalgebra $\mathcal{C}^K$ of $\mathcal{C} \equiv \mathcal{C}^0$ and declaring that the C^K - structure is the one for which $\mathcal{C}^K$ consists of K-times differentiable functions. [$\mathcal{C}^K$ is a $*-$, but not a C^*-, algebra for $K > 0$, as it is not complete..] The corresponding C^K-space is Q^K. For $K = \infty$, we get the manifold Q^∞.

We have the inclusions

$$\mathcal{C}^\infty \subset ... \subset \mathcal{C}^K ... \subset \mathcal{C}^0 \equiv \mathcal{C} \tag{6.3}$$

where

$$\overline{\mathcal{C}}^{(\infty)} = \overline{\mathcal{C}}^{(K)} = \overline{\mathcal{C}} \equiv \mathcal{C}, \tag{6.4}$$

the bar as usual denoting closure.In contrast, Q^∞ and Q^K are all the same as sets, being $\{x\}$.

A dense $*$-subalgebra of a C^*-algebra $\mathcal{C}$ will be denoted by $\mathcal{C}^\cdot$, the superscript highlighting some additional property. The algebras $\mathcal{C}^K$ are such examples.

Example 1: Consider the algebra $\mathcal{C}$ generated by the identity, an element u and its inverse u^{-1}. Its elements are $a = \sum_{N\in\mathbb{Z}} \alpha_N u^N$ where α_N's are complex numbers vanishing rapidly in N at ∞. The $*$ is defined by $u^* = u^{-1}, a^* = \sum a_N^* u^{-N}$. As $\mathcal{C}$ has identity $\mathbb{1}$, there is a natural way to define inverse a^{-1} too : a^{-1} is that element of $\mathcal{C}$ such that $a^{-1}a = aa^{-1} = \mathbb{1}$. There is also a canonical norm $||.||$ compatible with properties c) [8, 10]: $||a||$ = Maximum of $|\lambda|$ such that $a^*a - |\lambda|^2$ has no inverse.

The space Q for this $\mathcal{C}$ is just the circle S^1, u_c being the function with value $e^{i\theta}$ at $e^{i\theta} \in S^1$.

If similarly we consider the algebra associated with N commuting unitary elements, we get the N-torus T^N. If for $N = 2$, the generating unitary elements do not commute, but fulfill $u_1u_2 = \omega u_2u_1, \omega$ being any phase, we get the noncommutative torus [15, 8]. It is the "rational" or "fuzzy" torus if $\omega^K = 1$ for some $K \in \mathbb{Z}$, otherwise it is "irrational" [12, 16].

7 States and Observables

The formulation of quantum physics best suited for the current discussion is based on the algebra $\mathcal{B}$ of bounded observables and states ω on $\mathcal{B}$. $\mathcal{B}$ has a *-operation (anti-involution) and $\omega(b) \in \mathbb{C}$ for $b \in \mathcal{B}$ with $\omega(b^*b) \geq 0$, $\omega(\mathbb{1}) = 1$. ω can be thought of as the density matrix describing the ensemble and b the operator whose mean value is being measured. The Gel'fand-Naimark-Segal (GNS) construction lets us recover the Hilbert space formulation from ω and b.

8 Instantaneous Measurements and Classical Topology

Time in *conventional* quantum physics has a unique role.It is not a quantum variable, and all elementary quantum observations are instantaneous.

Now elementary measurements-those instantaneous in time- can only measure commuting observables.Thus the probabilty of finding the value a for the observable A at time $t-\epsilon$ and then b for B at $t+\epsilon$ is $\omega(P_a(t-\epsilon)P_b(t+\epsilon)P_a(t+\epsilon))$,where $P_{a,b}$ are projectors at indicated times. If the order is reversed, the answer is $\omega(P_a(t-\epsilon)P_b(t+\epsilon)P_a(t+\epsilon))$. They do not coincide as $\epsilon \to 0$ unless $P_a(t)P_b(t) = P_b(t)P_a(t)$, that is $AB = BA$.As experiments cannot resolve time sequence if ϵ is small enough, we cannot consistently assign joint probabilities to noncommuting observables in elementary measurements.

Thus from instantaneous measurements, we can extract commutative C^*-algebras and therefrom Hausdorff topological spaces.

If "commutation" is classical, then instantaneous measurements and Hausdorff spaces (the stuff of manifolds) are also partners in this classicality.

It is known that a state ω restricted to a commutative C^*-algebra is equivalent to a classical probabilty measure on its underlying topological space. As a wave function $|\psi>$ thus is equivalent to a classical probability measure for an instantaneous measurement (which any way is the only sort of measurement discussed in usual quantum physics), there is no need to invoke "collapse of wave packets" or similar hypotheses for its interpretation. The uniqueness of quantum measurement theory then consists in the special relations it predicts between outcomes of measurements of different commutative algebras $\mathcal{C}_1$ and $\mathcal{C}_2$. These relations are often universal, being independent of the state vector $|\psi>$.

Such a point of view of quantum physics, or at least a view close to it, has been advocated especially by Sorkin [17].

Thus we see that instantaneous measurements are linked both to classical topology and to classical measurement theory.

But surely the notion of *instantaneous measurements* can only be an idealization. Measurements must be extended in time too, just as they are extended in space. But we know of no fully articulated theory of measurements extended in time, and maintaining quantum coherence during its duration, although interesting research about these matters exists [18].

A quantum theory of measurements extended in time, with testable predictions, could be of fundamental importance. We can anticipate that it will involve noncommutative algebras $\mathcal{N}$ instead of commutative algebras, the hermitean form $\psi^\dagger\chi$ for the appropriate vectors ψ, χ in the Hilbert space being valued in $\mathcal{N}$. Such quantum theories were encountered in [2]. Mathematical tools for their further development are probably available in

Noncommutative Geometry [8, 9, 10, 11, 12].

But we are hardly done , we do not have Q as a manifold , or its dimension etc.

9 What Time Evolution Tells Us

Time evolution $U(t)$ evolves all observables $\mathcal{B} \equiv \mathcal{B}^{(0)}$ continuously in conventional quantum physics:$\omega(U(t)^{-1}bU(t))$ is continuous in t for all $b \in \mathcal{B}^{(0)}$.

Let $\mathcal{B}^{(1)} \subset \mathcal{B}^{(0)}$ be the subset of $\mathcal{B}^{(0)}$ with differentiable time evolution.The Hamiltonian H is defined only on $\mathcal{B}^{(1)}$: If $b(t) = U(t)^{-1}b(0)U(t) \in \mathcal{B}^{(1)}$, then $idb(t)/dt = [b(t), H]$. For example, for $H = p^2/2m$ plus a smooth potential $V(x)$,$\mathcal{B}^{(1)}$ contains twice-differentiable functions of x. For $D = -i\alpha.\partial$, it has C^1 functions.

K-times differentiabilty in this way gives $\mathcal{B}^{(K)}$ with inclusions $... \subset \mathcal{B}^{(K)} \subset \mathcal{B}^{(K-1)} \subset ... \subset \mathcal{B}^{(0)}$.

Let $\mathcal{B}^{(\infty)}$=$\bigcap \mathcal{B}^{(K)}$.From $\mathcal{B}^{(\infty)}$,we have to extract a subalgebra which helps us reconstruct the spatial slice Q with its differential structure,dimension etc. The criterion to do so may be a weak form of relativistic causality. In relativity,if an observable is localised in a spatial region D at time zero, its support D_t at time t is within the future light cone of D.This means in particular that as $t \to 0$, $D_t \to D_0 = D$.There is no spread all over in infinitesimal times. Such a constraint is compatible with H having a finite number of spatial derivatives. Relativistic causality for example is violated by the Hamiltonian $(p^2 + m^2)^{1/2}$ whereas the Dirac operator is of first order and causal.

If H is of first order and f and g are functions,then $[[H, f], g] = 0$. This is so for example for the Dirac Hamiltonian. More generally,if H is of finite order, $[[H, f_1], f_2], f_3]..., f_K]] = 0$ for a finite K.All this suggests the

Definition: A commutative subalgebra $\mathcal{C}^{(\infty)}$ of $\mathcal{B}^{(\infty)}$ is weakly causal if, for $f_i \in \mathcal{C}^{(\infty)}$, $[[H, f_1], f_2], f_3]..., f_K]] = 0$ for some K.

This pale form of causality can be valid *generically* only for functions on a spatial manifold M. For example, the Hamiltonian of a simple harmonic oscillator fulfills this criterion in both position and momentum space.

Conjecture: $\mathcal{C}^{(\infty)}$ determines M and its C^∞-structure by the analogue of a Gel'fand-Neumark construction.

If $\mathcal{B}^{(K)}$ is substituted for $\mathcal{B}^{(\infty)}$ and a corresponding $\mathcal{C}^{(K)}$ is extracted, the latter will fix only the C^K-structure of Q. Requiring just continuity , we can recover Q only as a topological space.

We can expand observables in eigenstates of H : $b(t) = \sum b_n e^{i\omega_n t}$, with $\parallel b(t) \parallel^2 \equiv$

$\omega(b(t)^* b(t)) < \infty$.From $d^K b(t)/dt^K = \sum (i\omega)^K b_n e^{i\omega_n t}$, we see that requiring convergence of r.h.s. in norm for high K suppresses high frequencies.(We are ignoring issues of null states of ω here.)Thus low energy observations recover Q with its C^∞- structure. But as higher and higher energies are observed, that is, as shorter and shorter time scales are resolved,Q gets more rough, retaining progressively less of its differentable structure. Eventually for nondifferentiable b, Q is just a topological space and retains no differentiable structure.

The situation is in fact more dramatic.The algebra giving Q as a topological space is the C^*-algebra of continuous functions $\mathcal{C}^{(0)}$.The maximum commutative C^*-algebra $\mathcal{C}^{(0)''}$ containing $\mathcal{C}^{(0)}$ does not give Q as a topological space modelled on a manifold.

Much of what we discussed above is based on spectral considerations, suggesting that more remarks are necessary as regards isospectral manifolds. We will not however undertake this task here.

10 Dimension and Metric

Suppose that Q has been recovered as a manifold. We can then find its dimension in the usual way.

There is also a novel manner to find its dimension d from the spectrum $\{\lambda_n\}$ of H: If H is of order N, $|\lambda_n|$ grows like $n^{N/d}$ as $n \to \infty$ [8, 9, 10, 19].

We can find a metric as well for Q [8, 10, 19]: It is specified by the distance

$$d(x,y) = \{\sup_a |a_c(x) - a_c(y)| : \frac{1}{N!} || \underbrace{[a,[a,\ldots[a,H]\ldots]}_{N\ a's} || \leq 1\}. \tag{10.1}$$

This remarkable formula gives the usual metric for the Dirac operator [$N = 1$] [8, 9] and the Laplacian [$N = 2$] [19].

11 What is Quantum Topology?

A question of the following sort often suggests itself when encountering discussions of topology in quantum gravity: If Q is a topological space, possibly with additional differential and geometric structures ["classical" data], what is meant by *quantizing* Q?

It is perhaps best understood as: *finding an algebra of operators on a Hilbert space from which Q and its attributes can be reconstructed* [much as in the Gel'fand-Naimark theorem].

12 Topology Change

We now use the preceding ideas to discuss topology change, following ref. 3. [See ref. [20] for related work.]

There are indications from theoretical considerations that spatial topology in quantum gravity cannot be a time-invariant attribute, and that its transmutations must be permitted in any eventual theory.

The best evidence for the necessity of topology change comes from the examination of the spin-statistics connection for the so-called geons [21, 22, 23, 24, 25]. Geons are solitonic excitations caused by twists in spatial topology. In the absence of topology change, a geon can neither annihilate nor be pair produced with a partner geon, so that no geon has an associated antigeon.

Now spin-statistics theorems generally emerge in theories admitting creation-annihilation processes [22, 23, 26]. It can therefore be expected to fail for geons in gravity theories with no topology change. Calculations on geon quantization in fact confirm this expectation [22, 27].

The absence of a universal spin-statistics connection in these gravity theories is much like its absence for a conventional nonrelativistic quantum particle which too cannot be pair produced or annihilated. Such a particle can obey any sort of statistics including parastatistics regardless of its intrinsic spin. But the standard spin-statistics connection can be enforced in nonrelativistic dynamics also by introducing suitable creation-annihilation processes [28].

There is now a general opinion that the spin-statistics theorem should extend to gravity as well. Just as this theorem emerges from even nonrelativistic physics once it admits pair production and annihilation [23], quantum gravity too can be expected to become compatible with this theorem after it allows suitable topology change [26]. In this manner, the desire for the usual spin-statistics connection leads us to look for a quantum gravity with transmuting topology.

Canonical quantum gravity in its elementary form is predicated on the hypothesis that spacetime topology is of the form $\Sigma \times \mathbf{R}$ (with $\mathbf{R}$ accounting for time) and has an eternal spatial topology. This fact has led to numerous suggestions that conventional canonical gravity is inadequate if not wrong, and must be circumvented by radical revisions of spacetime concepts [29], or by improved approaches based either on functional integrals and cobordism [26] or on alternative quantization methods.

Ideas on topology change were first articulated in quantum gravity, and more specifically in attempts at semiclassical quantization of classical gravity. Also it is an attribute intimately linked to gravity in the physicist's mind. These connections and the apparently revolutionary nature of topology change as an idea have led to extravagant speculations about twinkling topology in quantum gravity and their impact on fundamental concepts in physics.

Here we show that models of quantum particles exist which admit topology change or contain states with no well-defined classical topology. *This is so even though gravity does not have a central role in our ideas and is significant only to the extent that metric is important for a matter Hamiltonian.* These models use only known physical principles and have no revolutionary content, and at least suggest that topology change in quantum gravity too may be achieved with a modest physical input and no drastic alteration of basic laws.

We consider particle dynamics. The configuration space of a particle being ordinary space, we are thus imagining a physicist probing spatial topology using a particle.

Let us consider a particle with no internal degrees of freedom living on the union Q' of two intervals which are numbered as 1 and 2:

$$Q' = [0, 2\pi] \bigcup [0, 2\pi] \equiv Q'_1 \bigcup Q'_2 \ . \tag{12.1}$$

It is convenient to write its wave function ψ as (ψ_1, ψ_2), where each ψ_i is a function on $[0, 2\pi]$ and $\psi_i^* \psi_i$ is the probability density on Q'_i. The scalar product between ψ and another wave function $\chi = (\chi_1, \chi_2)$ is

$$(\psi, \chi) = \int_0^{2\pi} dx \sum_i (\psi_i^* \chi_i)(x) \ . \tag{12.2}$$

It is interesting that we can also think of this particle as moving on $[0, 2\pi]$ and having an internal degree of freedom associated with the index i.

After a convenient choice of units, we define the Hamiltonian formally by

$$(H\psi)_i(x) = -\frac{d^2\psi_i}{dx^2}(x) \tag{12.3}$$

[where ψ_i is assumed to be suitably differentiable in the interval $[0, 2\pi]$]. This definition is only formal as we must also specify its domain $\mathcal{H}^1$ [13]. The latter involves the statement of the boundary conditions (BC's) at $x = 0$ and $x = 2\pi$.

Arbitrary BC's are not suitable to specify a domain: A symmetric operator $\mathcal{O}$ with domain $D(\mathcal{O})$ will not be self-adjoint unless the following criterion is also fulfilled:

$$\mathcal{B}_{\mathcal{O}}(\psi,\chi) \equiv (\psi, \mathcal{O}\chi) - (\mathcal{O}^{\dagger}\psi, \chi) = 0 \ \ for \ all \ \chi \in D(\mathcal{O}) \Leftrightarrow \psi \in D(\mathcal{O}) \ . \tag{12.4}$$

For the differential operator H, the form $\mathcal{B}_H(\cdot,\cdot)$ is given by

$$\mathcal{B}_H(\psi,\chi) = \sum_{i=1}^{2}\left[-\psi_i^*(x)\frac{d\chi_i(x)}{dx} + \frac{d\psi_i^*(x)}{dx}\chi_i(x)\right]_0^{2\pi} \ . \tag{12.5}$$

It is not difficult to show that there is a $U(4)$ worth of $D(H) \equiv \mathcal{H}^1$ here compatible with (12.4).

We would like to restrict this enormous choice for $D(H)$, our intention not being to study all possible domains for $D(H)$. So let us restrict ourselves to the domains

$$D_u = \{\psi \in C^2(Q') : \psi_i(2\pi) = u_{ij}\psi_j(0), \ \ \frac{d\psi_i}{dx}(2\pi) = u_{ij}\frac{d\psi_j}{dx}(0), \ u \in U(2)\} \ . \tag{12.6}$$

These domains have the virtue of being compatible with the definition of momentum in the sense discussed in ref. 2.

There are two choices of u which are of particular interest:

$$a) \qquad u_a = \begin{pmatrix} 0 & e^{i\theta_{12}} \\ e^{i\theta_{21}} & 0 \end{pmatrix}, \tag{12.7}$$

$$b) \qquad u_b = \begin{pmatrix} e^{i\theta_{11}} & 0 \\ 0 & e^{i\theta_{22}} \end{pmatrix}. \tag{12.8}$$

In case a, the density functions $\psi_i^*\chi_i$ fulfill

$$(\psi_1^*\chi_1)(2\pi) = (\psi_2^*\chi_2)(0) \ , \tag{12.9}$$

$$(\psi_2^*\chi_2)(2\pi) = (\psi_1^*\chi_1)(0) \ . \tag{12.10}$$

Figure 1 displays (12.10), these densities being the same at the points connected by broken lines.

In case b, they fulfill, instead,

$$(\psi_1^*\chi_1)(2\pi) = (\psi_1^*\chi_1)(0) \ , \tag{12.11}$$

$$(\psi_2^*\chi_2)(2\pi) = (\psi_2^*\chi_2)(0) \tag{12.12}$$

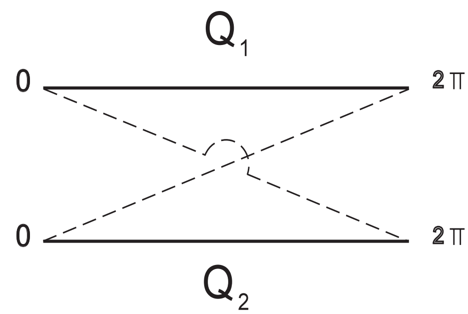

Figure 1: In case a, the density functions are the same at the points joined by broken lines in this Figure.

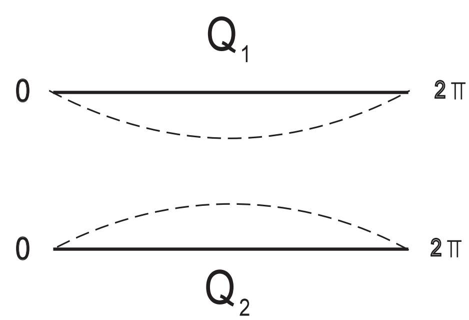

Figure 2: In case b, the density functions are the same at the points joined by broken lines in this Figure.

which fact is shown in a similar way in Figure 2.

Now if $\psi_i^*, \chi_i \in D_u$,then $\psi_i^* \chi_i \in \mathcal{C}^{(0)}$ in the operator-theoretic approach used earlier.Such probabilty densities in fact generate $\mathcal{C}^{(0)}$.Therefore their continuity properties determine the topology of the space to be identified as Q.It follows that we can identify the points joined by dots to get the classical configuration space Q if $u = u_a$ or u_b. It is *not* Q', but rather a circle S^1 in case a and the union $S^1 \bigcup S^1$ of two circles in case b.

The requirement $H^M D_u^\infty \subseteq D_u^\infty \subset D_u$ for $u = u_{a,b}$ and for all $M \in \mathbb{N}$ implies that arbitrary derivatives of $\psi_i^* \chi_i \in D_u^\infty$ are continuous at the points joined by broken lines, that is on S^1 and $S^1 \bigcup S^1$ for the two cases. We can prove this easily using (12.6). In this way, from D_u^∞, we also recover S^1 and $S^1 \bigcup S^1$ as manifolds.

When u has neither of the values (12.7) and (12.8), then Q becomes the union of two intervals. The latter happens for example for

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad . \tag{12.13}$$

In all such cases, Q can be regarded as a manifold with boundaries as shown by the argument above.

Dynamics for Boundary Conditions

We saw in the previous section that topology change can be achieved in quantum physics by treating the parameters in the BC's as suitable external parameters which can be varied.

However it is not quite satisfactory to have to regard u as an external parameter and not subject it to quantum rules. We now therefore promote it to an operator, introduce its conjugate variables and modify the Hamiltonian as well to account for its dynamics. The result is a closed quantum system. It has no state with a sharply defined u. We cannot therefore associate one or two circles with the quantum particle and quantum spatial topology has to be regarded as a superposition of classical spatial topologies. Depending on our choice of the Hamiltonian, it is possible to prepare states where topology is peaked at one or two S^1's for a long time, or arrange matters so that there is transmutation from one of these states to another.

Quantization of u is achieved as follows. Let $T(\alpha)$ be the antihermitean generators of the Lie algebra of $U(2)$ [the latter being regarded as the group of 2×2 unitary matrices] and normalized according to $Tr\ T(\alpha)T(\beta) = -N\delta_{\alpha\beta}$, N being a constant. Let $\hat{u}$ be the matrix of quantum operators representing the classical u. It fulfills

$$\hat{u}_{ij}\hat{u}_{ik}^\dagger = \mathbf{1}\delta_{jk}, \quad [\hat{u}_{ij}, \hat{u}_{kh}] = 0 \ , \tag{12.14}$$

$\hat{u}^{\dagger}_{ik}$ being the adjoint of $\hat{u}_{ik}$. The operators conjugate to $\hat{u}$ will be denoted by L_α. If

$$[T_\alpha, T_\beta] = c^\gamma_{\alpha\beta} T_\gamma, \tag{12.15}$$

$$c^\gamma_{\alpha\beta} = \text{structure constants of } U(2), \tag{12.16}$$

L_α has the commutators

$$[L_\alpha, \hat{u}] = -T(\alpha)\hat{u} \ , \tag{12.17}$$

$$[L_\alpha, L_\beta] = c^\gamma_{\alpha\beta} L_\gamma, \tag{12.18}$$

$$[T(\alpha)\hat{u}]_{ij} \equiv T(\alpha)_{ik}\hat{u}_{kj}. \tag{12.19}$$

If $\hat{V}$ is the quantum operator for a function V of u, $[L_\alpha, \hat{V}]$ is determined by (12.17) and (12.18).

The Hamiltonian for the combined particle-u system can be taken to be, for example,

$$\hat{H} = H + \frac{1}{2I}\sum_\alpha L^2_\alpha, \tag{12.20}$$

I being the moment of inertia.

Quantized BC's with a particular dynamics are described by (12.14), (12.17),(12.18) and (12.20).

The general state vector in the domain of $\hat{H}$ is a superposition of state vectors $\phi \otimes_{\mathbf{C}} |u\rangle$ where $\phi \in D_u$ and $|u\rangle$ is a generalized eigenstate of $\hat{u}$:

$$\hat{u}_{ij}|u\rangle = u_{ij}|u\rangle, \quad \langle u'|u\rangle = \delta(u'^{-1}u) \ . \tag{12.21}$$

The δ-function here is defined by

$$\int du f(u)\delta(u'^{-1}u) = f(u'), \tag{12.22}$$

du being the (conveniently normalized) Haar measure on $U(2)$.

It follows that the classical topology of one and two circles is recovered on the states $\sum_\lambda C_\lambda \phi^{(\lambda)}_{u_a} \otimes_{\mathbf{C}} |u_a\rangle$ and $\sum_\lambda D_\lambda \phi^{(\lambda)}_{u_b} \otimes_{\mathbf{C}} |u_b\rangle$, $[C_\lambda,\ D_\lambda \in \mathbb{C},\ \phi^{(\lambda)}_{u_{a,b}} \in D_{u_{a,b}}]$ with the two fixed values u_a, and u_b of (12.7) and (12.8) for u.

As the dynamical system has been enhanced by $U(2)$, the configuration space we recover is not Q in the strict sense,but rather $Q \times U(2)$. But we will refer to only Q as the configuration space below as a matter of convenience.

Now the above vectors are clearly idealized and unphysical , and with infinite norm. The best we can do with normalizable vectors to localize topology around one or two circles is to work with the wave packets

$$\int du f(u)\phi_u \otimes_{\mathbf{C}} |u\rangle \ , \tag{12.23}$$

$$\int \phi_u \in D_u, \tag{12.24}$$

$$\int du|f(u)|^2 < \infty \tag{12.25}$$

where f is sharply peaked at the u for the desired topology. The classical topology recovered from these states will only approximately be one or two circles, the quantum topology also containing admixtures from neighboring topologies of two intervals.

A localized state vector of the form (12.25) is not as a rule an eigenstate of a Hamiltonian like $\hat{H}$. Rather it will spread in course of time so that classical topology is likely to disintegrate mostly into that of two intervals. We can of course localize it around one or two S^1's for a very long time by choosing I to be large, the classical limit for topology being achieved by letting $I \to \infty$. By adding suitable potential terms, we can also no doubt arrange matters so that a wave packet concentrated around $u = u_a$ moves in time to one concentrated around $u = u_b$. This process would be thought of as topology change by a classical physicist.

The preceding considerations on topology change admit generalizations to higher dimensions as explained in ref.3.

13 Final Remarks

In this article we have touched upon several issues concerning quantum topology and showed their utility for research of current interest such as topology change and fuzzy topology. Our significant contribution, if any, here has been in formulating new fundamental problems with reasonable clarity. We have also sketched a few answers, but they are tentative and incomplete.

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References

[1] A.P.Balachandran in *Proceedings of the Workshop on Frontiers in Field Theory, Quantum Gravity and String Theory*,Puri,December 12-21 (1996) and quant-phys/9702055.

[2] A.P. Balachandran, G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano and P. Teotonio-Sobrinho, *Finite Quantum Physics and Noncommutative Geometry* [IC/94/38, DSF-T-2/94 SU-4240-567 (1994), hep-th/9403067], in *Proceedings of the XV Autumn School,“ Particle Physics in the 90's”* , eds. G. Branco and M. Pimenta, *Nucl. Phys. B* (*Proc. Suppl.*) **37C** (1995) 20; *J. Geom. Phys.* **18** (1996) 163 ; F.Lizzi, *Int.J.Mod.Phys.***A14**(1999),hep-th/9902187.

[3] A.P. Balachandran, G. Bimonte, G. Marmo and A. Simoni, *Nucl. Phys.* **B446** (1995) 299.

[4] A.P. Balachandran and L. Chandar, *Nucl. Phys.* **B428** (1994)435.

[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. I, *Functional Analysis* [Academic Press, 1972].

[6] A. Connes and C. Rovelli, *Class. Quant. Grav.* **11** (1994) 2899.

[7] J.M.G. Fell and R.S. Doran, *Representations of *-Algebras, Locally Compact Groups and Banach *-Algebraic Bundles* [Academic Press, 1988].

[8] A. Connes, *Noncommutative Geometry* [Academic Press, 1994].

[9] A. Connes and J. Lott, *Nucl. Phys.* **B** (Suppl.) **18** (1990) 29; J.C. Várilly and J.M. Gracia-Bondía, *J. Geom. Phys.* **12** (1993) 223.

[10] A. Connes,*Commun. Math. Phys.* **182**(1996) 155. There are several excellent accounts of Noncommutative Geometry besides refs. 7,8 and this paper. Two such are [11] and [12]. They also have references.

[11] G.Landi, *An Introduction to Noncommutative Spaces and their Geometry*, Springer-Verlag,Berlin,1977 and hep-th/9701078.

[12] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, London Mathematical Society Lecture Notes Series 206 [Cambridge University Press, 1995].

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. II:*Fourier Analysis; Self Adjointness* [Academic Press, 1975].

[14] A topological space T is Hausdorff iff there are open sets $\mathcal{O}(x), \mathcal{O}(y)$ containing any two points x, y in T such that $\mathcal{O}(x) \cap \mathcal{O}(y) = \emptyset$. See *Encyclopedic Dictionary of Mathematics*, edited by Shôkichi Iyanaga and Yukiyosi Kawada, translation reviewed by K.O. May [The M.I.T. Press, 1977].

[15] A. Connes and M.A. Rieffel, *Contemporary Mathematics* **62** (1987) 237.

[16] N.Seiberg and E.Witten, hep-th/9908142 and references therein.

[17] R.D. Sorkin, *Mod. Phys. Letts.* **A9** (1994) 3119.

[18] J. Hartle, *Phys. Rev.* **D51** (1995) 1800; M.D. Srinivas, *Pramāna* **47** (1996) 1.

[19] J. Fröhlich and K. Gawedski, CRM Proceedings and Lecture Notes, vol. 7 (1994) 57; J. Fröhlich, O. Grandjean and A. Recknagel, hep-th/9612205.

[20] D. Marolf, Phys. Lett. **B 392** (1997)287.

[21] J.L. Friedman and R.D. Sorkin, *Phys. Rev. Lett.* **44** (1990) 1100; ibid. **45** (1980) 148; *Gen. Rel. Grav.* **14** (1982) 615.

[22] R.D. Sorkin; *Introduction to Topological Geons*, in *Topological Properties and Global Structure of Space-Time*, eds. P.G.

Bergman and V. de Sabata (Plenum, 1986); *Classical Topology and Quantum Phases: Quantum Geons*, in *Geometrical and Algebraic Aspects of Nonlinear Field Theory*, eds. S. De Filippo, M. Marinaro, G. Marmo and G. Vilasi (North-Holland, 1989).

[23] A.P. Balachandran, A. Daughton, Z.-C. Gu, G. Marmo, R.D. Sorkin, and A.M. Srivastava, *Mod. Phys. Lett.* **A5** (1990) 1575 and *Int. J. Mod. Phys.* **A8** (1993) 2993; A.P. Balachandran, W.D. McGlinn, L. O'Raifeartaigh, S. Sen and R.D. Sorkin, *Int. J. Mod. Phys.* **A7** (1992) 6887 [Errata: *Int. J. Mod. Phys.* **A9** (1994) 1395]; A.P. Balachandran, W.D. McGlinn, L. O'Raifeartaigh, S. Sen, R.D. Sorkin and A.M. Srivastava, *Mod. Phys. Lett.* **A7** (1992) 1427.

[24] A. P. Balachandran,E.Batista,I.P.Costa y Silva and P. Teotonio-Sobrinho, *Quantum Topology Change in (2+1)d* hep-th/9905136 and Int.J.Mod.Phys.(in press); *The Spin-Statistics Connection in Quantum Gravity*, hep-th/9910127 and Nuclear Phys.(in press); *On the Spin-Statistics Connection for Geons in (2+1)- Gravity* and *Novel Spin-Statistics Theorem in (2+1)-Gravity* , Syracuse University preprints SU-4240-677 and 702 (00).

[25] A. P. Balachandran, G. Marmo, B. S. Skagerstam and A. Stern, *Classical Topology and Quantum States* [World Scientific, 1991].

[26] F. Dowker and R.D. Sorkin,*Class. Quantum Grav.***15** (1998)1153 , gr-qc/9609064 ; F.Dowker and S.Surya, *Phys.Rev.***D58** (1998)124019,gr-qc/9711070 ; A. Borde , H.F. Dowker, R.S. Garcia , R.D. Sorkin and S. Surya , *Class.Quant.Grav.***16**(1999)3457, gr-qc/9901063; H.F. Dowker, R.S. Garcia and S.Surya , *Class.Quant.Grav.***17**(2000)697, gr-qc/9910034.

[27] C.Aneziris, A.P.Balachandran,M.Bourdeau, S.Jo, T.R.Ramadas and R.D.Sorkin, *Mod.Phys.Lett.* **A4** (1989)331; *Int.J.Mod.Phys.* **A4** (1989)5459.

[28] See especially refs. [22] and [23] in this regard.

[29] L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, *Phys. Rev. Lett.* **59** (1987) 521; R.D. Sorkin, *Int. J. Theor. Phys.* **30** (1991) 923. See also U. Yurtsever, *Class.Quant. Grav.* **11** (1994) 1013.

[30] N.P. Landsman, *Int. J. Mod. Phys.* **A6** (1991) 5349.

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