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On different results for new three step iteration process in Banach spaces

Kifayat Ullah[*] and Muhammad Arshad

*Correspondence:
kifayatmath@yahoo.com
Department of Mathematics,
International Islamic
University, Sector H-10,
Islamabad 44000, Pakistan

Abstract

In this paper we propose a new iteration process, called *AK* iteration process, for approximation of fixed points for contraction mappings. We show that our iteration process is faster than the leading Vatan Two-step iteration process for contraction mappings. Numerical examples are given to support the analytic proofs. Stability of *AK* iteration process and data dependence result for contraction mappings by employing *AK* iteration process are also discussed.

Keywords: Banach space, Iteration process, Stability, Data dependence

Mathematics Subject Classification: Primary 47H09, Secondary 47H10

Introduction and preliminaries

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. However, once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task that is why we use iteration processes for computing them. By time, many iteration processes have been developed and it is impossible to cover them all. The well-known Banach contraction theorem use Picard iteration process for approximation of fixed point. Some of the other well-known iteration processes are Mann (Mann 1953), Ishikawa (Ishikawa 1974), Agarwal (Agarwal et al. 2007), Noor (Noor 2000), Abbas (Abbas and Nazir 2014), SP (Phuengrattana and Suantai 2011), S* (Karahan and Ozdemir 2013), CR (Chugh et al. 2012), Normal-S (Sahu and Petrusel 2011), Picard Mann (Khan 2013), Picard-S (Gursoy and Karakaya 2014), Thakur New (Thakur et al. 2016) and Vatan Two-step (Karakaya et al. 2015).

Fastness and stability play important role for an iteration process to be preferred on another iteration process. In 1991 , Rhoades mentioned that the Mann iteration process for decreasing function converge faster than the Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann iteration process. Also the Mann iteration process appears to be independent of the initial guess (see also Rhoades 1977). In Agarwal et al. (2007), the authors claimed that Agarwal iteration process converges at a rate same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings. In Abbas and Nazir (2014),

the authors claimed that Abbas iteration process converge faster than Agarwal iteration process. In Chugh et al. (2012), the authors claimed that CR iteration process is equivalent to and faster than Picard, Mann, Ishikawa, Agarwal, Noor and SP iteration processes for quasi-contractive operators in Banach spaces. Also in Karakaya et al. (2014) the authors proved that CR iteration process converge faster than the S^* iterative process for the class of contraction mappings. In Gursoy and Karakaya (2014), authors claimed that Picard-S iteration process converge faster than all Picard, Mann, Ishikawa, Noor, SP, CR, Agarwal, S^*, Abbas and Normal-S iteration processes for contraction mappings. In Thakur et al. (2016), the authors proved with the help of numerical example that Thakur New iteration process converge faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes for the class of Suzuki generalized nonexpansive mappings. Similarly, in Karakaya et al. (2015), the authors proved that Vatan Two-step iteration process is faster than Picard-S, CR, SP and Picard-Mann iteration processes for weak contraction mappings. For fragmentation models and processes see Goufo (2014), Goufo and Noutchie (2013). Similarly, for local convergence of Chebyshev–Halley methods with six and eight order of convergence to approximate a locally unique solution of a nonlinear equation see Magrenan and Argyros (2016).

Motivated by above, in this paper, we introduce a new iteration process known as *AK* iteration process and prove analytically that our process is stable. Then we prove that *AK* iteration process converges faster than Vatan Two-step iteration process which is faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, S^*, Abbas, Normal-S and Two-step Mann iteration processes for contraction mappings. Numerically we compare the convergence of the *AK* iteration process with the three most leading iteration processes in the existing literature for contraction mappings. The data dependence result for fixed point of contraction mappings by employing *AK* iteration process is also proved.

We now recall some definitions, propositions and lemmas to be used in the next two sections.

A point p is called fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of a mapping T. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called contraction if there exists $\theta \in (0, 1)$ such that $\left\| Tx - Ty \right\| \leq \theta \left\| x - y \right\|$, for all $x, y \in C$.

Definition 1 (Berinde 2007) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two real convergent sequences with limits a and b, respectively. Then we say that $\{a_n\}_{n=0}^{\infty}$ converge faster than $\{b_n\}_{n=0}^{\infty}$ if

$$\lim_{n \to \infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0.$$

Definition 2 (Berinde 2007) Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same fixed point p. If $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$, for all $n \geq 0$, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). Then we say that $\{u_n\}_{n=0}^{\infty}$ converge faster than $\{v_n\}_{n=0}^{\infty}$ to p if $\{a_n\}_{n=0}^{\infty}$ converge faster than $\{b_n\}_{n=0}^{\infty}$.

Definition 3 (Harder [1987]) Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be *T*-stable or stable with respect to *T*, if for $\epsilon_n = \left\| t_{n+1} - f(T, t_n) \right\|, n = 0, 1, 2, 3, \ldots$, we have

$$\lim_{n \to \infty} \epsilon_n = 0 \iff \lim_{n \to \infty} t_n = p.$$

Definition 4 (Berinde [2007]) Let $T, \widetilde{T} : C \to C$ be two operators. We say that $\widetilde{T}$ is an approximate operator for *T* if, for some $\varepsilon > 0$, we have

$$\left\| Tx - \widetilde{T}x \right\| \le \varepsilon,$$

for all $x \in C$.

Lemma 1 (Weng [1991]) *Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\psi_{n+1} \le (1 - \phi_n)\psi_n + \varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$, then $\lim_{n \to \infty} \psi_n = 0$.

Lemma 2 (Soltuz and Grosan [2008]) *Let $\{\psi_n\}_{n=0}^{\infty}$ be nonnegative real sequence for which one assumes there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, the following inequality satisfies:*

$$\psi_{n+1} \le (1 - \phi_n)\psi_n + \phi_n\varphi_n,$$

where $\phi_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\varphi_n \ge 0$ for all $n \in \mathbb{N}$, then

$$0 \le \limsup_{n \to \infty} \psi_n \le \limsup_{n \to \infty} \varphi_n.$$

AK iteration process and its convergence analysis

Throughout this section we have $n \ge 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$, C is any subset of Banach space X and $T : C \to C$ is any mapping.

Gursoy and Karakaya ([2014]) introduced new iteration process called Picard-S iteration process, as follow:

$$\begin{cases} u_0 \in C \\ w_n = (1 - \beta_n)u_n + \beta_n Tu_n \\ v_n = (1 - \alpha_n)Tu_n + \alpha_n Tw_n \\ u_{n+1} = Tv_n. \end{cases} \tag{1}$$

They proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. Also, by providing a numerical example, it is shown that the Picard-S iteration process converge faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas, Normal-S and Two-step Mann iteration process.

After this Karakaya et al. (2015) introduced a new two step iteration process, we will call it Vatan Two-step iteration process, with the claim that it is even faster than Picard-S iteration process, as follow:

$$\begin{cases} u_0 \in C \\ v_n = T((1 - \beta_n)u_n + \beta_n Tu_n) \\ u_{n+1} = T((1 - \alpha_n)v_n + \alpha_n Tv_n). \end{cases} \tag{2}$$

Recently Thakur et al. (2016) used a new iteration process for approximation of fixed points, defined by:

$$\begin{cases} u_0 \in C \\ w_n = (1 - \beta_n)u_n + \beta_n Tu_n \\ v_n = T((1 - \alpha_n)u_n + \alpha_n w_n) \\ u_{n+1} = Tv_n. \end{cases} \tag{3}$$

With the help of numerical example, they proved that their new iteration process is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes for some class of mappings. we will call it Thakur New iteration process.

Problem 1 *Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes* (1), (2) *and* (3)?

To answer this, we introduce the following new iteration process (4), known as *AK* Iteration Process:

$$\begin{cases} x_0 \in C \\ z_n = T((1 - \beta_n)x_n + \beta_n Tx_n) \\ y_n = T((1 - \alpha_n)z_n + \alpha_n Tz_n) \\ x_{n+1} = Ty_n. \end{cases} \tag{4}$$

We have to prove that our new iteration process (4) is stable and have a good speed of convergence comparatively to other iteration processes.

Theorem 1 *Let C be a nonempty closed convex subset of a Banach space X and T : C → C be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (4) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in* [0, 1] *satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converge strongly to a unique fixed point of T.*

Proof The well-known Banach theorem guarantees the existence and uniqueness of fixed point p. We will show that $x_n \to p$ for $n \to \infty$. From (4) we have

$$\begin{aligned} \|z_n - p\| &= \|T((1 - \beta_n)x_n + \beta_n Tx_n) - p\| \\ &\leq \theta \|(1 - \beta_n)x_n + \beta_n Tx_n - (1 - \beta_n + \beta_n)p\| \\ &\leq \theta((1 - \beta_n)\|x_n - p\| + \beta_n \|Tx_n - Tp\|) \\ &\leq \theta((1 - \beta_n)\|x_n - p\| + \beta_n \theta \|x_n - p\|) \\ &= \theta(1 - \beta_n(1 - \theta))\|x_n - p\|. \end{aligned} \tag{5}$$

Similarly,

$$
\begin{aligned}
\left\|y_n - p\right\| &= \|T((1 - \alpha_n)z_n + \alpha_n T z_n) - Tp\| \\
&\leq \theta \|(1 - \alpha_n)z_n + \alpha_n T z_n - p\| \\
&\leq \theta[(1 - \alpha_n)\|z_n - p\| + \alpha_n \|T z_n - p\|] \\
&\leq \theta[(1 - \alpha_n)\|z_n - p\| + \alpha_n \theta \|z_n - p\|] \\
&\leq \theta(1 - \alpha_n(1 - \theta))\|z_n - p\| \\
&\leq \theta^2(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\|x_n - p\|.
\end{aligned}
\tag{6}
$$

Hence

$$
\begin{aligned}
\|x_{n+1} - p\| &= \left\|T y_n - p\right\| \\
&\leq \theta \left\|y_n - p\right\| \\
&\leq \theta^3(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\|x_n - p\| \\
&\leq \theta^3(1 - \alpha_n(1 - \theta))\|x_n - p\|,
\end{aligned}
\tag{7}
$$

by using the fact that $(1 - \beta_n(1 - \theta)) < 1$, for $\theta \in (0, 1)$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$.

From (7) we have the following inequalities:

$$
\begin{cases}
\|x_{n+1} - p\| \leq \theta^3(1 - \alpha_n(1 - \theta))\|x_n - p\|; \\
\|x_n - p\| \leq \theta^3(1 - \alpha_{n-1}(1 - \theta))\|x_{n-1} - p\|; \\
\|x_{n-1} - p\| \leq \theta^3(1 - \alpha_{n-2}(1 - \theta))\|x_{n-2} - p\|; \\
\vdots \\
\vdots \\
\|x_1 - p\| \leq \theta^3(1 - \alpha_0(1 - \theta))\|x_0 - p\|.
\end{cases}
\tag{8}
$$

From (8) we can easily derive

$$
\|x_{n+1} - p\| \leq \|x_0 - p\| \theta^{3(n+1)} \prod_{k=0}^{n}(1 - \alpha_k(1 - \theta)),
\tag{9}
$$

where $1 - \alpha_k(1 - \theta) \in (0, 1)$ because $\theta \in (0, 1)$ and $\alpha_n \in [0, 1]$, for all $n \in \mathbb{N}$. Since we know that $1 - x \leq e^{-x}$ for all $x \in [0, 1]$, so from (9) we get

$$
\|x_{n+1} - p\| \leq \frac{\|x_0 - p\| \theta^{3(n+1)}}{e^{(1-\theta)\sum_{k=0}^{n} \alpha_k}}.
\tag{10}
$$

Taking the limit of both sides of inequality (10) yields $\lim_{n \to \infty} \|x_n - p\| = 0$, i.e. $x_n \to p$ for $n \to \infty$, as required. $\qquad\square$

Theorem 2 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (4) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the iteration process (4) is T-stable.*

Proof Let $\{t_n\}_{n=0}^{\infty} \subset X$ be any arbitrary sequence in C. Let the sequence generated by (4) is $x_{n+1} = f(T, x_n)$ converging to unique fixed point p (by Theorem 1) and $\epsilon_n = \left\|t_{n+1} - f(T, t_n)\right\|$. We will prove that $\lim_{n \to \infty} \epsilon_n = 0 \Longleftrightarrow \lim_{n \to \infty} t_n = p$.

Let $\lim_{n \to \infty} \epsilon_n = 0$. By using (7) we get

$$\|t_{n+1} - p\| \le \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\|$$

$$= \epsilon_n + \left\| \begin{array}{l} T(T((1 - \alpha_n)T((1 - \beta_n)t_n + \beta_n Tt_n) \\ + \alpha_n T(T((1 - \beta_n)t_n + \beta_n Tt_n)))) - p \end{array} \right\|$$

$$\le \theta^3 (1 - \alpha_n(1 - \theta))\|t_n - p\| + \epsilon_n.$$

Define $\psi_n = \|t_n - p\|$, $\phi_n = \alpha_n(1 - \theta) \in (0, 1)$ and $\varphi_n = \epsilon_n$. Since $\lim_{n \to \infty} \epsilon_n = 0$, which implies that $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all conditions of Lemma 1 are fulfilled by above inequality. Hence by Lemma 1 we get $\lim_{n \to \infty} t_n = p$.

Conversely let $\lim_{n \to \infty} t_n = p$, we have

$$\epsilon_n = \|t_{n+1} - f(T, t_n)\|$$

$$\le \|t_{n+1} - p\| + \|f(T, t_n) - p\|$$

$$\le \|t_{n+1} - p\| + \theta^3 (1 - \alpha_n(1 - \theta))\|t_n - p\|.$$

This implies that $\lim_{n \to \infty} \epsilon_n = 0$.

Hence (4) is stable with respect to T. □

Theorem 3 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with fixed point p. Let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequences generated by (2) and (4) respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in [0, 1] satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the following are equivalent:*

(i) the AK iteration process (4) converges to the fixed point p of T;
(ii) the Vatan two-step iteration process (2) converges to the fixed point p of T.

Proof First we prove $(i) \Longrightarrow (ii)$. Let the iteration method (4) converges to the fixed point p of T i.e. $\lim_{n \to \infty} \|x_n - p\| = 0$. Now using (2) and (4) we have

$$\|z_n - v_n\| = \|T((1 - \beta_n)x_n + \beta_n Tx_n) - T((1 - \beta_n)u_n + \beta_n Tu_n)\|$$

$$\le \theta\{\|(1 - \beta_n)x_n + \beta_n Tx_n - (1 - \beta_n)u_n - \beta_n Tu_n\|\}$$

$$\le \theta\{(1 - \beta_n)\|x_n - u_n\| + \beta_n\|Tx_n - Tu_n\|\}$$

$$\le \theta\{(1 - \beta_n)\|x_n - u_n\| + \theta\beta_n\|x_n - u_n\|\}$$

$$= \theta(1 - \beta_n(1 - \theta))\|x_n - u_n\|. \tag{11}$$

Similarly, using (2) and (4) together with (11) we have

$$\|x_{n+1} - u_{n+1}\| = \|Ty_n - u_{n+1}\|$$

$$\le \|Ty_n - y_n\| + \|y_n - u_{n+1}\|$$

$$= \|T((1 - \alpha_n)z_n + \alpha_n Tz_n) - T((1 - \alpha_n)v_n + \alpha_n Tv_n)\|$$

$$+ \|Ty_n - y_n\|$$

$$\le \theta\|(1 - \alpha_n)z_n + \alpha_n Tz_n - (1 - \alpha_n)v_n - \alpha_n Tv_n\|$$

$$+ \|Ty_n - y_n\|$$

$$\le \theta\{(1 - \alpha_n)\|z_n - v_n\| + \alpha_n\|Tz_n - Tv_n\|\} + \|Ty_n - y_n\|$$

$$\le \theta\{(1 - \alpha_n)\|z_n - v_n\| + \theta\alpha_n\|z_n - v_n\|\} + \|Ty_n - y_n\|$$

$$\le \theta(1 - \alpha_n(1 - \theta))\|z_n - v_n\| + \|Ty_n - y_n\|$$

$$\le \theta^2 (1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\|x_n - u_n\|$$

$$+ \|Ty_n - y_n\|. \tag{12}$$

For $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ and $\theta \in (0, 1)$, we have

$$(1 - \beta_n(1 - \theta)) < 1. \tag{13}$$

By using (13) together with (12) we get

$$\|x_{n+1} - u_{n+1}\| \leq (1 - \alpha_n(1 - \theta))\|x_n - u_n\| + \|Ty_n - y_n\|. \tag{14}$$

Define $\psi_n = \|x_n - u_n\|, \phi_n = \alpha_n(1 - \theta) \in (0, 1)$ and $\varphi_n = \|Ty_n - y_n\|$.

Since $\lim_{n \to \infty} \|x_n - p\| = 0$ and $Tp = p$ so

$$\lim_{n \to \infty} \|Ty_n - y_n\| = \lim_{n \to \infty} \|Ty_n - Tp + p - y_n\|$$
$$\leq (1 + \theta) \lim_{n \to \infty} \|y_n - p\|$$
$$= 0,$$

which implies that $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all conditions of Lemma 1 are fulfilled by (14), so we get

$$\lim_{n \to \infty} \psi_n = \lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{15}$$

Using (15) we get $\|u_n - p\| \leq \|x_n - u_n\| + \|x_n - p\| \to 0$ as $n \to \infty$. Hence $\lim_{n \to \infty} \|u_n - p\| = 0$ i.e. the Vatan two-step iteration process (2) converges to the fixed point p of T.

Next we will prove $(ii) \implies (i)$. Let $\lim_{n \to \infty} \|u_n - p\| = 0$.

By using (2) and (4) we have

$$\|v_n - z_n\| = \|T((1 - \beta_n)u_n + \beta_n Tu_n) - T((1 - \beta_n)x_n + \beta_n Tx_n)\|$$
$$\leq \theta\{\|(1 - \beta_n)u_n + \beta_n Tu_n - (1 - \beta_n)x_n - \beta_n Tx_n\|\}$$
$$\leq \theta\{(1 - \beta_n)\|u_n - x_n\| + \beta_n\|Tu_n - Tx_n\|\}$$
$$\leq \theta\{(1 - \beta_n)\|u_n - x_n\| + \theta\beta_n\|u_n - x_n\|\}$$
$$= \theta(1 - \beta_n(1 - \theta))\|u_n - x_n\|. \tag{16}$$

Similarly, using (2) and (4) together with (16) we have

$$\|u_{n+1} - x_{n+1}\| = \|T((1 - \alpha_n)v_n + \alpha_n Tv_n) - Ty_n\|$$
$$\leq \theta\|(1 - \alpha_n)v_n + \alpha_n Tv_n - y_n\|$$
$$= \theta\|(1 - \alpha_n)v_n + \alpha_n Tv_n - (1 - \alpha_n + \alpha_n)y_n\|$$
$$\leq \theta(1 - \alpha_n)\|v_n - y_n\| + \theta\alpha_n\|Tv_n - y_n\|$$
$$\leq \theta(1 - \alpha_n)\|v_n - Tv_n\| + \theta(1 - \alpha_n)\|Tv_n - y_n\|$$
$$\quad + \theta\alpha_n\|Tv_n - y_n\|$$
$$= \theta\|Tv_n - y_n\| + \theta(1 - \alpha_n)\|v_n - Tv_n\|$$
$$= \theta\|Tv_n - T((1 - \alpha_n)z_n + \alpha_n Tz_n)\| + \theta(1 - \alpha_n)\|v_n - Tv_n\|$$
$$\leq \theta^2\{(1 - \alpha_n)\|v_n - z_n\| + \alpha_n\|v_n - Tz_n\|\}$$
$$\quad + \theta(1 - \alpha_n)\|v_n - Tv_n\|$$
$$\leq \theta^2\{(1 - \alpha_n)\|v_n - z_n\| + \alpha_n\|v_n - Tv_n\| + \alpha_n\|Tv_n - Tz_n\|\}$$
$$\quad + \theta(1 - \alpha_n)\|v_n - Tv_n\|$$
$$\leq \theta^2\{(1 - \alpha_n)\|v_n - z_n\| + \theta\alpha_n\|v_n - z_n\|\}$$
$$\quad + \theta^2\alpha_n\|v_n - Tv_n\| + \theta(1 - \alpha_n)\|v_n - Tv_n\|$$
$$= \theta^2(1 - \alpha_n(1 - \theta))\|v_n - z_n\| + \theta(1 - \alpha_n(1 - \theta))\|v_n - Tv_n\|$$
$$\leq \theta^3(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\|u_n - x_n\|$$
$$\quad + \theta(1 - \alpha_n(1 - \theta))\|v_n - Tv_n\|. \tag{17}$$

By using (12) together with (17) we get

$$\|u_{n+1} - x_{n+1}\| \le (1 - \alpha_n(1 - \theta))\|u_n - x_n\| + (1 - \alpha_n(1 - \theta))\|v_n - Tv_n\|. \qquad (18)$$

Define $\psi_n = \|u_n - x_n\|, \phi_n = \alpha_n(1 - \theta) \in (0, 1)$ and $\varphi_n = (1 - \alpha_n(1 - \theta))\|v_n - Tv_n\|$.

Since $\lim_{n \to \infty} \|u_n - p\| = 0$ and $Tp = p$, so

$$\begin{aligned}
\lim_{n \to \infty} \|Tv_n - v_n\| &= \lim_{n \to \infty} \|Tv_n - Tp + p - v_n\| \\
&\le (1 + \theta) \lim_{n \to \infty} \|v_n - p\| \\
&= 0.
\end{aligned} \qquad (19)$$

From (19) we have $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all conditions of Lemma 1 are fulfilled by (18), and so

$$\lim_{n \to \infty} \psi_n = \lim_{n \to \infty} \|u_n - x_n\| = 0. \qquad (20)$$

Using (20) we get $\|x_n - p\| \le \|u_n - x_n\| + \|u_n - p\| \to 0$ as $n \to \infty$. Hence $\lim_{n \to \infty} \|x_n - p\| = 0$ i.e. the *AK* iteration process (4) converges to the fixed point p of T. □

Theorem 4 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping with fixed point p. For given $u_0 = x_0 \in C$, let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequences generated by (2) and (4) respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ such that $\alpha \le \alpha_n < 1$, for some $\alpha > 0$ and for all $n \in \mathbb{N}$. Then $\{x_n\}_{n=0}^{\infty}$ converge to p faster than $\{u_n\}_{n=0}^{\infty}$ does.*

Proof From (9) we have

$$\|x_{n+1} - p\| \le \|x_0 - p\|\theta^{3(n+1)} \prod_{k=0}^{n} (1 - \alpha_k(1 - \theta)). \qquad (21)$$

From iteration process (2), also converging to unique fixed point p (Karakaya et al. 2015, Theorem 2) , we have

$$\begin{aligned}
\|u_{n+1} - p\| &= \|T((1 - \alpha_n)v_n + \alpha_n Tv_n) - p\| \\
&\le \theta(1 - \alpha_n(1 - \theta))\|v_n - p\| \\
&= \theta(1 - \alpha_n(1 - \theta))\|T((1 - \beta_n)u_n + \beta_n Tu_n) - p\| \\
&\le \theta^2(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\|u_n - p\| \\
&\le \theta^2(1 - \alpha_n(1 - \theta))\|u_n - p\|.
\end{aligned}$$

Repeating this process n times, we get

$$\|u_{n+1} - p\| \le \|u_0 - p\|\theta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k(1 - \theta)). \qquad (22)$$

Since $\alpha \le \alpha_n < 1$ for some $\alpha > 0$ and for all $n \in \mathbb{N}$, so (21) implies that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \|x_0 - p\|\theta^{3(n+1)} \prod_{k=0}^{n} (1 - \alpha(1 - \theta)) \\
&= \|x_0 - p\|\theta^{3(n+1)}(1 - \alpha(1 - \theta))^{n+1}.
\end{aligned} \qquad (23)$$

Similarly, (22) together with assumption $\alpha \leq \alpha_n < 1$, for some $\alpha > 0$ and for all $n \in \mathbb{N}$, leads to

$$\|u_{n+1} - p\| \leq \|u_0 - p\|\theta^{2(n+1)} \prod_{k=0}^{n}(1 - \alpha(1 - \theta))$$
$$= \|u_0 - p\|\theta^{2(n+1)}(1 - \alpha(1 - \theta))^{n+1}. \tag{24}$$

Define

$$a_n = \|x_0 - p\|\theta^{3(n+1)}(1 - \alpha(1 - \theta))^{n+1},$$

and

$$b_n = \|u_0 - p\|\theta^{2(n+1)}(1 - \alpha(1 - \theta))^{n+1},$$

then

$$\Psi_n = \frac{a_n}{b_n}$$
$$= \frac{\|x_0 - p\|\theta^{3(n+1)}(1 - \alpha(1 - \theta))^{n+1}}{\|u_0 - p\|\theta^{2(n+1)}(1 - \alpha(1 - \theta))^{n+1}} \tag{25}$$
$$= \theta^{n+1}.$$

Since $\lim_{n \to \infty} \frac{\Psi_{n+1}}{\Psi_n} = \lim_{n \to \infty} \frac{\theta^{n+2}}{\theta^{n+1}} = \theta < 1$, so by ratio test $\sum_{n=0}^{\infty} \Psi_n < \infty$. Hence from (25) we have,

$$\lim_{n \to \infty} \frac{\|x_{n+1} - p\|}{\|u_{n+1} - p\|} = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \Psi_n = 0,$$

which implies that $\{x_n\}_{n=0}^{\infty}$ is faster than $\{u_n\}_{n=0}^{\infty}$. □

Following are some numerical examples to support analytical proof of Theorem 4 and to illustrate the efficiency of *AK* iteration process (4).
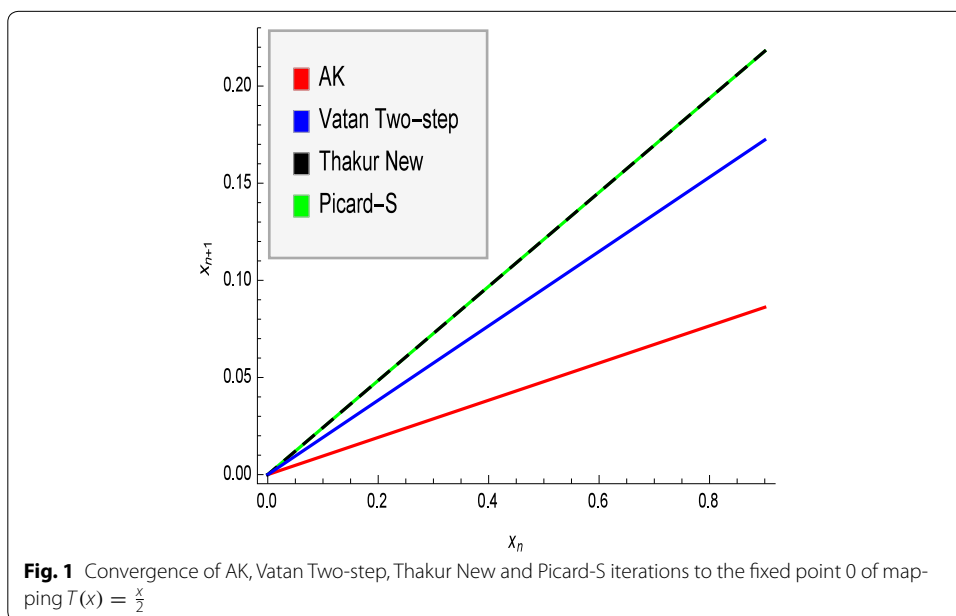
Example 1 Let $T : [0, 1] \to [0, 1]$ defined by $T(x) = \frac{x}{2}$, be any mapping. It is easy to see that T is a contraction mapping. Hence T has a unique fixed point 0.

In Table 1, iterative values generated by our new *AK*, Vatan Two-step, Thakur New and Picard-S iteration processes are given, where $x_0 = u_0 = 0.9$, $\alpha_n = \beta_n = \frac{1}{4}$ for all n and $n = \overline{0, 9}$. Graphic representation is given in Fig. 1, where sequence of each iteration process for graph is represented by x_n. We can easily see that the new *AK* iterations are the first converging one than the Vatan Two-step, the Thakur New and the Picard-S iterations.

Example 2 Define a mapping $T : [0, 4] \to [0, 4]$ by $T(x) = (x + 2)^{\frac{1}{3}}$. Clearly T is a contraction mapping. Hence T has a unique fixed point. In Table 2, iterative values generated by our new *AK*, Vatan Two-step, Thakur New and Picard-S iteration processes are given, where $x_0 = u_0 = 1.99$, $\alpha_n = \beta_n = \frac{1}{4}$ for all n and $n = \overline{0, 10}$.

Table 1 Iterative values of AK, Vatan Two-step, Thakur New and Picard-S iteration processes for mapping $T(x) = \frac{x}{2}$, where $\alpha_n = \beta_n = \frac{1}{4}$, for all n

	AK	Vatan Two-step	Thakur New	Picard-S
x_0	0.9	0.9	0.9	0.9
x_1	8.6133×10^{-2}	1.7227×10^{-1}	2.1797×10^{-1}	2.1797×10^{-1}
x_2	8.2432×10^{-3}	3.2973×10^{-2}	5.2789×10^{-2}	5.2789×10^{-2}
x_3	7.889×10^{-4}	6.3112×10^{-3}	1.2785×10^{-2}	1.2785×10^{-2}
x_4	7.55×10^{-5}	1.208×10^{-3}	3.0963×10^{-3}	3.0963×10^{-3}
x_5	7.2256×10^{-6}	2.3122×10^{-4}	7.499×10^{-4}	7.499×10^{-4}
x_6	6.9151×10^{-7}	4.4257×10^{-5}	1.8162×10^{-4}	1.8162×10^{-4}
x_7	6.618×10^{-8}	8.471×10^{-6}	4.3985×10^{-5}	4.3985×10^{-5}
x_8	6.3336×10^{-9}	1.6214×10^{-6}	1.0653×10^{-5}	1.0653×10^{-5}
x_9	6.0615×10^{-10}	3.1035×10^{-7}	2.5799×10^{-6}	2.5799×10^{-6}
x_{10}	5.801×10^{-11}	5.9402×10^{-8}	6.2483×10^{-7}	6.2483×10^{-7}

Fig. 1 Convergence of AK, Vatan Two-step, Thakur New and Picard-S iterations to the fixed point 0 of mapping $T(x) = \frac{x}{2}$

We can easily see that our new *AK* iterations are the first converging one than the Vatan Two-step, the Thakur New and the Picard-S iterations.
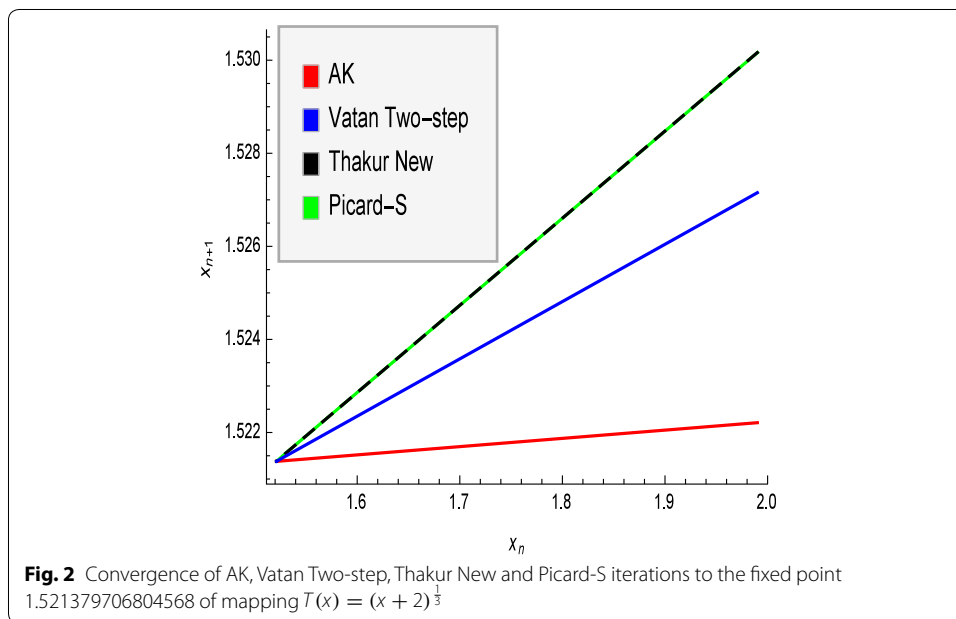
Graphic representation is given in Fig. 2, where sequence of each iteration process is represented by x_n.

Data dependence result

Fixed point theory is concerned with investigating a wide variety of issues such as the existence (and uniqueness) of fixed points, the construction of fixed points, etc. One of these themes is data dependency of fixed points. Data dependency of fixed points has been the subject of research in fixed point theory for some time now, and data dependence research is an important theme in its own right. Some work in this direction are Rus and Muresan (1998), Rus et al. (2001, 2003), Berinde (2003), Espínola and Petrusel (2005), Markin (1973), Chifu and Petrusel (2007), Olatinwo (2009, 2010), Soltuz (2001,

Table 2 Iterative values of AK, Vatan Two-step, Thakur New and Picard-S iteration processes for $\alpha_n = \beta_n = \frac{1}{4}$, for all n and mapping $T(x) = (x+2)^{\frac{1}{3}}$

	AK	Vatan Two-step	Thakur New	Picard-S
x_0	1.99	1.99	1.99	1.99
x_1	1.522210596157901	1.527152378405542	1.530163443560674	1.530160376515624
x_2	1.521381239904628	1.521453635507796	1.521551978236029	1.521551916843118
x_3	1.521379709633547	1.521380654057891	1.521383088492668	1.521383087287047
x_4	1.521379706809788	1.521379718941864	1.521379773188262	1.521379773164595
x_5	1.521379706804577	1.521379706960085	1.521379708107703	1.521379708107238
x_6	1.521379706804568	1.521379706806560	1.521379706830149	1.521379706830139
x_7	1.521379706804568	1.521379706804593	1.521379706805070	1.521379706805069
x_8	1.521379706804568	1.521379706804568	1.521379706804577	1.521379706804577
x_9	1.521379706804568	1.521379706804568	1.521379706804568	1.521379706804568
x_{10}	1.521379706804568	1.521379706804568	1.521379706804568	1.521379706804568
x_{11}	1.521379706804568	1.521379706804568	1.521379706804568	1.521379706804568

Fig. 2 Convergence of AK, Vatan Two-step, Thakur New and Picard-S iterations to the fixed point 1.521379706804568 of mapping $T(x) = (x+2)^{\frac{1}{3}}$

2004), Soltuz and Grosan (2008), Chugh and Kumar (2011) and the references therein. In this section, we establish the following data dependence result.

Theorem 5 *Let $\widetilde{T}$ be an approximate operator of a contraction mapping T. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (4) for T and define an iterative sequence $\{\widetilde{x}_n\}_{n=0}^{\infty}$ as follows*

$$\begin{cases} \widetilde{x}_0 \in C \\ \widetilde{z}_n = \widetilde{T}((1-\beta_n)\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{x}_n) \\ \widetilde{y}_n = \widetilde{T}((1-\alpha_n)\widetilde{z}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \\ \widetilde{x}_{n+1} = \widetilde{T}\widetilde{y}_n, \end{cases} \tag{26}$$

with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying (i). $\frac{1}{2} \leq \alpha_n$, for all $n \in \mathbb{N}$, and (ii). $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $Tp = p$ and $\widetilde{T}\widetilde{p} = \widetilde{p}$ such that $\lim_{n \to \infty} \widetilde{x}_n = \widetilde{p}$, then we have

$$\left\| p - \widetilde{p} \right\| \leq \frac{9\varepsilon}{1 - \theta},$$

where $\varepsilon > 0$ is a fixed number.

Proof It follows from (4) and (26) that

$$
\begin{aligned}
\left\| z_n - \widetilde{z}_n \right\| &= \left\| T((1 - \beta_n)x_n + \beta_n Tx_n) - \widetilde{T}((1 - \beta_n)\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{x}_n) \right\| \\
&\leq \left\| T((1 - \beta_n)x_n + \beta_n Tx_n) - T((1 - \beta_n)\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{x}_n) \right\| \\
&\quad + \left\| T((1 - \beta_n)\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{x}_n) - \widetilde{T}((1 - \beta_n)\widetilde{x}_n + \beta_n \widetilde{T}\widetilde{x}_n) \right\| \\
&\leq \theta \left((1 - \beta_n)\left\| x_n - \widetilde{x}_n \right\| + \beta_n \left\| Tx_n - \widetilde{T}\widetilde{x}_n \right\| \right) + \varepsilon \\
&\leq \theta \left((1 - \beta_n)\left\| x_n - \widetilde{x}_n \right\| + \beta_n \left(\left\| Tx_n - T\widetilde{x}_n \right\| + \left\| T\widetilde{x}_n - \widetilde{T}\widetilde{x}_n \right\| \right) \right) \\
&\quad + \varepsilon \\
&\leq \theta(1 - \beta_n(1 - \theta))\left\| x_n - \widetilde{x}_n \right\| + \theta\beta_n\varepsilon + \varepsilon.
\end{aligned}
\tag{27}
$$

Using (27), we have

$$
\begin{aligned}
\left\| y_n - \widetilde{y}_n \right\| &= \left\| T((1 - \alpha_n)z_n + \alpha_n Tz_n) - \widetilde{T}((1 - \alpha_n)\widetilde{z}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \right\| \\
&\leq \left\| T((1 - \alpha_n)z_n + \alpha_n Tz_n) - T((1 - \alpha_n)\widetilde{z}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \right\| \\
&\quad + \left\| T((1 - \alpha_n)\widetilde{z}_n + \alpha_n \widetilde{T}\widetilde{z}_n) - \widetilde{T}((1 - \alpha_n)\widetilde{z}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \right\| \\
&\leq \theta \left(\left\| (1 - \alpha_n)z_n + \alpha_n Tz_n - (1 - \alpha_n)\widetilde{z}_n - \alpha_n \widetilde{T}\widetilde{z}_n \right\| \right) + \varepsilon \\
&\leq \theta \left((1 - \alpha_n)\left\| z_n - \widetilde{z}_n \right\| + \alpha_n \left\| Tz_n - \widetilde{T}\widetilde{z}_n \right\| \right) + \varepsilon \\
&\leq \theta \left((1 - \alpha_n)\left\| z_n - \widetilde{z}_n \right\| + \alpha_n \left(\begin{array}{c} \left\| Tz_n - T\widetilde{z}_n \right\| \\ + \left\| T\widetilde{z}_n - \widetilde{T}\widetilde{z}_n \right\| \end{array} \right) \right) + \varepsilon \\
&\leq \theta \left((1 - \alpha_n)\left\| z_n - \widetilde{z}_n \right\| + \alpha_n \left(\theta \left\| z_n - \widetilde{z}_n \right\| + \varepsilon \right) \right) + \varepsilon \\
&= \theta \left((1 - \alpha_n(1 - \theta))\left\| z_n - \widetilde{z}_n \right\| + \alpha_n\varepsilon \right) + \varepsilon \\
&\leq \theta \left((1 - \alpha_n(1 - \theta)) \left(\begin{array}{c} \theta(1 - \beta_n(1 - \theta))\left\| x_n - \widetilde{x}_n \right\| \\ + \theta\beta_n\varepsilon + \varepsilon \end{array} \right) + \alpha_n\varepsilon \right) \\
&\quad + \varepsilon \\
&= \theta^2(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\left\| x_n - \widetilde{x}_n \right\| + \theta^2\beta_n\varepsilon \\
&\quad - \theta^2\alpha_n\beta_n\varepsilon + \theta^3\alpha_n\beta_n\varepsilon + \theta\varepsilon - \theta\alpha_n\varepsilon + \theta^2\alpha_n\varepsilon + \theta\alpha_n\varepsilon + \varepsilon \\
&= \theta^2(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))\left\| x_n - \widetilde{x}_n \right\| \\
&\quad + \theta^2\beta_n\varepsilon + \theta^2\alpha_n\beta_n\varepsilon(\theta - 1) + \theta\varepsilon + \theta^2\alpha_n\varepsilon + \varepsilon
\end{aligned}
\tag{28}
$$

Similarly, using (28), we have

$$
\begin{aligned}
\left\| x_{n+1} - \tilde{x}_{n+1} \right\| &= \left\| Ty_n - \tilde{T}\tilde{y}_n \right\| \\
&\leq \theta \left\| y_n - \tilde{y}_n \right\| + \varepsilon \\
&\leq \theta^3 (1 - \alpha_n(1-\theta))(1 - \beta_n(1-\theta)) \left\| x_n - \tilde{x}_n \right\| + \theta^3 \beta_n \varepsilon \\
&\quad + \theta^3 \alpha_n \beta_n \varepsilon (\theta - 1) + \theta^2 \varepsilon + \theta^3 \alpha_n \varepsilon + \theta \varepsilon + \varepsilon.
\end{aligned}
\tag{29}
$$

For $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ and $\theta \in (0, 1)$, we have

$$
\begin{cases}
(1 - \beta_n(1-\theta)) < 1 \\
\theta^2, \theta^3 < 1 \\
\theta - 1 < 0 \\
\theta^3 \beta_n < 1,
\end{cases}
\tag{30}
$$

and it follows from assumption (*i*) that

$$
1 - \alpha_n < \alpha_n
\tag{31}
$$

Using (30) and (31) together with (29), we get

$$
\begin{aligned}
\left\| x_{n+1} - \tilde{x}_{n+1} \right\| &\leq (1 - \alpha_n(1-\theta)) \left\| x_n - \tilde{x}_n \right\| + \alpha_n \varepsilon + 4\varepsilon \\
&= (1 - \alpha_n(1-\theta)) \left\| x_n - \tilde{x}_n \right\| + \alpha_n \varepsilon + 4(1 - \alpha_n + \alpha_n)\varepsilon \\
&\leq (1 - \alpha_n(1-\theta)) \left\| x_n - \tilde{x}_n \right\| + \alpha_n(1-\theta)\frac{9\varepsilon}{1-\theta}.
\end{aligned}
\tag{32}
$$

Let $\psi_n = \left\| x_n - \tilde{x}_n \right\|$, $\phi_n = \alpha_n(1-\theta)$, $\varphi_n = \frac{9\varepsilon}{1-\theta}$, then from Lemma 2 together with (32), we get

$$
0 \leq \limsup_{n \to \infty} \left\| x_n - \tilde{x}_n \right\| \leq \limsup_{n \to \infty} \frac{9\varepsilon}{1-\theta}.
\tag{33}
$$

Since by Theorem 1 we have $\lim_{n \to \infty} x_n = p$ and by assumption we have $\lim_{n \to \infty} \tilde{x}_n = \tilde{p}$. Using these together with (33), we get

$$
\left\| p - \tilde{p} \right\| \leq \frac{9\varepsilon}{1-\theta},
$$

as required. $\qquad\square$

Conclusions

New iteration process (4) namely *AK* iteration process is introduced for approximating fixed points of contraction mappings. Theorem 1 shows that our new iteration process is also converging to fixed point like other existing iteration processes for contraction mappings. In Theorem 4 we show that our new iteration process is moving faster than the leading Vatan two-step iteration process (2), which was developed by Karakaya et al. (2015). Examples 1 and 2 are given to verify our claim. Our new iteration process is now available for the engineers, computer scientists, physicists as well as mathematicians to solve different problems more efficiently.

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