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On the method of finding periodic solutions of second-order neutral differential equations with piecewise constant arguments

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Abstract

This paper provides a method of finding periodical solutions of the second-order neutral delay differential equations with piecewise constant arguments of the form $x''(t) + px''(t-1) = qx(2[\frac{t+1}{2}]) + f(t)$, where $[\cdot]$ denotes the greatest integer function, p and q are nonzero constants, and f is a periodic function of t. This reduces the $2n$-periodic solvable problem to a system of $n+1$ linear equations. Furthermore, by applying the well-known properties of a linear system in the algebra, all existence conditions are described for $2n$-periodical solutions that render explicit formula for these solutions.

Keywords: differential equation; piecewise constant argument; periodic solution

1 Introduction

Certain functional differential equation of neutral delay type with piecewise constant arguments exists in the form of

$$\big(x(t) + px(t-1)\big)'' = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \tag{1}$$

where $[\cdot]$ denotes the greatest integer function, p and q are nonzero constants, and $f(t)$ is a periodic function with positive integer period of $2n$.

In the past, many useful methods such as Hale [1], Fink [2] and [3] were developed to study the almost periodic differential equations. Such equations have diversified application in the field of biology, neural networks, physics, chemistry, engineering, and so on [4–7]. Besides, these equations have combined properties of both differential and difference type. The solutions of these equations are continuous with the continuous dynamical systems structure. Certain biomedical and disease dynamics models exploited these equations due to their resemblance with sequential continuous models [4].

The natural occurrence of these equations in approximating the partial differential equations via piecewise constant arguments has already been demonstrated [8]. Meanwhile, the uniqueness of almost periodic solutions to the second order neutral delay differential equations of the form (1) was studied in depth [9, 10]. Despite these studies, the uniqueness of the solution on such equation remains debatable.

In this view, this paper reports all conditions for the uniqueness, infiniteness and emptiness of $2n$-periodic solutions of (1) for f with $2n$-periodicity. Thus, the works of [9–14] are revisited for further improvement to achieve the correct uniqueness conditions. Furthermore, an explicit formula for the exact periodic solutions of the equation is provided. The equivalence of equation (1) to the system of $n+1$ linear equations is also demonstrated. The existence condition for the periodic solution of (1) is described easily using the properties of a linear algebraic system. Some equations having a unique and infinite number of periodic solutions are emphasized as examples to authenticate the incorrectness of uniqueness results that were provided with other studies.

Throughout this paper, we use the following notations: $\mathbf{R}$ as the set of reals; $\mathbf{Z}$ as the set of integers and $\mathbf{C}$ as the set of complex numbers.

2 Definition of solution. Example

A function x is said to be a solution of (1) if the following conditions are satisfied:

(i) x is differentiable on $\mathbf{R}$;

(ii) the second order derivative of $x(t) + px(t-1)$ exists on $\mathbf{R}$ except possibly at the points $t = 2k+1$, $k \in \mathbf{Z}$, where one-sided second order derivatives of $x(t) + px(t-1)$ exist;

(iii) x satisfies (1) on each interval $(2k-1, 2k+1)$ with integer $k \in \mathbf{Z}$.

Example 1 Let $p = 0.5$ and $q = 3$. One can easily check, that in (1), when $f(t) = \cos \pi t$, the 2-periodic continuous function

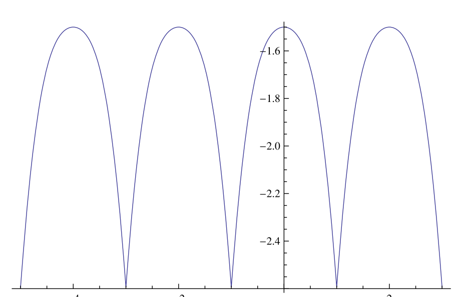
$$x_\alpha(t) = \frac{2}{\pi^2} - \frac{1+t^2}{2}\alpha - \frac{2\cos \pi t}{\pi^2}, \quad t \in [-1,1],$$

satisfies (1) on each interval $(2k-1, 2k+1)$ with integer $k \in \mathbf{Z}$ for any number α. Note that this function is not differentiable at the points $t = 2k-1$, $k \in \mathbf{Z}$ for any $\alpha \neq 0$ (see Figure 1). To be differentiable, x should satisfy the equality $x'(2k-1) = x'(2k+1)$, $k \in \mathbf{Z}$, which is equivalent to $\alpha = 0$. In this case

$$x_0(t) = \frac{2}{\pi^2} - \frac{2\cos \pi t}{\pi^2}$$

is the solution of (1).

Figure 1 The graph of $x_\alpha(t)$ with $\alpha = 3$.



Example 1 shows that for the uniqueness of solution, it is natural for the solution to be differentiable. This condition is omitted in many works (see [10] and its references), where the uniqueness of solution does not hold. A similar comment was first given in [9].

3 2- and 4-periodic solutions

In this section we give the uniqueness conditions of periodic solutions of equation (1) for the cases when f are 2- and 4-periodic functions.

The case $n = 1$. Let f be a 2-periodic continuous function and x be a 2-periodic solution of (1). Then by the definition of solution

$$\begin{aligned} &x'(t) = x'(t+2) \quad \text{for all } t \in \mathbf{R}, \\ &x''(t) = x''(t+2) \quad \text{on each interval } (2k-1, 2k+1) \text{ with integer } k \in \mathbf{Z}. \end{aligned} \tag{2}$$

It follows from here and (1) that

$$\begin{aligned} &\bigl(x(t) + px(t-1)\bigr)'' = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \\ &\bigl(x(t+1) + px(t)\bigr)'' = qx\left(2\left[\frac{t+2}{2}\right]\right) + f(t+1) \end{aligned} \tag{3}$$

or

$$\bigl(1-p^2\bigr)x''(t) = qx\left(2\left[\frac{t+1}{2}\right]\right) - pqx\left(2\left[\frac{t+2}{2}\right]\right) + f(t) - pf(t+1). \tag{4}$$

Since $2[\frac{t+1}{2}] = 0$ as $t \in [-1, 1)$ and $2[\frac{t+2}{2}] = 2$ as $t \in (0, 1]$, taking into account the periodicity of x, from (4) we have

$$x''(t) = \frac{q}{1+p}x(0) + \frac{1}{1-p^2}\bigl(f(t) - pf(t+1)\bigr). \tag{5}$$

Integrating (5) on $[-1, t)$, $t \le 1$, we obtain

$$x(t) = x(-1) + x'(-1)(t+1) + \frac{q}{1+p}x(0)\frac{(t+1)^2}{2} + F_1(p;t), \tag{6}$$

where

$$F_1(p;t) = \frac{1}{1-p^2}\int_{-1}^{t}\int_{-1}^{t_1}\bigl(f(s) - pf(s+1)\bigr)\,ds\,dt_1.$$

To find the unknown numbers $x(0)$, $x(-1)$ and $x'(-1)$, from (6) we have

$$\begin{aligned} &x(0) = x(-1) + x'(-1) + \frac{1}{2}\frac{q}{1+p}x(0) + F_1(p;0), \\ &x(1) = x(-1) + 2x'(-1) + \frac{2q}{1+p}x(0) + F_1(p;1), \\ &x'(1) = x'(-1) + \frac{2q}{1+p}x(0) + F_1'(p;1). \end{aligned} \tag{7}$$

It follows from the periodicity of x and the continuity of x' that $x(-1) = x(1)$ and $x'(-1) = x'(1)$. Then the system of equations (7) has a unique solution $(x(0), x(1), x'(-1))$ if and only if

$$D_1(p,q) := \begin{vmatrix} \frac{1}{2}\frac{q}{1+p} - 1 & 1 & 1 \\ \frac{2q}{1+p} & 0 & 2 \\ \frac{2q}{1+p} & 0 & 0 \end{vmatrix} = \frac{4q}{1+p} \neq 0.$$

Conversely, if (x_1, x_2, x_3) is the solution of (7), then the function

$$x(t) = x_2 + x_3(t+1) + \frac{q}{1+p} x_1 \frac{(t+1)^2}{2} + F_1(p;t), \quad t \in [-1,1],$$

is a 2-periodic solution of (1) with $x(0) = x_1$, $x(-1) = x_2$, $x'(-1) = x_3$.

Summarizing, we have the following.

Theorem 1 *Let f be a 2-periodic continuous function and $p^2 \neq 1$. Then equation* (1) *has a unique* 2*-periodic solution x having the form* (6), *where $(x(0), x(1), x'(-1))$ is the solution of* (7).

The case $n = 2$. Let f be a continuous 4-periodic function and x be a 4-periodic solution of (1). It follows from (1) and 4-periodicity of $x(t)$ that

$$\begin{aligned} x''(t) + px''(t-1) &= qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \\ x''(t+1) + px''(t) &= qx\left(2\left[\frac{t+2}{2}\right]\right) + f(t+1), \\ x''(t+2) + px''(t+1) &= qx\left(2\left[\frac{t+3}{2}\right]\right) + f(t+2), \\ x''(t-1) + px''(t+2) &= qx\left(2\left[\frac{t+4}{2}\right]\right) + f(t+3). \end{aligned} \tag{8}$$

This system of equations with respect to $x''(t-1)$, $x''(t)$, $x''(t+1)$, $x''(t+2)$ is solvable if and only if

$$\Delta(p) := \begin{vmatrix} p & 1 & 0 & 0 \\ 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 1 & 0 & 0 & p \end{vmatrix} = p^4 - 1 \neq 0.$$

Then

$$x''(t) = \frac{\Delta(p,q)}{\Delta(p)},$$

where

$$\Delta(p,q) := \begin{vmatrix} p & Q_4 & 0 & 0 \\ 0 & Q_1 & 1 & 0 \\ 0 & Q_2 & p & 1 \\ 1 & Q_3 & 0 & p \end{vmatrix}, \qquad Q_k = qx\left(2\left[\frac{t+k+1}{2}\right]\right) + f(t+k), \quad k = 1,2,3,4.$$

Simple calculations give

$$\begin{aligned} \Delta(p,q) &= \sum_{k=1}^{4}(-1)^{k+1}p^{4-k}Q_k \\ &= \frac{q}{\Delta(p)}\sum_{k=1}^{4}(-1)^{k+1}p^{4-k}x\left(2\left[\frac{t+k+1}{2}\right]\right) + \frac{1}{\Delta(p)}\sum_{k=1}^{4}(-1)^{k+1}p^{4-k}f(t+k). \end{aligned}$$

Thus, when f is a 4-periodic function, equation (1) is equivalent to the equation

$$x(t) = x(-2) + x'(-2)(t+2) + \frac{q}{\Delta(p)}\Phi_2(p;t) + F_2(p;t), \tag{9}$$

where

$$\begin{aligned} \Phi_2(p;t) &= \sum_{k=1}^{4}(-1)^{k+1}p^{4-k}\int_{-2}^{t}\int_{-2}^{t_1}x\left(2\left[\frac{s+k+1}{2}\right]\right)ds\,dt_1, \\ F_2(p;t) &= \frac{1}{\Delta(p)}\sum_{k=1}^{4}(-1)^{k+1}p^{4-k}\int_{-2}^{t}\int_{-2}^{t_1}f(s+k)\,ds\,dt_1. \end{aligned}$$

We set

$$X[s] = \sum_{k=1}^{4}(-1)^{k+1}p^{4-k}x\left(2\left[\frac{s+k+1}{2}\right]\right).$$

Then

$$\begin{aligned} &X[s] = X[-2] \quad \text{as } -2 \le s < -1, \\ &X[s] = X[-1] \quad \text{as } -1 \le s < 0, \\ &X[s] = X[0] \quad \text{as } 0 \le s < 1, \\ &X[s] = X[1] \quad \text{as } 1 \le s < 2. \end{aligned}$$

Therefore

$$\begin{aligned} \Phi_2(p;t) &= \int_{-2}^{t}\int_{-2}^{t_1}X[s]\,ds\,dt_1 = X[-2]\frac{(t+2)^2}{2} \quad \text{for } -2 \le t < -1, \\ \Phi_2(p;t) &= \int_{-2}^{-1}\int_{-2}^{t_1}X[s]\,ds\,dt_1 + \int_{-1}^{t}\int_{-2}^{-1}X[s]\,ds\,dt_1 + \int_{-1}^{t}\int_{-1}^{t_1}X[s]\,ds\,dt_1 \\ &= \Phi_2(p;-1-0) + X[-2](t+1) + X[-1]\frac{(t+1)^2}{2} \quad \text{for } -1 \le t < 0, \end{aligned}$$

$$\begin{aligned}
\Phi_2(p;t) &= \int_{-2}^{0}\int_{-2}^{t_1} X[s]\,ds\,dt_1 + \int_0^t\left(\int_{-2}^{-1} + \int_{-1}^{0}\right)X[s]\,ds\,dt_1 + \int_0^t\int_0^{t_1} X[s]\,ds\,dt_1 \\
&= \Phi_2(p;0-0) + \bigl(X[-2]+X[-1]\bigr)t + X[0]\frac{t^2}{2} \quad \text{for } 0 \le t < 1, \\
\Phi_2(p;t) &= \int_{-2}^{1}\int_{-2}^{t_1} X[s]\,ds\,dt_1 + \int_1^t\left(\int_{-2}^{-1} + \int_{-1}^{0} + \int_0^1\right)X[s]\,ds\,dt_1 + \int_1^t\int_1^{t_1} X[s]\,ds\,dt_1 \\
&= \Phi_2(p;1-0) + \bigl(X[-2]+X[-1]+X[0]\bigr)(t-1) + X[1]\frac{(t-1)^2}{2} \quad \text{for } 1 \le t < 2.
\end{aligned}$$

The value of the function $X[s]$ depends on $x(-2)$, $x(0)$, $x'(-2)$. Therefore the right-hand side of (9) depends on unknowns $x(-2)$, $x(0)$, $x'(-2)$. To find these unknown numbers, we use the periodicity property of the continuous and differentiable function x, i.e., $x(-2) = x(2+0)$ and $x'(-2) = x'(2+0)$.

From (9) we get a system of linear equations with respect to $x(-2)$, $x(0)$, $x'(-2)$, i.e.,

$$\begin{aligned}
x(0) &= x(-2) + x'(-2) + \frac{q}{\Delta(p)}\Phi_2(p;0) + F_2(p;0), \\
x(2) &= x(-2) + 4x'(-2) + \frac{q}{\Delta(p)}\Phi_2(p;2) + F_2(p;2), \\
x'(2) &= x'(-2) + \frac{q}{\Delta(p)}\Phi_2'(p;2) + F_2'(p;2).
\end{aligned} \tag{10}$$

The values of $\Phi_2(p;t)$ at the points -1, 0, 1 and 2 have the form

$$\begin{aligned}
\Phi_2(p;-1) &= \frac{1}{2}\bigl(p^3x(0) - p^2x(0) + px(2) - x(-2)\bigr), \\
\Phi_2(p;0) &= \frac{3}{2}\bigl(p^3x(0) - p^2x(0) + px(2) - x(-2)\bigr) + \frac{1}{2}\bigl(p^3x(0) - p^2x(2) + px(2) - x(0)\bigr), \\
\Phi_2(p;1) &= \frac{5}{2}\bigl(p^3x(0) - p^2x(0) + px(2) - x(-2)\bigr) \\
&\quad + \frac{3}{2}\bigl(p^3x(0) - p^2x(2) + px(2) - x(0)\bigr) + \frac{1}{2}\bigl(p^3x(2) - p^2x(2) + px(0) - x(0)\bigr), \\
\Phi_2(p;2) &= \frac{7}{2}\bigl(p^3x(0) - p^2x(0) + px(2) - x(-2)\bigr) + \frac{5}{2}\bigl(p^3x(0) - p^2x(2) + px(2) - x(0)\bigr) \\
&\quad + \frac{3}{2}\bigl(p^3x(2) - p^2x(2) + px(0) - x(0)\bigr) + \frac{1}{2}\bigl(p^3x(2) - p^2x(0) + px(0) - x(2)\bigr).
\end{aligned}$$

Hence equation (10) can be rewritten as

$$\begin{aligned}
&\left(\frac{q}{\Delta(p)}\left(2p^3 - \frac{3}{2}p^2 - \frac{1}{2}\right) - 1\right)x(0) + \left(\frac{q}{\Delta(p)}\left(2p - \frac{1}{2}p^2 - \frac{3}{2}\right) + 1\right)x(-2) \\
&\qquad + 2x'(-2) = -F_2(p;0), \\
&\frac{q}{\Delta(p)}\bigl(6p^3 - 4p^2 + 2p - 4\bigr)x(0) + \frac{q}{\Delta(p)}\bigl(2p^3 - 4p^2 + 6p - 4\bigr)x(-2) \\
&\qquad + 4x'(-2) = -F_2(p;2), \\
&\frac{2q}{\Delta(p)}\bigl(p^3 - p^2 + p - 1\bigr)x(0) + \frac{2q}{\Delta(p)}\bigl(p^3 - p^2 + p - 1\bigr)x(-2) = -F_2'(p;2).
\end{aligned} \tag{11}$$

We denote by $D_2(p,q)$ a determinant of the matrix $M_2(p,q)$, where

$$M_2(p,q) := \begin{pmatrix} \frac{q}{\Delta(p)}(2p^3 - \frac{3}{2}p^2 - \frac{1}{2}) - 1 & \frac{q}{\Delta(p)}(2p - \frac{1}{2}p^2 - \frac{3}{2}) + 1 & 2 \\ \frac{q}{\Delta(p)}(6p^3 - 4p^2 + 2p - 4) & \frac{q}{\Delta(p)}(2p^3 - 4p^2 + 6p - 4) & 4 \\ \frac{2q}{\Delta(p)}(p^3 - p^2 + p - 1) & \frac{2q}{\Delta(p)}(p^3 - p^2 + p - 1) & 0 \end{pmatrix}.$$

One can check that

$$D_2(p,q) = \frac{8q(2 + 2p^2 + q)}{(1 + p)(p^2 + 2)}.$$

Now we are able to describe existence conditions of the 4-periodic solutions of (1), which are different from the result of Theorem 1.

Theorem 2 *Let f be a 4-periodic function and $p^4 \neq 1$. Then*

(i) *Equation* (1) *has a unique 4-periodic solution x if and only if $D_2(p,q) \neq 0$. The 4-periodic solution x has the form* (9), *where $(x(0), x(-2), x'(-2))$ is the solution of* (11).

(ii) *If $D_2(p,q) = 0$ and $(F_2(p;0), F_2(p;2), F_2'(p;2)) = (0,0,0)$, then equation* (1) *has an infinite number of 4-periodic solutions having the form*

$$x_\alpha(t) = \alpha\left(x(-2) + x'(-2)(t+2) + \frac{q}{\Delta(p)}\Phi_2(p;t)\right) + F_2(p;t), \quad \text{as } t \in [-2,2), \tag{12}$$

where $(x(0), x(-2), x'(-2))$ is an eigenfunction of $M_2(p,q)$ corresponding to 0, α is any number.

(iii) *If $D_2(p,q) = 0$ and $(F_2(p;0), F_2(p;2), F_2'(p;2)) \neq (0,0,0)$, then equation* (1) *has no 4-periodic solution.*

Proof (i) Let x be a 4-periodic solution of (1). Then x can be presented by (9), where $(x(0), x(-2), x'(-2))$ is the solution of (11). The linear system (11) is solvable if and only if $D_2(p,q) \neq 0$. Hence $D_2(p,q) \neq 0$. Conversely, if $D_2(p,q) \neq 0$, equation (11) has a unique solution $(x(0), x(-2), x'(-2))$. One can check that the function x having the form (9) is the solution of (1).

The uniqueness of solution of (1) is trivial.

(ii) Let $F_2(p;0) = F_2(p;2) = F_2'(p;2) = 0$. Then equation (11) reduces to a non-homogeneous equation. This equation has a non-trivial solution if and only if $D_2(p,q) = 0$. This non-trivial solution $(x(0), x(-2), x'(-2))$ is an eigenvector of $M_2(p,q) = 0$ corresponding to the number 0. Then the 4-periodic function

$$x_\alpha(t) = \alpha\left(x(-2) + x'(-2)(t+2) + \frac{q}{\Delta(p)}\Phi_2(p;t)\right) + F_2(p;t)$$

is a solution of (1), where α is any number.

(iii) If $D_2(p,q) = 0$ and $(F_2(p;0), F_2(p;2), F_2'(p;2)) \neq (0,0,0)$, then equation (11) has no solution. Therefore (1) has no 4-periodic solution.

This completes the proof. □

4 Remarks and examples

We remark that (iii) of Theorem 2 says only non-existence of 4-periodic solutions. For example, it does not give non-existence for 2-periodic solutions of (1), when f is 2-periodic.

We give an example for (ii) of Theorem 2.

Example 2 Let $p = 2$ and $q = -10$. In this case

$$M_2(p,q) = \begin{pmatrix} -\frac{22}{3} & \frac{2}{3} & 2 \\ -\frac{64}{3} & -\frac{16}{3} & 4 \\ -\frac{20}{3} & -\frac{20}{3} & 0 \end{pmatrix},$$

and $D_2(p,q) = 0$. The eigenfunction of $M_2(p,q)$ corresponding to the eigenvalue 0 is $(1,-1,4)$.

Let

$$f(t) = \begin{cases} \sin \pi t & \text{for } t \in [-2,-1), \\ -\sin \pi t & \text{for } t \in [-1,0), \\ 3\sin \pi t & \text{for } t \in [0,1), \\ 5\sin \pi t & \text{for } t \in [1,2]. \end{cases}$$

Then

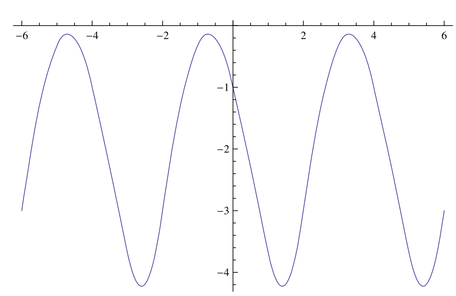
$$F_2(2;t) = \begin{cases} -\frac{\pi(2+t)+\sin \pi t}{\pi^2} & \text{for } t \in [-2,-1), \\ \frac{\pi t+3\sin \pi t}{\pi^2} & \text{for } t \in [-1,1), \\ \frac{2\pi-\pi t+\sin \pi t}{\pi^2} & \text{for } t \in [1,2]. \end{cases}$$

Direct calculations show that $F_2(2,0) = F_2(2;2) = F_2'(2;2) = 0$. The solution of the corresponding equation (1) is 4-periodic function x_α, $\alpha \in \mathbf{C}$, defined on $[-2,2]$ as

$$x_\alpha = \begin{cases} \alpha(3-t^2) + \frac{-\pi(2+t)+\sin \pi t}{\pi^2} & \text{for } t \in [-2,-1), \\ \alpha(1-4t-3t^2) + \frac{\pi t+3\sin \pi t}{\pi^2} & \text{for } t \in [-1,0), \\ \alpha(1-4t+t^2) + \frac{\pi t+3\sin \pi t}{\pi^2} & \text{for } t \in [0,1), \\ \alpha(3-8t+3t^2) + \frac{\pi(2-t)+\sin \pi t}{\pi^2} & \text{for } t \in [1,2]. \end{cases}$$

The graphs of $x_\alpha(t)$ as $\alpha = 1$ and $\alpha = -2$ are shown in Figures 2 and 3, respectively.

Figure 2 The graph of the 4-periodic solution $x_\alpha(t)$ with $\alpha = 1$.



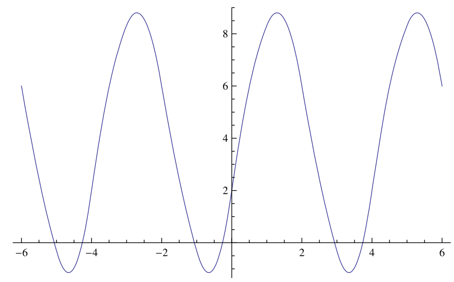


Figure 3 The graph of the 4-periodic solution $x_\alpha(t)$ with $\alpha = -2$.

Note that in this example the parameters of the equation satisfy the conditions of the main results of the papers [9, 11, 12]. Example 2 shows incorrectness of the results Theorem 17 in [9], Theorem 3.1 in [12] and Theorem 2.2 in [11], that claim the uniqueness of the almost periodic solutions of (1).

Since any 2-periodic function can be considered as a 4-periodic function, a question arises:

Do 4-periodic solutions of (1) exist in the case when f is a 2-periodic function?

The answers of this question, by Theorem 2, can be given via three cases:

(i) The case $D_2(p,q) \neq 0$. For this case, by (i) of Theorem 2, equation (1) has the unique 4-periodic solution $x_4(t)$. But by Theorem 1, equation (1) has the unique 2-periodic solution $x_2(t)$. Hence, we must have $x_2(t) = x_4(t)$ (see Example 3).

(ii) An interesting case is when $D_2(p,q) = 0$ and a 2-periodic function f satisfies the equality $(F_2(p;0), F_2(p;2), F_2'(p;2)) = (0,0,0)$. For this case, by (ii) of Theorem 2, equation (1) has an infinite number of 4-periodic solutions. Moreover, there exists a 2-periodic function f such that (1) has unique 2-periodic solutions and an infinite number of 4-periodic solutions (see Example 4).

(iii) In the case when the parameters of (1) satisfy the conditions in (iii) of Theorem 2, then equation (1), with 2-periodic function f, has no 4-periodic solutions.

Example 3 Let $p = 3$, $q = 1$ and a 2-periodic function be given as

$$f(t) = \begin{cases} t+1 & \text{for } t \in [-1,0), \\ 1-t & \text{for } t \in [0,1]. \end{cases}$$

For this case $D_2(3,1) = 21/5$. By using MATHEMATICA, we applied both Theorems 1 and 2 and obtained $x_2(t) = x_4(t)$, where the 2-periodic solution $x_2(t)$ of (1) is

$$x_2(t) = \begin{cases} \frac{1}{24}(-12 - 3t^2 - 2t^3) & \text{for } t \in [-1,0), \\ \frac{1}{24}(-12 - 3t^2 + 2t^3) & \text{for } t \in [0,1]. \end{cases}$$

Example 4 Let $p = 2$ and $q = -10$ and f be a 2-periodic function as

$$f(t) = \begin{cases} \sin \pi t + 11 \sin 2\pi t & \text{for } t \in [-1,0), \\ \sin \pi t + \sin 2\pi t & \text{for } t \in [0,1]. \end{cases}$$

For this case, $D_2(2,-10) = 0$. Then

$$F_2(2;t) = \begin{cases} \frac{10\pi(2+t)+4\sin\pi t-7\sin 2\pi t}{4\pi^2} & \text{for } t \in [-2,-1), \\ \frac{-10\pi t+4\sin\pi t+3\sin 2\pi t}{4\pi^2} & \text{for } t \in [-1,0), \\ \frac{10\pi t+4\sin\pi t-7\sin 2\pi t}{4\pi^2} & \text{for } t \in [-1,0), \\ \frac{-10\pi(-2+t)+4\sin\pi t+3\sin 2\pi t}{4\pi^2} & \text{for } t \in [1,2]. \end{cases}$$

Direct calculations show that $F_2(2,0) = F_2(2;2) = F_2'(2;2) = 0$. The solution of the corresponding equation (1) is a 4-periodic function x_α, $\alpha \in \mathbf{C}$, defined on $[-2,2]$ as

$$x_\alpha = \begin{cases} \alpha(3-t^2) + \frac{10\pi(2+t)+4\sin\pi t-7\sin 2\pi t}{4\pi^2} & \text{for } t \in [-2,-1), \\ \alpha(1-4t-3t^2) + \frac{-10\pi t+4\sin\pi t+3\sin 2\pi t}{4\pi^2} & \text{for } t \in [-1,0), \\ \alpha(1-4t+t^2) + \frac{10\pi t+4\sin\pi t-7\sin 2\pi t}{4\pi^2} & \text{for } t \in [0,1), \\ \alpha(3-8t+3t^2) + \frac{10\pi(2-t)+4\sin\pi t+3\sin 2\pi t}{4\pi^2} & \text{for } t \in [1,2]. \end{cases}$$

5 The case $n \in \mathbf{N}$

Let f be a $2n$-periodic continuous function and x be a $2n$-periodic solution of (1). We describe the function x on $[-n,n]$. Without loss of generality, we can assume n is a positive even number. Otherwise, if n is an odd number, we seek a function x on $[-n+1,n+1]$.

Using the definition of solution from (1), we write the following system of $2n$ equations:

$$\begin{aligned} &x''(t) + px''(t-1) = qx\left(2\left[\frac{t+1}{2}\right]\right) + f(t), \\ &x''(t+1) + px''(t) = qx\left(2\left[\frac{t+2}{2}\right]\right) + f(t+1), \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &x''(t+2n-2) + px''(t+2n-3) = qx\left(2\left[\frac{t+n}{2}\right]\right) + f(t+2n-2), \\ &x''(t-1) + px''(t+2n-2) = qx\left(2\left[\frac{t+n+1}{2}\right]\right) + f(t+2n-). \end{aligned} \tag{13}$$

Assuming the right-hand sides of (13) are known, we consider this system of equations with respect to

$$x''(t-1), x''(t), \dots, x''(t+2n-1).$$

It is solvable if and only if $\Delta(p) \neq 0$, where $\Delta(p) = \det \mathbf{P}$, $\mathbf{P}$ is $2n \times 2n$ matrix

$$\mathbf{P} = \begin{pmatrix} p & 1 & 0 & \dots & 0 & 0 \\ 0 & p & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & 1 \\ 1 & 0 & 0 & \dots & 0 & p \end{pmatrix}.$$

Observe that

$$\Delta(p) = p^{2n} - 1.$$

Assuming $p^{2n} \neq 1$, we find $x''(t)$ from (3)

$$x''(t) = \frac{\Delta(p;t)}{\Delta(p)}, \tag{14}$$

where $\Delta(p;t) = \det \mathbf{Q}$, $\mathbf{Q}$ is $2n \times 2n$ matrix

$$\mathbf{Q} = \begin{pmatrix} p & Q_{2n} & 0 & \dots & 0 & 0 \\ 0 & Q_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & Q_{2n-2} & 0 & \dots & p & 1 \\ 1 & Q_{2n-1} & 0 & \dots & 0 & p \end{pmatrix},$$

$$Q_k = qx\left(2\left[\frac{t+k+1}{2}\right]\right) + f(t+k), \quad k = 1, 2, \dots, 2n.$$

Using the properties of determinant, we have

$$\begin{aligned} \det \mathbf{Q} &= -Q_{2n} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & p & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & 1 \\ 1 & 0 & 0 & \dots & 0 & p \end{pmatrix} + Q_1 \begin{pmatrix} p & 0 & 0 & \dots & 0 & 0 \\ 0 & p & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & 1 \\ 1 & 0 & 0 & \dots & 0 & p \end{pmatrix} + \cdots \\ &\quad - Q_{2n-2} \begin{pmatrix} p & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & p \end{pmatrix} + Q_{2n-1} \begin{pmatrix} p & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \\ &= \sum_{k=1}^{2n} (-1)^{k+1} Q_k p^{2n-k} \end{aligned}$$

or

$$\Delta(p,q) = \frac{q}{\Delta(p)} \sum_{k=1}^{2n} (-1)^{k+1} p^{2n-k} x\left(2\left[\frac{t+k+1}{2}\right]\right) + \frac{1}{\Delta(p)} \sum_{k=1}^{2n} (-1)^{k+1} p^{2n-k} f(t+k).$$

Since f is a $2n$-periodic function, equation (1) is equivalent to the equation

$$x(t) = x(-n) + x'(-n)(t+n) + \frac{q}{\Delta(p)} \Phi_n(p;t) + F_n(p;t), \tag{15}$$

where

$$\Phi_n(p;t) = \sum_{k=1}^{2n}(-1)^{k+1}p^{2n-k}\int_{-n}^{t}\int_{-n}^{t_1} x\left(2\left[\frac{s+k+1}{2}\right]\right)ds\,dt_1,$$

$$F_n(p;t) = \frac{1}{\Delta(p)}\sum_{k=1}^{2n}(-1)^{k+1}p^{2n-k}\int_{-n}^{t}\int_{-n}^{t_1} f(s+k)\,ds\,dt_1.$$

We set

$$X[s] = \sum_{k=1}^{2n}(-1)^{k+1}p^{2n-k}x\left(2\left[\frac{s+k+1}{2}\right]\right).$$

Since $2[\frac{t+1}{2}] = 2k$ for $t \in [2k, 2k+2)$, $k \in \mathbf{Z}$,

$$X[s] = X[k] \quad \text{for } k \le s < k+1, k = -n, \dots, n-1.$$

Therefore $\Phi_n(p;t)$ can be represented as

$$\Phi_n(p;t) = \int_{-n}^{t}\int_{-n}^{t_1} X[s]\,ds\,dt_1 = X[-n]\frac{(t+n)^2}{2} \quad \text{for } -n \le t < -n+1,$$

$$\begin{aligned}\Phi_n(p;t) &= \int_{-n}^{-n+1}\int_{-n}^{t_1} X[s]\,ds\,dt_1 + \int_{-n+1}^{t}\int_{-n}^{-n+1} X[s]\,ds\,dt_1 + \int_{-n+1}^{t}\int_{-n+1}^{t_1} X[s]\,ds\,dt_1 \\ &= \Phi_n(p;-n+1-0) + X[-n](t+n-1) + X[-n+1]\frac{(t+n-1)^2}{2} \\ &\quad \text{for } -n+1 \le t < -n+2,\end{aligned}$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\begin{aligned}\Phi_n(p;t) &= \int_{-n}^{n-2}\int_{-n}^{t_1} X[s]\,ds\,dt_1 + \int_{n-2}^{t}\int_{-n}^{n-2} X[s]\,ds\,dt_1 + \int_{n-2}^{t}\int_{n-2}^{t_1} X[s]\,ds\,dt_1 \\ &= \Phi_n(p;n-2-0) + \sum_{k=0}^{2n-3}\int_{n-2}^{t}\int_{-n+k}^{-n+k+1} X[s]\,ds\,dt_1 + \int_{n-2}^{t}\int_{n-2}^{t_1} X[s]\,ds\,dt_1 \\ &= \Phi_n(p;n-2) + \sum_{k=0}^{2n-3} X[-n+k](t-n+2) + X[n-2]\frac{(t-n+2)^2}{2} \\ &\quad \text{for } n-2 \le t < n-1,\end{aligned}$$

$$\begin{aligned}\Phi_n(p;t) &= \Phi_n(p;n-1-0) + \sum_{k=0}^{2n-2}\int_{n-1}^{t}\int_{-n+k}^{-n+k+1} X[s]\,ds\,dt_1 + \int_{n-1}^{t}\int_{n-1}^{t_1} X[s]\,ds\,dt_1 \\ &= \Phi_n(p;n-1) + \sum_{k=0}^{2n-2} X[-n+k](t-n+1) + X[n-1]\frac{(t-n+1)^2}{2} \\ &\quad \text{for } n-1 \le t < n.\end{aligned}$$

These equations show that the right-hand side of (15) depends on $n+1$ unknowns $x(-n+2), x(-n+4), \dots, x(n), x'(-n)$, where n is an even number. Hence equation (15) is equivalent to the following system of $n+1$ equations with respect to $x(-n+2), x(-n+4), \ldots, x(n), x'(-n)$ (see Lemma 1):

$$\begin{aligned}
&x(-n+2) = x(-n) + x'(-n) + \frac{q}{\Delta(p)} \sum_{k=1}^{n} P_{1k}(p)x(-n+2k) + F_n(p;-n+2), \\
&x(-n+4) = x(-n) + 2x'(-n) + \frac{q}{\Delta(p)} \sum_{k=1}^{n} P_{2k}(p)x(-n+2k) + F_n(p;-n+4), \\
&\cdots \qquad \cdots \qquad \cdots \\
&x(n) = x(-n) + 2nx'(-n) + \frac{q}{\Delta(p)} \sum_{k=1}^{n} P_{nk}(p)x(-n+2k) + F_n(p;n), \\
&x'(n) = x'(-n) + \frac{q}{\Delta(p)} \sum_{k=1}^{n} P_{n+1,k}(p)x(-n+2k) + F'_n(p;n),
\end{aligned} \tag{16}$$

where the polynomials $P_{ij}(p)$ are defined by (19).

We denote by $D(p,q)$ the determinant of the matrix

$$\mathbf{B} = \begin{pmatrix} \frac{q}{\Delta(p)}P_{11}(p) - 1 & \frac{q}{\Delta(p)}P_{12}(p) & \ldots & \frac{q}{\Delta(p)}P_{1,n}(p) + 1 & 1 \\ \frac{q}{\Delta(p)}P_{11}(p) & \frac{q}{\Delta(p)}P_{22}(p) - 1 & \ldots & \frac{q}{\Delta(p)}P_{2,n-1}(p) + 1 & 2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \frac{q}{\Delta(p)}P_{n1}(p) & \frac{q}{\Delta(p)}P_{n2}(p) & \ldots & \frac{q}{\Delta(p)}P_{n,n}(p) & 2n \\ \frac{q}{\Delta(p)}P_{n+1,1}(p) & \frac{q}{\Delta(p)}P_{n+1,2}(p) & \ldots & \frac{q}{\Delta(p)}P_{n+1,n}(p) & 0 \end{pmatrix}. \tag{17}$$

The main result of this section is the following theorem.

Theorem 3 *Let $p^{2n} \neq 1$ and f be a $2n$-periodic continuous function. Then*

(i) *if $D(p,q) \neq 0$, equation* (1) *has a unique $2n$-periodic solution having the form* (15), *where $(x(-n+2), x(-n+2), \ldots x(n), x'(-n))$ is the unique solution of* (17);

(ii) *if $D(p,q) = 0$ and $F_n(p;-n+2) = \cdots = F_n(p;n) = F'_n(p;n) = 0$, then equation* (1) *has an infinite number of $2n$-periodic solutions having the form*

$$x_\alpha(t) = \alpha\left(x(-n) + x'(-n)(t+n) + \frac{q}{\Delta(p)}\Phi_n(p;t)\right) + F_n(p;t),$$

where $(x(-n+2), \ldots, x(n), x'(-n))$ is an eigenfunction of $\mathbf{B}$ corresponding to the eigenvalue 0, α is any number;

(iii) *if $D(p,q) = 0$ and $(F_n(p;-n+2), \ldots, F_n(p;n), F'_n(p;n)) \neq (0, \ldots, 0)$, then equation* (1) *does not have any $2n$-periodic solution.*

Proof The proof of the theorem is similar to the proof of Theorem 2. □

Lemma 1 *Equation* (15) *is equivalent to the system of equations* (17).

Proof Since x is a $2n$ periodic solution of (1), it satisfies equations $x(-n) = x(n)$ and $x'(-n) = x'(n)$. From (15) we can describe the values of $x(-n+2), x(-n+4), \ldots, x(n), x'(-n)$. Therefore we get the $n+1$ linear system of equations

$$\begin{aligned}
&x(-n+2) = x(-n) + x'(-n) + \frac{q}{\Delta(p)}\Phi_n(p;-n+2) + F_n(p;-n+2),\\
&x(-n+4) = x(-n) + 2x'(-n) + \frac{q}{\Delta(p)}\Phi_n(p;-n+4) + F_n(p;-n+4),\\
&\dots \quad \dots \quad \dots\\
&x(n) = x(-n) + 2nx'(-n) + \frac{q}{\Delta(p)}\Phi_n(p;n) + F_n(p;n),\\
&x'(n) = x'(-n) + \frac{q}{\Delta(p)}\Phi'_n(p;n) + F'_n(p;n).
\end{aligned} \tag{18}$$

Note that

$$\begin{aligned}
X[r] &= \sum_{k=1}^{n}\bigl(p^{2n-2k+1} - p^{2n-2k}\bigr)x(r+2k) = \sum_{k=1}^{n}\mathbf{p}_k^0 x(r+2k) \quad \text{for even } |r|,\\
X[r] &= \bigl(p^{2n-1} - 1\bigr)x(r+1) + \sum_{k=1}^{n-1}\bigl(-p^{2n-2k} + p^{2n-2k-1}\bigr)x(r+2k+1)\\
&= \sum_{k=1}^{n}\mathbf{p}_k^1 x(r+2k+1) \quad \text{for odd } |r|,
\end{aligned}$$

where

$$\mathbf{p}_k^0 = p^{2n-2k+1} - p^{2n-2k}, \qquad \mathbf{p}_k^1 = -p^{2n-2k} + p^{2n-2k-1} \quad \text{and} \quad \mathbf{p}_n^1 = p^{2n-1} - 1.$$

The values of $\Phi_n(p;\cdot)$ at the points $-n+2, -n+4, \dots, n$ are given by

$$\begin{aligned}
\Phi_n(p;-n+2) &= \frac{3}{2}X[-n] + \frac{1}{2}X[-n+1] = \frac{3}{2}\sum_{k=1}^{n}\mathbf{p}_k^0 x(-n+2k) + \frac{1}{2}\sum_{k=1}^{n}\mathbf{p}_k^1 x(-n+2+2k),\\
\Phi_n(p;-n+4) &= \frac{7}{2}X[-n] + \frac{5}{2}X[-n+1] + \frac{3}{2}X[-n+2] + \frac{1}{2}X[-n+3]\\
&= \frac{7}{2}\sum_{k=1}^{n}\mathbf{p}_k^0 x(-n+2k) + \frac{5}{2}\sum_{k=1}^{n}\mathbf{p}_k^1 x(-n+2+2k)\\
&\quad + \frac{3}{2}\sum_{k=1}^{n}\mathbf{p}_k^0 x(-n+2+2k) + \frac{1}{2}\sum_{k=1}^{n}\mathbf{p}_k^1 x(-n+4+2k),\\
&\dots \quad \dots \quad \dots\\
\Phi_n(p;n) &= \left(2n - \frac{1}{2}\right)X[-n] + \left(2n - \frac{3}{2}\right)X[-n+1] + \dots + \frac{3}{2}X[n-2] + \frac{1}{2}X[n-1]\\
&= \left(2n - \frac{1}{2}\right)\sum_{k=1}^{n}\mathbf{p}_k^0 x(-n+2k) + \left(2n - \frac{3}{2}\right)\sum_{k=1}^{n}\mathbf{p}_k^1 x(-n+2+2k)\\
&\quad + \dots + \frac{3}{2}\sum_{k=1}^{n}\mathbf{p}_k^0 x(n-2+2k) + \frac{1}{2}\sum_{k=1}^{n}\mathbf{p}_k^1 x(n+2k),
\end{aligned}$$

$$\begin{aligned}\Phi'_n(p;n) &= \sum_{r=0}^{2n-1} X[-n+r] = \sum_{k=1}^{n} \mathbf{p}_k^0 x(-n+2k) + \sum_{k=1}^{n} \mathbf{p}_k^1 x(-n+2+2k)\\ &\quad + \sum_{k=1}^{n} \mathbf{p}_k^0 x(-n+2+2k) + \cdots + \sum_{k=1}^{n} \mathbf{p}_k^0 x(n-2+2k) + \sum_{k=1}^{n} \mathbf{p}_k^1 x(n+2k),\end{aligned}$$

or, equivalently,

$$\begin{aligned}\Phi_n(p;-n+2) &= \left(\frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1\right)x(-n+2) + \sum_{k=2}^{n}\left(\frac{3}{2}\mathbf{p}_k^0 + \frac{1}{2}\mathbf{p}_{k-1}^1\right)x(-n+2k),\\
\Phi_n(p;-n+4) &= \left(\frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1\right)x(-n+2)\\
&\quad + \left(\frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1\right)x(-n+4)\\
&\quad + \sum_{k=3}^{n}\left(\frac{7}{2}\mathbf{p}_k^0 + \frac{5}{2}\mathbf{p}_{k-1}^1 + \frac{3}{2}\mathbf{p}_{k-1}^0 + \frac{1}{2}\mathbf{p}_{k-2}^1\right)x(-n+2k),\\
\Phi_n(p;-n+6) &= \left(\frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_n^1 + \frac{7}{2}\mathbf{p}_n^0 + \frac{5}{2}\mathbf{p}_{n-1}^1 + \frac{3}{2}\mathbf{p}_{n-1}^0 + \frac{1}{2}\mathbf{p}_{n-2}^1\right)x(-n+2)\\
&\quad + \left(\frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_1^1 + \frac{7}{2}\mathbf{p}_1^1 + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1\right)x(-n+4)\\
&\quad + \left(\frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_1^1 + \frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1\right)x(-n+6)\\
&\quad + \sum_{k=4}^{n}\left(\frac{11}{2}\mathbf{p}_k^0 + \frac{9}{2}\mathbf{p}_{k-1}^1 + \frac{7}{2}\mathbf{p}_{k-1}^0 + \frac{5}{2}\mathbf{p}_{k-2}^1 + \frac{3}{2}\mathbf{p}_{k-2}^0 + \frac{1}{2}\mathbf{p}_{k-3}^1\right)x(-n+2k),\\
\cdots \quad &\cdots \quad \cdots\\
\Phi_n(p;n) &= \left(\left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_n^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_n^0 + \cdots\right.\\
&\quad \left. + \frac{5}{2}\mathbf{p}_2^1 + \frac{3}{2}\mathbf{p}_2^0 + \frac{1}{2}\mathbf{p}_1^1\right)x(-n+2)\\
&\quad + \left(\left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_1^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{7}{2}\right)\mathbf{p}_n^1\right.\\
&\quad \left. + \left(2n-\frac{9}{2}\right)\mathbf{p}_1^0 + \cdots + \frac{5}{2}\mathbf{p}_3^1 + \frac{3}{2}\mathbf{p}_3^0 + \frac{1}{2}\mathbf{p}_2^1\right)x(-n+4)\\
&\quad + \cdots + \left(\left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_1^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_1^0 + \cdots\right.\\
&\quad \left. + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1\right)x(n-2)\\
&\quad + \left(\left(2n-\frac{1}{2}\right)\mathbf{p}_n^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_{n-1}^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_{n-1}^0 + \cdots\right.\\
&\quad \left. + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1\right)x(n),\\
\Phi'_n(p;n) &= \left(\mathbf{p}_1^0 + \mathbf{p}_n^1 + \mathbf{p}_n^0 + \cdots + \mathbf{p}_2^1 + \mathbf{p}_2^0 + \mathbf{p}_1^1\right)x(-n+2)\\
&\quad + \left(\mathbf{p}_2^0 + \mathbf{p}_1^1 + \mathbf{p}_1^0 + \mathbf{p}_n^1 + \mathbf{p}_n^0 + \cdots + \mathbf{p}_3^1 + \mathbf{p}_3^0 + \mathbf{p}_2^1\right)x(-n+4)\\
&\quad + \cdots + \left(\mathbf{p}_n^0 + \mathbf{p}_{n-1}^1 + \mathbf{p}_{n-1}^0 + \cdots + \mathbf{p}_1^1 + \mathbf{p}_1^0 + \mathbf{p}_n^1\right)x(n).\end{aligned}$$

We denote

$$
\begin{aligned}
P_{11}(p) &= \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1,\\
P_{1k}(p) &= \frac{3}{2}\mathbf{p}_k^0 + \frac{1}{2}\mathbf{p}_{k-1}^1, \quad k = 2,\dots,n,\\
P_{21}(p) &= \frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1,\\
P_{22}(p) &= \frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1,\\
P_{2k}(p) &= \frac{7}{2}\mathbf{p}_k^0 + \frac{5}{2}\mathbf{p}_{k-1}^1 + \frac{3}{2}\mathbf{p}_{k-1}^0 + \frac{1}{2}\mathbf{p}_{k-2}^1, \quad k = 3,\dots,n,\\
P_{31}(p) &= \frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_n^1 + \frac{7}{2}\mathbf{p}_n^0 + \frac{5}{2}\mathbf{p}_{n-1}^1 + \frac{3}{2}\mathbf{p}_{n-1}^0 + \frac{1}{2}\mathbf{p}_{n-2}^1,\\
P_{32}(p) &= \frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_1^1 + \frac{7}{2}\mathbf{p}_1^1 + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1,\\
P_{33}(p) &= \frac{11}{2}\mathbf{p}_1^0 + \frac{9}{2}\mathbf{p}_1^1 + \frac{7}{2}\mathbf{p}_1^0 + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1,\\
P_{3k}(p) &= \frac{11}{2}\mathbf{p}_k^0 + \frac{9}{2}\mathbf{p}_{k-1}^1 + \frac{7}{2}\mathbf{p}_{k-1}^0 + \frac{5}{2}\mathbf{p}_{k-2}^1 + \frac{3}{2}\mathbf{p}_{k-2}^0 + \frac{1}{2}\mathbf{p}_{k-3}^1, \quad k = 4,\dots,n, \qquad (19)\\
&\dots \quad \dots \quad \dots\\
P_{n1}(p) &= \left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_n^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_n^0 + \cdots + \frac{5}{2}\mathbf{p}_2^1 + \frac{3}{2}\mathbf{p}_2^0 + \frac{1}{2}\mathbf{p}_1^1,\\
P_{n2}(p) &= \left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_1^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{7}{2}\right)\mathbf{p}_n^1 + \left(2n-\frac{9}{2}\right)\mathbf{p}_1^0\\
&\quad + \cdots + \frac{5}{2}\mathbf{p}_3^1 + \frac{3}{2}\mathbf{p}_3^0 + \frac{1}{2}\mathbf{p}_2^1,\\
&\dots \quad \dots \quad \dots\\
P_{n,n-1}(p) &= \left(2n-\frac{1}{2}\right)\mathbf{p}_1^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_1^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_1^0 + \cdots + \frac{5}{2}\mathbf{p}_n^1 + \frac{3}{2}\mathbf{p}_n^0 + \frac{1}{2}\mathbf{p}_{n-1}^1,\\
P_{nn}(p) &= \left(2n-\frac{1}{2}\right)\mathbf{p}_n^0 + \left(2n-\frac{3}{2}\right)\mathbf{p}_{n-1}^1 + \left(2n-\frac{5}{2}\right)\mathbf{p}_{n-1}^0 + \cdots + \frac{5}{2}\mathbf{p}_1^1 + \frac{3}{2}\mathbf{p}_1^0 + \frac{1}{2}\mathbf{p}_n^1,\\
P_{n+1,k}(p) &= \sum_{r=1}^{n}\left(\mathbf{p}_r^0 + \mathbf{p}_r^1\right), \quad k = 1,\dots,n.
\end{aligned}
$$

From these notations we obtain equivalence of the system of equations (18) to the system of equations (17).

This completes the proof. □

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