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Variable KM-like algorithms for fixed point problems and split feasibility problems

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Abstract

The convergence analysis of a variable KM-like method for approximating common fixed points of a possibly infinitely countable family of nonexpansive mappings in a Hilbert space is proposed and proved to be strongly convergent to a common fixed point of a family of nonexpansive mappings. Our variable KM-like technique is applied to solve the split feasibility problem and the multiple-sets split feasibility problem. Especially, the minimum norm solutions of the split feasibility problem and the multiple-sets split feasibility problem are derived. Our results can be viewed as an improvement and refinement of the previously known results.

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1 Introduction

Problems of image reconstruction from projections can be represented by a system of linear equations

$$Ax = b. \tag{1.1}$$

In practice, the system (1.1) is often inconsistent, and one usually seeks a point which minimizes $x \in \mathbb{R}^n$ by some predetermined optimization criterion. The problem is frequently ill-posed and there may be more than one optimal solution. The standard approach to dealing with that problem is via regularization. The well-known convex feasibility problem is to find a point x^* satisfying the following:

$$\text{to find a point } x \in \bigcap_{i=1}^{m} C_i,$$

where $m \geq 1$ is an integer, and each C_i is a nonempty closed convex subset of a Hilbert space H. A special case of the convex feasibility problem is the split feasibility problem given by:

Let C, Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is

$$\text{to find a point } x \in C \text{ such that } Ax \in Q. \tag{1.2}$$

The SFP is said to be consistent if (1.2) has a solution. It is easy to see that SFP (1.2) is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \text{ such that } x = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x, \tag{1.3}$$

where P_C and P_Q are the projections onto C and Q, respectively, and A^* is the adjoint of A. Let L denote the spectral radius of A^*A. It is well known that if $\gamma \in (0, 2/L)$, the operator $T = P_C(I - \gamma A^*(I - P_Q)A)$ in the operator equation (1.3) is nonexpansive [1].

It has been extensively studied during the last decade because of its applications in modeling inverse problems which arise in phase retrievals and in medical image reconstruction. It has also been applied to modeling intensity-modulated radiation therapy; see, for example [2–7] and the references therein.

Several iterative methods have been proposed and analyzed to solve the SFP (1.2); see, for example [1, 3, 6, 8–14] and the references therein. Byrne [3] introduced the CQ algorithm

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}, \tag{1.4}$$

and proved that the sequence $\{x_n\}$ generated by the CQ algorithm (1.4) converges weakly to a solution of SFP (1.2), where $T = P_C(I - \gamma A^*(I - P_Q)A)$ and $0 < \gamma < 2/L$.

In view of the fixed point formulation (1.3) of the SFP (1.2), Xu [1] and Yang [14] applied the following perturbed Krasnosel'skiĭ-Mann CQ algorithm to solve the SFP (1.2):

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \in \mathbb{N}. \tag{1.5}$$

Here $\{T_n\}$ is a sequence of operators defined by

$$T_n = P_{C_n}\bigl(I - \gamma A^*(I - P_{Q_n})A\bigr), \quad n \in \mathbb{N},$$

where $\{C_n\}$ and $\{Q_n\}$ are sequences of nonempty closed convex subsets in H_1 and H_2, respectively, which obey the following assumption:

(C0) $\sum_{n=1}^{\infty} \alpha_n d_\rho(C_n, C) < \infty$ and $\sum_{n=1}^{\infty} \alpha_n d_\rho(Q_n, Q) < \infty$ for each $\rho > 0$, where d_ρ is the ρ-distance between Q_n and Q (see Section 3.2).

It is not very easy to verify condition (C0) for each $\rho > 0$. Thus, the condition (C0) is quite restrictive even for weak convergence of the sequence $\{x_n\}$ defined by (1.5). One of our objectives is to relax the condition (C0).

Many practical problems can be formulated as a *fixed point problem* (FPP): finding an element x such that

$$x = Tx, \tag{1.6}$$

where T is a nonexpansive self-mapping defined on a closed convex subset C of a Hilbert space H. The solution set of FPP (1.6) is denoted by $F(T)$. It is well known that if $F(T) \neq \emptyset$, then $F(T)$ is closed and convex. The fixed point problem (1.6) is ill-posed (it may fail to have a solution, nor uniqueness of solution) in general. Regularization by contractions can removed such illness. We replace the nonexpansive mapping T by a family of contractions $T_t^f := tf + (1 - t)T$, with $t \in (0, 1)$ and $f : C \to C$ a fixed contraction. We call f an anchoring function. The regularized problem of fixed point for T is the fixed point problem for T_t^f. The mapping T_t^f has a unique fixed point, namely, $x_t \in C$. Therefore, x_t is the solution of the fixed point problem

$$x_t = tfx_t + (1-t)Tx_t. \tag{1.7}$$

We now discretize the regularization (1.7) to define an explicit iterative algorithm:

$$x_{n+1} = t_n fx_n + (1-t_n)Tx_n, \quad n \in \mathbb{N}. \tag{1.8}$$

The iterative algorithm (1.8) is due to Moudafi [15], by generalizing Browder's and Halpern's methods, who introduced viscosity approximation methods. Suzuki [16] established a strong convergence theorem by using Halpern's method to averaged mapping $T_\lambda = \lambda I + (1-\lambda)T$, $\lambda \in (0,1)$ for nonexpansive mappings T in certain Banach spaces. Takahashi [17] proved a strong convergence theorem of the following iterative algorithm for countable families of nonexpansive mappings in certain Banach spaces:

$$x_{n+1} = t_n fx_n + (1-t_n)T_n x_n, \quad n \in \mathbb{N}. \tag{1.9}$$

Recently, Yao and Xu [18] introduced and studied strong convergence of the following modified methods:

$$x_{n+1} = P_C\bigl[t_n fx_n + (1-t_n)Tx_n\bigr], \quad n \in \mathbb{N}, \tag{1.10}$$

where $f : C \to H$ is a fixed non-self contraction and $\{t_n\}$ is a sequence in $(0,1)$ satisfying the conditions:

(S1) $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} t_n = \infty$,

(S2) either $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$ or $t_{n+1}/t_n \to 0$ as $n \to \infty$.

One can easily see that (1.10) is a regularized iterative algorithm.

Motivated by [1, 11, 14], we study the following more general non-regularized algorithm, called *variable KM-like algorithm* which generates a sequence $\{x_n\}$ according to the recursive formula:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n f_n x_n + (1-\alpha_n)T_n x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n \operatorname{Proj}_C[y_n] \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{T_n\}$ is a sequence of nonexpansive self-mappings of C and $\{f_n\}$ is a sequence of (not necessarily contraction) mappings from C into H.

In the present paper, we will study the strong convergence of the proposed variable KM-like algorithm in the framework of Hilbert spaces. The paper is organized as follows. The next section contains preliminaries. In Section 3, we will study the convergence analysis of our variable KM-like algorithm for fixed point problem (1.6) without the assumption (S2). This result will be applied to prove convergence of some perturbed algorithms for the SFP (1.2) and the multiple-sets split feasibility problem under some weaker assumptions.

As special cases, we obtain algorithms which converge strongly to the minimum norm solutions of the split feasibility problem and the multiple-sets split feasibility problem. Our results are new and interesting in the following contexts:

(i) Our algorithm (3.1) is not regularized by contractions.
(ii) f_n is not necessarily contraction. In the existing literature, anchoring function f is a fixed contraction mapping [15, 17–19] or strongly pseudo-contraction mapping [20].
(iii) In the convergence analysis of (3.1) for fixed point problem (1.6), the assumption (S2) is not required.
(iv) A fixed $\rho > 0$ for a (C0)-like condition is adopted.

2 Preliminaries

Let C be a nonempty subset of a Hilbert space H. Throughout the paper, we denote by $B_r[x]$ the closed ball defined by $B_r[x] = \{y \in C : \|y - x\| \le r\}$. Let $T_1, T_2 : C \to H$ be two mappings. We denote by $\mathscr{B}(C)$ the collection of all bounded subsets of C. The *deviation* between T_1 and T_2 on $B \in \mathscr{B}(C)$ [21], denoted by $\mathcal{D}_B(T_1, T_2)$, is defined by

$$\mathcal{D}_B(T_1, T_2) = \sup\{\|T_1x - T_2x\| : x \in B\}.$$

Let $T : C \to H$ be a mapping. Then T is said to be a κ*-contraction* if there exists $\kappa \in [0,1)$ such that $\|Tx - Ty\| \le \kappa\|x - y\|$ for all $x, y \in C$. Furthermore, it is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$.

Let $\{f_n\}$ be a sequence of mappings from C into H. Following [20–22], we say $\{f_n\}$ is a sequence of *nearly contraction* mappings with sequence $\{(k_n, a_n)\}$ if there exist a sequence $\{k_n\}$ in $[0,1)$ and a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ such that

$$\|f_nx - f_ny\| \le k_n\|x - y\| + a_n \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

One can observe that a sequence of contraction mappings is essentially a sequence of nearly contraction mappings.

We now construct a sequence of nearly contractions.

Example 2.1 Let $H = \mathbb{R}$ and $C = [0,1]$. Let $\{f_n\}$ be a sequence of mappings $f_n : C \to H$ defined by

$$f_n(x) = \begin{cases} \frac{x}{n+1}, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{3}{n+1}, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{2.1}$$

Set $k_n := \frac{1}{n+1}$. We consider the following cases:

Case 1: If $x, y \in [0, \frac{1}{2}]$, then

$$f_nx - f_ny = k_n(x - y) \quad \text{for all } n \in \mathbb{N}.$$

Case 2: If $x, y \in (\frac{1}{2}, 1]$, then

$$f_nx - f_ny = 0 \quad \text{for all } n \in \mathbb{N}.$$

Case 3: If $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, then

$$|f_n x - f_n y| = \left| \frac{x}{n+1} - \frac{3}{n+1} \right| \leq \frac{1}{n+1}|x-y| + \frac{1}{n+1}|y-3| \leq k_n|x-y| + \frac{5}{2(n+1)}, \quad n \in \mathbb{N}.$$

Therefore, for all $x, y \in [0,1]$, we have

$$|f_n x - f_n y| \leq k_n|x-y| + a_n \quad \text{for all } n \in \mathbb{N},$$

where $a_n := \frac{5}{2(n+1)}$. Therefore, $\{f_n\}$ is a sequence of nearly contraction mappings with sequence $\{(k_n, a_n)\}$.

Let C be a nonempty closed convex subset of a Hilbert space H. We use P_C to denote the (*metric*) *projection* from H onto C; namely, for $x \in H$, $P_C(x)$ is the unique point in C with the property

$$\|x - P_C(x)\| = \inf\{\|x - z\| : z \in C\}.$$

The following is a useful characterization of projections.

Lemma 2.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$. Then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z \rangle \leq 0 \quad \textit{for all } y \in C.$$

Lemma 2.3 [23, Corollary 5.6.4] *Let C be a nonempty closed convex subset of H and $T : C \to C$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to zero, then $(I-T)x = 0$.*

Lemma 2.4 [24] *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers and let $\{b_n\}$ be a sequence in $\mathbb{R}$ satisfying the following condition*:

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n + c_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following statements hold:

(a) *If $b_n \leq K\alpha_n$ for all $n \in \mathbb{N}$ and for some $K \geq 0$, then*

$$a_{n+1} \leq \delta_n a_1 + (1-\delta_n)K + \sum_{j=1}^{n} c_j \quad \textit{for all } n \in \mathbb{N},$$

where $\delta_n = \Pi_{j=1}^{n}(1-\alpha_j)$ and hence $\{a_n\}$ is bounded.

(b) *If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty}(b_n/\alpha_n) \leq 0$, then $\{a_n\}_{n=1}^{\infty}$ converges to zero.*

3 Convergence analysis of a variable KM-like algorithm

First, we prove the following.

Proposition 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to H$ a κ-contraction. Then there exists a unique point $x^* \in C$ such that $x^* = P_{F(T)}f(x^*)$.*

Proof Since $f : C \to H$ is a κ-contraction, it follows that $P_{F(T)}f$ is a κ-contraction from C onto itself. Then there exists a unique point $x^* \in C$ such that $x^* = P_{F(T)}f(x^*)$. □

3.1 A variable KM-like algorithm

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{f_n\}$ be a sequence of nearly contractions from C into H such that $\{f_n\}$ converges pointwise to f and let $\{T_n\}$ be a sequence of nonexpansive self-mappings of C which are viewed as perturbations. For computing a common fixed point of the sequence $\{T_n\}$ of nonexpansive mappings, we propose the following *variable KM-like algorithm:*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n f_n x_n + (1-\alpha_n)T_n x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[y_n] \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

We investigate the asymptotic behavior of the sequence $\{x_n\}$ generated, from an arbitrary $x_1 \in C$, by the algorithm (3.1) to a common fixed point of the sequence $\{T_n\}$.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, and let $f : C \to H$ be a κ-contraction with $\kappa \in [0,1)$ such that $P_{F(T)}f(x^*) = x^* \in F(T)$. Let $\{f_n\}$ be a sequence of nearly contraction mappings from C into H with the sequence $\{(k_n, a_n)\}$ in $[0,1) \times [0,\infty)$ such that $k_n \to \kappa$, and let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (3.1), where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1)$. Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\lim_{n\to\infty} f_n x^* = f x^*$,

(C4) $\sum_{n=1}^{\infty}(1-\alpha_n)\|T_n x^* - x^*\| < \infty$.

Define

$$R := \max\{\|x_1 - x^*\|, K^*\} + \sum_{n=1}^{\infty}(1-\alpha_n)\|T_n x^* - x^*\| \quad \textit{and} \\ K^* := \sup_{n\in\mathbb{N}} \frac{\|f_n x^* - x^*\| + a_n}{1-k_n}. \tag{3.2}$$

Then the following statements hold:

(a) *The sequence $\{x_n\}$ generated by (3.1) remains in the closed ball $B_R[x^*]$.*

(b) *If the following assumption holds:*

(C5) $\lim_{n\to\infty}\|T_nv_n - Tv_n\| = 0$ *for all* $\{v_n\}$ *in* $B_R[x^*]$,
then $\{x_n\}$ *converges strongly to* x^*.

Proof (a) Set $z_n := P_C[y_n]$. Observe that

$$\begin{aligned}
\|z_n - x^*\| &= \|P_C[y_n] - P_C[x^*]\| \\
&\le \|y_n - x^*\| \\
&\le \alpha_n\|f_nx_n - x^*\| + (1-\alpha_n)\|T_nx_n - x^*\| \\
&\le \alpha_n(\|f_nx_n - f_nx^*\| + \|f_nx^* - x^*\|) + (1-\alpha_n)(\|T_nx_n - T_nx^*\| + \|T_nx^* - x^*\|) \\
&\le \alpha_n(k_n\|x_n - x^*\| + \|f_nx^* - x^*\| + a_n) + (1-\alpha_n)(\|x_n - x^*\| + \|T_nx^* - x^*\|) \\
&= (1-(1-k_n)\alpha_n)\|x_n - x^*\| + \alpha_n(\|f_nx^* - x^*\| + a_n) + (1-\alpha_n)\|T_nx^* - x^*\|.
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n(x_n - x^*) + \beta_n(z_n - x^*)\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|z_n - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n[(1-(1-k_n)\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n(\|f_nx^* - x^*\| + a_n) + (1-\alpha_n)\|T_nx^* - x^*\|] \\
&= (1-(1-k_n)\alpha_n\beta_n)\|x_n - x^*\| + \alpha_n\beta_n(\|f_nx^* - x^*\| + a_n) \\
&\quad + (1-\alpha_n)\beta_n\|T_nx^* - x^*\| \\
&\le (1-(1-k_n)\alpha_n\beta_n)\|x_n - x^*\| + (1-k_n)\alpha_n\beta_nK^* + (1-\alpha_n)\|T_nx^* - x^*\| \\
&\le \max\{\|x_1 - x^*\|, K^*\} + \sum_{j=1}^{n}(1-\alpha_j)\|T_jx^* - x^*\|.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n)\|T_nx^* - x^*\| < \infty$, by Lemma 2.4, we find that $\{\|x_n - x^*\|\}$ is bounded. Moreover,

$$\|x_{n+1} - x^*\| \le \max\{\|x_1 - x^*\|, K^*\} + \sum_{n=1}^{\infty}(1-\alpha_n)\|T_nx^* - x^*\| = R, \quad \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is well defined in the ball $B_R[x^*]$.

(b) Assume that $\lim_{n\to\infty}\|T_nv_n - Tv_n\| = 0$ for all $\{v_n\}$ in $B_R[x^*]$. Set $\gamma_n := \langle x^* - fx^*, x^* - z_n\rangle$. We now proceed with the following steps:

Step 1: $\{f_nx_n\}$ *and* $\{T_nx_n\}$ *are bounded.*

Without loss of generality, we may assume that $\beta \le \beta_n$ for all $n \in \mathbb{N}$ for some $\beta > 0$. From (C3), we have

$$\sum_{n=1}^{\infty}(1-\alpha_n)\|T_nx^* - x^*\| < \infty,$$

which implies that $\lim_{n\to\infty}(1-\alpha_n)\|T_nx^* - x^*\| = 0$. Since $\alpha_n \to 0$, it follows that

$$\lim_{n\to\infty}\|T_nx^* - x^*\| = 0.$$

Since

$$\begin{aligned}\|T_nx_n - x^*\| &\le \|T_nx_n - T_nx^*\| + \|T_nx^* - x^*\| \\ &\le \|x_n - x^*\| + \|T_nx^* - Tx^*\|,\end{aligned}$$

and $\{\|T_nx^* - x^*\|\}$ converges to 0, we conclude that $\{T_nx_n\}$ is bounded. Moreover,

$$\begin{aligned}\|f_nx_n - x^*\| &\le \|f_nx_n - f_nx^*\| + \|f_nx^* - x^*\| \\ &\le k_n\|x_n - x^*\| + \|f_nx^* - x^*\| + a_n,\end{aligned}$$

it follows that $\{f_nx_n\}$ is bounded.

Step 2: $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$.

Set $u_n := f_nx_n$. We write

$$x_{n+1} = (1-\beta_n)x_n + \beta_nz_n.$$

Observe that

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \|y_{n+1} - y_n\| \\ &= \|\alpha_{n+1}u_{n+1} + (1-\alpha_{n+1})T_{n+1}x_{n+1} - (\alpha_nu_n + (1-\alpha_n)T_nx_n)\| \\ &= \|\alpha_{n+1}u_{n+1} - \alpha_nu_n + (1-\alpha_{n+1})(T_{n+1}x_{n+1} - T_{n+1}x_n) \\ &\quad + (1-\alpha_{n+1})T_{n+1}x_n - (1-\alpha_n)T_nx_n\| \\ &= (1-\alpha_{n+1})\|x_{n+1} - x_n\| + \|T_{n+1}x_n - T_nx_n\| \\ &\quad + \alpha_{n+1}(\|T_{n+1}x_n\| + \|u_{n+1}\|) + \alpha_n(\|T_nx_n\| + \|u_n\|),\end{aligned}$$

which gives

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le{}& \|T_{n+1}x_n - T_nx_n\| + \alpha_{n+1}(\|T_{n+1}x_n\| + \|u_{n+1}\|) \\ &+ \alpha_n(\|T_nx_n\| + \|u_n\|).\end{aligned}$$

As we have shown in Step 1, $\{T_nx_n\}$ and $\{u_n\}$ are bounded. Observe that

$$\begin{aligned}\|T_{n+1}x_n - x^*\| &\le \|T_{n+1}x_n - T_{n+1}x^*\| + \|T_{n+1}x^* - x^*\| \\ &\le \|x_n - x^*\| + \|T_{n+1}x^* - x^*\|\end{aligned}$$

and

$$\begin{aligned}\|f_{n+1}x_n - x^*\| &\le \|f_{n+1}x_n - f_{n+1}x^*\| + \|T_{n+1}x^* - x^*\| \\ &\le \|x_n - x^*\| + \|f_{n+1}x^* - x^*\| + a_n.\end{aligned}$$

Thus, $\{f_{n+1}x_n\}$ and $\{T_{n+1}x_n\}$ are bounded. Hence,

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

By [25, Lemma 2.2], we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0,$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \beta_n \|z_n - x_n\| = 0.$$

Step 3: $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Note

$$\|y_n - T_n x_n\| = \big\|\alpha_n f_n x_n + (1-\alpha_n) T_n x_n - T_n x_n\big\| = \alpha_n \|f_n x_n - T_n x_n\|,$$

and hence

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| + \|T_n x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \beta_n \|z_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \beta_n \|y_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \alpha_n \beta_n \|f_n x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\big(\|x_n - Tx_n\| + \|Tx_n - T_n x_n\|\big) \\
&\quad + \alpha_n \beta_n \|f_n x_n - T_n x_n\| + \|T_n x_n - Tx_n\|,
\end{aligned}$$

which implies that

$$\beta_n \|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \alpha_n \beta_n \|f_n x_n - T_n x_n\| + 2\|T_n x_n - Tx_n\|.$$

Note $\alpha_n \to 0$ and $\|T_n x_n - Tx_n\| \to 0$, we conclude that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Step 4: $\limsup_{n\to\infty} \gamma_n \le 0$.

Note that

$$\begin{aligned}
\|z_n - Tz_n\| &\le \|z_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tz_n\| \\
&\le 2\|z_n - x_n\| + \|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

We take a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle fx^* - x^*, z_n - x^*\rangle = \lim_{n\to\infty} \langle fx^* - x^*, z_{n_i} - x^*\rangle \quad \text{and} \quad z_{n_i} \rightharpoonup z \in C \quad \text{as } i \to \infty.$$

Since $\{z_n\}$ is in C and $\|z_n - Tz_n\| \to 0$, we conclude, from Lemma 2.3 that $z \in F(T)$. Since $x^* = P_{F(T)} f(x^*)$, we obtain from Lemma 2.2 that

$$\limsup_{n\to\infty} \gamma_n = \limsup_{n\to\infty} \langle fx^* - x^*, z_{n_i} - x^*\rangle = \langle fx^* - x^*, z - x^*\rangle \le 0.$$

Step 5: $x_n \to x^*$.

Since $\{\|z_n - x^*\|\}$ is bounded, there exists $R_1 > 0$ such that $\|z_n - x^*\| \le R_1$ for all $n \in \mathbb{N}$. Noting that $z_n = P_C[y_n]$. Hence, from (3.1), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \langle z_n - y_n + y_n - x^*, z_n - x^*\rangle \\
&\le \langle y_n - x^*, z_n - x^*\rangle \\
&= \langle \alpha_n(f_n x_n - f_n x^* + f_n x^* - fx^* + fx^* - x^*) \\
&\quad + (1-\alpha_n)(T_n x_n - T_n x^* + T_n x^* - x^*), z_n - x^*\rangle \\
&\le [\alpha_n(\|f_n x_n - f_n x^*\| + \|f_n x^* - fx^*\|) \\
&\quad + (1-\alpha_n)(\|T_n x_n - T_n x^*\| + \|T_n x^* - Tx^*\|)]\|z_n - x^*\| \\
&\quad + \alpha_n\langle fx^* - x^*, z_n - x^*\rangle \\
&\le [\alpha_n(k_n\|x_n - x^*\| + \|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)(\|x_n - x^*\| + \|T_n x^* - Tx^*\|)]\|z_n - x^*\| + \alpha_n\gamma_n \\
&= (1-(1-k_n)\alpha_n)\|x_n - x^*\|\|z_n - x^*\| + [\alpha_n(\|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)\|T_n x^* - Tx^*\|]\|z_n - x^*\| + \alpha_n\gamma_n \\
&\le \frac{1-(1-k_n)\alpha_n}{2}(\|x_n - x^*\|^2 + \|z_n - x^*\|^2) + [\alpha_n(\|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)\|T_n x^* - Tx^*\|]R_1 + \alpha_n\gamma_n \\
&\le \frac{1-(1-k_n)\alpha_n}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|z_n - x^*\|^2 + [\alpha_n(\|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)\|T_n x^* - Tx^*\|]R_1 + \alpha_n\gamma_n,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - x^*\|^2 &\le (1-(1-k_n)\alpha_n)\|x_n - x^*\|^2 + 2[\alpha_n(\|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)\|T_n x^* - Tx^*\|]R_1 + 2\alpha_n\gamma_n.
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)(x_n - x^*) + \beta_n(z_n - x^*)\|^2 \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 \\
&\le (1-(1-k_n)\alpha_n\beta_n)\|x_n - x^*\|^2 + 2[\alpha_n\beta_n(\|f_n x^* - fx^*\| + a_n) \\
&\quad + (1-\alpha_n)\|T_n x^* - Tx^*\|]R_1 + 2\alpha_n\beta_n\gamma_n.
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\|f_n x^* - fx^*\|}{1-k_n} = 0$ and $\sum_{n=1}^{\infty}(1-\alpha_n)\|T_n x^* - Tx^*\| < \infty$, we conclude from Lemma 2.4(b) that $x_n \to x^*$. □

Remark 3.3 Theorem 3.2 has the following characterization for convergence analysis of (3.1):

(a) Iterates of (3.1) remains in the closed ball $B_R[x^*]$.
(b) The assumption (S2) is not required.

(c) (C4) is adopted for only for $x^* \in F(T)$. In particular, the condition '$\sum_{n=1}^{\infty} \|T_n z - Tz\| < \infty$ for all $z \in F(T)$' is adopted in [26, Theorem 3.1].

Thus, Theorem 3.2 is more general by nature. Therefore, Theorem 3.2 significantly extends and improves [26, Theorem 3.1] and [18, Theorem 3.2].

Theorem 3.2 remains true if condition (C4) is replaced with the condition that the mappings $\{T_n\}$ and T have common fixed points. In fact, we have

Theorem 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$, and $f : C \to H$ be a κ-contraction with $\kappa \in [0,1)$ such that $P_{F(T)} f(x^*) = x^* \in F(T)$. Let $\{f_n\}$ be a sequence of nearly contraction mappings from C into H with sequence $\{(k_n, a_n)\}$ in $[0,1) \times [0,\infty)$ such that $k_n \to \kappa$. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $F(T) = \bigcap_{n\in\mathbb{N}} F(T_n)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by* (3.1), *where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying* (C1), (C2), *and* (C3). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.1) *remains in the closed ball $B_R[x^*]$, where $R = \max\{\|x_1 - x^*\|, K^*\}$ and K^* is given in* (3.2).

(b) *If the assumption* (C5) *holds, then $\{x_n\}$ converges strongly to x^*.*

We now prove strong convergence of the sequence $\{x_n\}$ generated by (3.1) under condition (C6).

Theorem 3.5 *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, and $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself. Let $f : C \to H$ be a κ-contraction with $\kappa \in [0,1)$ such that $P_{F(T)} f(x^*) = x^* \in F(T)$ and $\{f_n\}$ be a sequence of k_n-contraction mappings from C into H such that $k_n \to \kappa$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by* (3.1), *where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying* (C1), (C2), (C3), *and* (C4). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.1) *remains in the closed ball $B_R[x^*]$, where*

$$R = \max\left\{ \|x_1 - x^*\|, \sup_{n\in\mathbb{N}} \frac{\|f_n x^* - x^*\|}{1 - k_n} \right\} + \sum_{n=1}^{\infty} (1 - \alpha_n) \|T_n x^* - x^*\|.$$

(b) *If the following assumption holds*:
(C6) $\sum_{n=1}^{\infty} \mathcal{D}_{B_R[x^*]}(T_n, T) < \infty$,
then $\{x_n\}$ converges strongly to x^*.

Proof We show that $\sum_{n=1}^{\infty} \mathcal{D}_{B_R[x^*]}(T_n, T) < \infty$ implies that $\lim_{n\to\infty} \|T_n v_n - T v_n\| = 0$ for all $\{v_n\}$ in $B_R[x^*]$. Let $\{w_n\}$ be a sequence in $B_R[x^*]$. Then

$$\sum_{n=1}^{\infty} \|T_n w_n - T w_n\| \le \sum_{n=1}^{\infty} \mathcal{D}_{B_R[x^*]}(T_n, T) < \infty.$$

It follows that $\lim_{n\to\infty} \|T_n w_n - T w_n\| = 0$. Thus, the condition (C5) in Theorem 3.2 holds. Therefore, Theorem 3.5 follows from Theorem 3.2. □

For a sequence $\{u_n\}$ in H with $u_n \to u \in H$, define $f_n : C \to H$ by

$$f_n x = u_n, \quad \forall x \in C.$$

Then each f_n is 0-contraction with $f_n x \to fx = u$. In this case algorithm (3.1) with $T_n = T$ reduces to

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C\left[\alpha_n u_n + (1 - \alpha_n)Tx_n\right], \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Corollary 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u \in H$ and $P_{F(T)}(u) = x^* \in F(T)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by* (3.3), *where $\{\alpha_n\}$ is a sequence in* $(0,1]$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying* (C1) *and* (C2). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.3) *remains in the closed ball $B_R[x^*]$, where* $R = \max\{\|x_1 - x^*\|, \sup_{n\in\mathbb{N}} \|u_n - x^*\|\}$.

(b) *$\{x_n\}$ converges strongly to x^*.*

Remark 3.7 If $u = 0$ in Corollary 3.6, then $\{x_n\}$ generated by Algorithm 3.3 converges strongly to the minimum norm solution of the FPP (1.6). Corollary 3.6 also provides a closed ball in which $\{x_n\}$ lies. Therefore, Corollary 3.6 significantly extends and improves [27, Theorem 3.1].

3.2 The split feasibility problem

In this section we apply Theorem 3.5 to solve the SFP (1.2). We begin with the ρ-distance:

Definition 3.8 Let C and Q be two closed convex subsets of a Hilbert space H and let ρ be a positive constant. The *ρ-distance* between C and Q is defined by

$$d_\rho(C, Q) = \sup_{\|x\|\leq\rho} \|P_C x - P_Q x\|.$$

By employing Theorem 3.5, we present a variable KM-like CQ algorithm (3.6) for finding solutions of the SFP (1.2) and prove its strong convergence.

Theorem 3.9 *Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $\{C_n\}$ and $\{Q_n\}$ be sequences of closed convex subsets of H_1 and H_2, respectively. Let $f : C \to H_1$ be a κ-contraction and $\{f_n\}$ be a sequence of k_n-contraction mappings from C into H_1 such that $k_n \to \kappa$. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint A^*. For $\gamma \in (0, 2/L)$, define*

$$T = P_C\big(I - \gamma A^*(I - P_Q)A\big), \tag{3.4}$$

and

$$T_n = P_{C_n}\big(I - \gamma A^*(I - P_{Q_n})A\big), \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Assume that SFP (1.2) *is consistent with* $P_{F(T)}fx^* = x^* \in F(T)$. *For given* $x_1 \in C$, *let* $\{x_n\}$ *be a sequence in C generated by the following variable KM-like CQ algorithm*:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\bigl[\alpha_n f_n x_n + (1-\alpha_n)P_{C_n}\bigl(I-\gamma A^*(I-P_{Q_n})A\bigr)x_n\bigr], \quad \forall n \in \mathbb{N}, \quad (3.6)$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *and* $\{\beta_n\}$ *is a sequence in* $(0,1)$ *satisfying* (C1), (C2), (C3), *and* (C4). *Then the following statements hold*:

(a) *The sequence* $\{x_n\}$ *generated by* (3.6) *remains in the closed ball* $B_R[x^*]$, *where*

$$R = \max\Bigl\{\|x_1 - x^*\|, \sup_{n\in\mathbb{N}}\bigl(\|f_n x^* - x^*\|/(1-k_n)\bigr)\Bigr\} + \sum_{n=1}^{\infty}(1-\alpha_n)\|T_n x^* - x^*\|.$$

(b) *If* $\overline{\rho} = \max\{\|Ax\|, \|(I-\gamma A^*(I-P_Q)A)x\| : x \in B_R[x^*]\}$ *and the following assumption holds*:
(C7) $\sum_{n=1}^{\infty} d_{\overline{\rho}}(C_n, C) < \infty$ *and* $\sum_{n=1}^{\infty} d_{\overline{\rho}}(Q_n, Q) < \infty$,
then $\{x_n\}$ *converges strongly to* x^*.

Proof (a) Since $\gamma \in (0, 2/L)$, T and T_n for all $n \in \mathbb{N}$ are nonexpansive mappings and $F(T) \neq \emptyset$ because SFP (1.2) is consistent. Hence this part follows from Theorem 3.5(a).

(b) Assume that

$$\overline{\rho} = \max\bigl\{\|Ax\|, \bigl\|\bigl(I-\gamma A^*(I-P_Q)A\bigr)x\bigr\| : x \in B_R\bigl[x^*\bigr]\bigr\}.$$

Now, let $x \in H_1$ be such that $x \in B_R[x^*]$. Since each P_{C_n} is the nonexpansive, we have

$$\begin{aligned}
\|T_n x - Tx\| &= \bigl\|P_{C_n}\bigl(I-\gamma A^*(I-P_{Q_n})A\bigr)x - P_C\bigl(I-\gamma A^*(I-P_Q)A\bigr)x\bigr\| \\
&\le \bigl\|P_{C_n}\bigl(I-\gamma A^*(I-P_{Q_n})A\bigr)x - P_{C_n}\bigl(I-\gamma A^*(I-P_Q)A\bigr)x\bigr\| \\
&\quad + \bigl\|P_{C_n}\bigl(I-\gamma A^*(I-P_Q)A\bigr)x - P_C\bigl(I-\gamma A^*(I-P_Q)A\bigr)x\bigr\| \\
&\le \gamma\bigl\|A^*(P_{Q_n}Ax - P_Q Ax)\bigr\| \\
&\quad + \bigl\|P_{C_n}\bigl(I-\gamma A^*(I-P_Q)A\bigr)x - P_C\bigl(I-\gamma A^*(I-P_Q)A\bigr)x\bigr\| \\
&\le \gamma\|A\|\|P_{Q_n}Ax - P_Q Ax\| + d_{\overline{\rho}}(C_n, C) \\
&\le \gamma\|A\| d_{\overline{\rho}}(Q_n, Q) + d_{\overline{\rho}}(C_n, C).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty}\mathcal{D}_{B_R[x^*]}(T_n, T) &= \sum_{n=1}^{\infty}\sup_{x\in B_R[x^*]}\|T_n x - Tx\| \\
&\le \sum_{n=1}^{\infty} d_{\overline{\rho}}(C_n, C) + \gamma\|A\|\sum_{n=1}^{\infty} d_{\overline{\rho}}(Q_n, Q) < \infty.
\end{aligned}$$

Hence condition (C6) in Theorem 3.5 holds. Therefore, Theorem 3.9(b) follows from Theorem 3.5(b). □

For a sequence $\{u_n\}$ in H_1 with $u_n \to 0 \in H_1$, define $f_n : C \to H_1$ by

$$f_n x = u_n, \quad \forall x \in C.$$

Then each f_n is 0-contraction with $f_n x \to fx = 0$. In this case variable KM-like CQ algorithm (3.6) reduces to the following *variable KM-like CQ algorithm*:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\bigl[\alpha_n u_n + (1-\alpha_n)P_{C_n}\bigl(I - \gamma A^*(I - P_{Q_n})A\bigr)x_n\bigr], \quad \forall n \in \mathbb{N}. \tag{3.7}$$

We now present strong convergence of the variable KM-like CQ algorithm (3.7) to the minimum norm solution of the SFP (1.2).

Corollary 3.10 *Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $\{C_n\}$ and $\{Q_n\}$ be sequences of closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with the adjoint A^*. For $\gamma \in (0, 2/L)$, define T and T_n by* (3.4) *and* (3.5), *respectively. Assume that the SFP* (1.2) *is consistent with $P_{F(T)}(0) = x^* \in F(T)$. For given $x_1 \in C$ and a sequence $\{u_n\}$ in H_1 with $u_n \to 0 \in H_1$, let $\{x_n\}$ be a sequence in C generated by a variable KM-like CQ algorithm* (3.7), *$\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying* (C1), (C2), *and* (C4). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.7) *remains in the closed ball $B_R[x^*]$, where*

$$R = \max\bigl\{\|x_1 - x^*\|, \|x^*\|\bigr\} + \sum_{n=1}^{\infty}(1-\alpha_n)\|T_n x^* - x^*\|.$$

(b) *If $\overline{\rho} = \max\{\|Ax\|, \|(I - \gamma A^*(I - P_Q)A)x\| : x \in B_R[x^*]\}$ and the assumption* (C7) *holds, then $\{x_n\}$ converges strongly to x^*.*

Corollary 3.10 significantly extends and improves [11, Theorem 3.1].

3.3 The constrained multiple-sets split feasibility problem

In this section, we consider the following *multiple-sets split feasibility problem* which models the intensity-modulated radiation therapy [6] and has recently been investigated by many researchers, see, for example, [1, 3, 6, 8–14] and the references therein.

Let H_1 and H_2 be two Hilbert spaces and let r and p be two natural numbers. For each $i \in \{1, 2, \dots, p\}$, let C_i be a nonempty closed convex subset of H_1 and for each $j \in \{1, 2, \dots, r\}$, let Q_j be a nonempty closed convex subset of H_2. Further, for each $j \in \{1, 2, \dots, r\}$, let $A_j : H_1 \to H_2$ be a bounded linear operator and Ω be a closed convex subset of H_1. The (constrained) multiple-sets split feasibility problem (MSSFP) is to find a point $x^* \in \Omega$ such that

$$x^* \in C := \bigcap_{i=1}^{p} C_i \quad \text{and} \quad A_j x^* \in Q_j, \quad j \in \{1, 2, \dots, r\}. \tag{3.8}$$

When $p = r = 1$, then the MSSFP (3.8) reduces to the SFP (1.2).

The split feasibility problem (SFP) and multiples-set split feasibility problem (MSSFP) model image retrieval [28] and intensity-modulated radiation therapy [6], and they have recently been investigated by many researchers.

For each $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,r\}$, let $\overline{\alpha}_i$ and $\overline{\beta}_j$ be two positive numbers. Let $B : H_1 \to H_1$ be the gradient $\nabla\psi$ of a convex and continuously differentiable function $\psi : H_1 \to \mathbb{R}$ defined by

$$\psi(x) := \frac{1}{2}\sum_{i=1}^{p}\overline{\alpha}_i\|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{r}\overline{\beta}_j\|A_jx - P_{Q_j}A_jx\|^2, \quad \forall x \in H_1. \tag{3.9}$$

Following [28], we see that

$$Bx := \sum_{i=1}^{p}\overline{\alpha}_i(I - P_{C_i})x + \sum_{j=1}^{r}\overline{\beta}_jA_j^*(I - P_{Q_j})A_jx, \quad \forall x \in H_1, \tag{3.10}$$

where A_j^* is the adjoint of A_j, $j \in \{1,2,\dots,r\}$. The nonexpansivity of $I - P_C$ implies that B is a Lipschitzian mapping with Lipschitz constant

$$\mathcal{L}^* := \sum_{i=1}^{p}\overline{\alpha}_i + \sum_{j=1}^{r}\overline{\beta}_j\|A_j\|^2. \tag{3.11}$$

Thus, variable KM-like CQ algorithm can be developed to solve the MSSFP (3.8). Let $\{\Omega_n\}$, $\{C_{n,i}\}$ and $\{Q_{n,j}\}$ be the sequences of closed convex sets, which are viewed as perturbations for the closed convex sets Ω, $\{C_i\}$ and $\{Q_j\}$, respectively.

We now present an iterative algorithm for solving the MSSFP (3.8).

Theorem 3.11 *Let $f : \Omega \to H_1$ be a κ-contraction and let $\{f_n\}$ be a sequence of k_n-contraction mappings from Ω into H_1 such that $k_n \to \kappa$. For $\gamma \in (0, 2/L^*)$, define*

$$Tx = P_{\Omega_n}\left(x - \gamma\left(\sum_{i=1}^{p}\overline{\alpha}_i(I - P_{C_i})x + \sum_{j=1}^{r}\overline{\beta}_jA_j^*(I - P_{Q_j})A_jx\right)\right) \tag{3.12}$$

and

$$T_nx = P_{\Omega_n}\left(x - \gamma\left(\sum_{i=1}^{p}\overline{\alpha}_i(I - P_{C_{i,n}})x + \sum_{j=1}^{r}\overline{\beta}_jA_j^*(I - P_{Q_{j,n}})A_jx\right)\right), \quad n \in \mathbb{N}. \tag{3.13}$$

Assume that the MSSFP (3.8) *is consistent with $P_{F(T)}fx^* = x^* \in F(T)$. For given $x_1 \in \Omega$, let $\{x_n\}$ be a sequence in Ω generated by*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_nP_C\big[\alpha_nf_nx_n + (1 - \alpha_n)T_nx_n\big], \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying* (C1), (C2), (C3), *and* (C4). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.12) *remains in the closed ball $B_R[x^*]$, where*

$$R = \max\left\{\|x_1 - x^*\|, \sup_{n\in\mathbb{N}}\big(\|f_nx^* - x^*\|/(1 - k_n)\big)\right\} + \sum_{n=1}^{\infty}(1 - \alpha_n)\|T_nx^* - x^*\|.$$

(b) *If* $\overline{\rho} = \max\{\max_{1\le j\le p}\|A_j x\|, \|(I-\gamma\nabla\psi)x\| : x\in B_R[x^*]\}$ *and for each* $i\in\{1,2,\ldots,p\}$ *and* $j\in\{1,2,\ldots,r\}$, *the following assumption holds*:
(C8) $\sum_{n=1}^{\infty} d_{\overline{\rho}}(\Omega_n,\Omega) < \infty$, $\sum_{n=1}^{\infty} D_{B_R[x^*]}(P_{C_{i,n}}, P_{C_i}) < \infty$ *and* $\sum_{n=1}^{\infty} d_{\overline{\rho}}(Q_{n,i}, Q) < \infty$,
then $\{x_n\}$ *converges strongly to* x^*.

Proof (a) Define

$$\psi_n(x) := \frac{1}{2}\sum_{i=1}^{p}\overline{\alpha}_i\|x-P_{i,n}x\|^2 + \frac{1}{2}\sum_{j=1}^{r}\overline{\beta}_j\|A_jx-P_{Q_{j,n}}A_jx\|^2.$$

The gradients of ψ and ψ_n are given by

$$\nabla\psi(x) = \sum_{i=1}^{p}\overline{\alpha}_i(I-P_{C_i})x + \sum_{j=1}^{r}\overline{\beta}_jA_j^*(I-P_{Q_j})A_jx$$

and

$$\nabla\psi_n(x) = \sum_{i=1}^{p}\overline{\alpha}_i(I-P_{C_{i,n}})x + \sum_{j=1}^{r}\overline{\beta}_jA_j^*(I-P_{Q_{j,n}})A_jx.$$

Hence, from (3.12) and (3.13), we have

$$Tx = P_\Omega\bigl(x-\gamma\nabla\psi(x)\bigr),$$

and

$$T_nx = P_{\Omega_n}\bigl(x-\gamma\nabla\psi_n(x)\bigr), \quad n\in\mathbb{N}.$$

Since $\gamma\in(0,2/\mathcal{L}^*)$, T and T_n, for all $n\in\mathbb{N}$, are nonexpansive mappings, and $F(T)\neq\emptyset$ because the MSSFP (3.8) is consistent. Hence, this part follows from Theorem 3.5(a).

(b) Assume that

$$\overline{\rho} = \max\Bigl\{\max_{1\le j\le p}\|A_jx\|, \bigl\|(I-\gamma\nabla\psi)x\bigr\| : x\in B_R\bigl[x^*\bigr]\Bigr\}.$$

Let $x\in H_1$ be such that $x\in B_R[x^*]$. Since each P_{C_n} is the nonexpansive, we have

$$\begin{aligned}
\|T_nx-Tx\| &= \bigl\|P_{\Omega_n}(I-\gamma\nabla\psi_n)x - P_\Omega(I-\gamma\nabla\psi)x\bigr\| \\
&\le \bigl\|P_{\Omega_n}(I-\gamma\nabla\psi_n)x - P_{\Omega_n}(I-\gamma\nabla\psi)x\bigr\| \\
&\quad + \bigl\|P_{\Omega_n}(I-\nabla\psi)x - P_\Omega(I-\gamma\nabla\psi)x\bigr\| \\
&\le \gamma\bigl\|\nabla\psi_n(x)-\nabla\psi(x)\bigr\| + \bigl\|P_{\Omega_n}(I-\gamma\nabla\psi)x - P_\Omega(I-\gamma\nabla\psi)x\bigr\| \\
&\le \gamma\sum_{i=1}^{p}\overline{\alpha}_i\|P_{C_{i,n}}x-P_{C_i}x\| + \sum_{j=1}^{p}\overline{\beta}_j\bigl\|A_j^*\bigr\|\|P_{Q_{j,n}}A_jx-P_{Q_j}A_jx\| + d_{\overline{\rho}}(\Omega_n,\Omega) \\
&\le \gamma\sum_{i=1}^{p}\overline{\alpha}_iD_{B_R[x^*]}(P_{C_{i,n}},P_{C_i}) + \sum_{j=1}^{p}\|A_j\|\overline{\beta}_jd_{\overline{\rho}}(P_{Q_{j,n}},P_{Q_j}) + d_{\overline{\rho}}(\Omega_n,\Omega).
\end{aligned}$$

By the assumptions, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\mathcal{D}_{B_R[x^*]}(T_n,T) &= \sum_{n=1}^{\infty}\sup_{x\in B_R[x^*]}\|T_nx-Tx\| \\ &\le \gamma\sum_{i=1}^{p}\overline{\alpha}_i\sum_{n=1}^{\infty}D_{B_R[x^*]}(P_{C_{i,n}},P_{C_i}) \\ &\quad+\sum_{j=1}^{p}\|A_j\|\overline{\beta}_j\sum_{n=1}^{\infty}d_{\overline{\rho}}(P_{Q_{j,n}},P_{Q_j})+\sum_{n=1}^{\infty}d_{\overline{\rho}}(\Omega_n,\Omega)<\infty.\end{aligned}$$

Hence condition (C6) in Theorem 3.5 holds. Therefore, Theorem 3.9(b) follows from Theorem 3.5(b). □

Theorem 3.11 significantly extends and improves [12, Theorem 1].

Finally, we present strong convergence of variable KM-like CQ algorithm (3.7) to the minimum norm solution of the MSSFP (3.8).

Corollary 3.12 *Define T and T_n by* (3.12) *and* (3.13), *respectively. Assume that the MSSFP* (3.8) *is consistent with $P_{F(T)}(0)=x^*\in F(T)$. For given $x_1\in C$ and a sequence $\{u_n\}$ in H_1 with $u_n\to 0\in H_1$, let $\{x_n\}$ be a sequence in C generated by the following variable KM-like CQ algorithm*:

$$x_{n+1}=(1-\beta_n)x_n+\beta_nP_C\big[\alpha_nu_n+(1-\alpha_n)T_nx_n\big]\quad \textit{for all } n\in\mathbb{N},$$

where $0<\gamma<2/\mathcal{L}^*$, $\{\alpha_n\}$ is a sequence in $(0,1]$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying* (C1), (C2), *and* (C4). *Then the following statements hold*:

(a) *The sequence $\{x_n\}$ generated by* (3.12) *remains in the closed ball $B_R[x^*]$, where*

$$R=\max\big\{\|x_1-x^*\|,\|x^*\|\big\}+\sum_{n=1}^{\infty}(1-\alpha_n)\|T_nx^*-x^*\|.$$

(b) *If $\overline{\rho}=\max\{\max_{1\le j\le p}\|A_jx\|,\|(I-\gamma\nabla\psi)x\|:x\in B_R[x^*]\}$ and for each $i\in\{1,2,\dots,p\}$ and $j\in\{1,2,\dots,r\}$, the assumption* (C8) *holds, then $\{x_n\}$ converges strongly to x^*.*

4 Numerical examples

In order to demonstrate the effectiveness, realization, and convergence of algorithm of Theorem 3.2, we consider the following example.

Example 4.1 Let $H=\mathbb{R}$ and $C=[0,1]$. Let T be a self-mapping on C defined by $Tx=1-x$ for all $x\in C$. Define $\{\alpha_n\}$ in $(0,1)$ by $\alpha_n=\frac{1}{n+1}$ and $\{\beta_n\}$ by $\beta_n=\frac{1}{2}$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, define $f_n:C\to H$ by (2.1). It is shown in Example 2.1 that $\{f_n\}$ is a sequence of nearly contraction mappings from C into H with sequence $\{(k_n,a_n)\}$, where $k_n=\frac{1}{n+1}$ and $a_n=\frac{5}{2(n+1)}$. It is easy to see that $\{f_n\}$ converges pointwise to f, where $f(x)=0$ for all $x\in C$. Note $k_n\to\kappa=0$, $F(T)=\{x^*\}=\{1/2\}$, and $\lim_{n\to\infty}f_nx^*=fx^*$. It can be observed that all the assumptions of Theorem 3.2 are satisfied and the sequence $\{x_n\}$ generated by (3.1) with

Table 1 The numerical results for initial guess $x_1 = 0, 0.2, 0.8, 1$

n	$x_1 = 0$	$x_1 = 0.2$	$x_1 = 0.8$	$x_1 = 1$
5	0.452291666666667	0.452604166666667	0.514843750000000	0.514947916666667
10	0.476041066961784	0.476041067553836	0.476047944746260	0.476047944894273
15	0.483829591828978	0.483829591828978	0.483829591829845	0.483829591829845
20	0.487787940564059	0.487787940564059	0.487787940564059	0.487787940564059
25	0.490187554131032	0.490187554131032	0.490187554131032	0.490187554131032
30	0.491798400218960	0.491798400218960	0.491798400218960	0.491798400218960
35	0.492954698188619	0.492954698188619	0.492954698188619	0.492954698188619
40	0.493825130132048	0.493825130132048	0.493825130132048	0.493825130132048
45	0.494504074844059	0.494504074844059	0.494504074844059	0.494504074844059
50	0.495048473664881	0.495048473664881	0.495048473664881	0.495048473664881

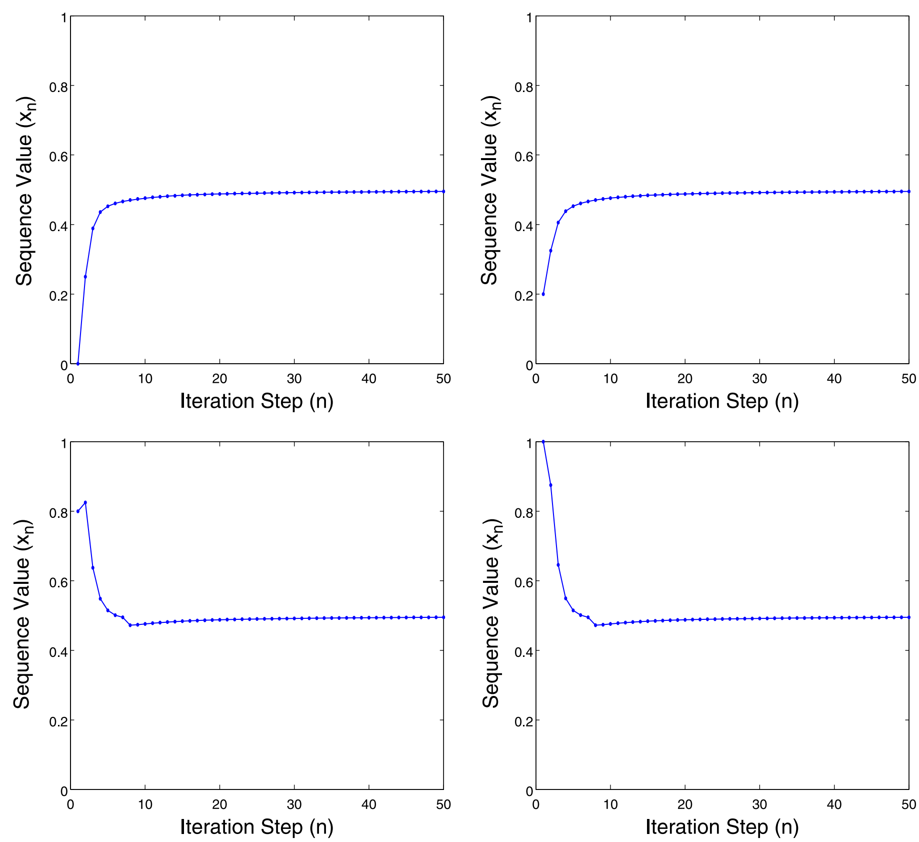

Figure 1 The convergence comparison of different initial values $x_1 = 0, 0.2, 0.8, 1$.

$T_n = T$ converges to $\frac{1}{2}$. In fact, under the above assumptions, the algorithm (3.1) can be simplified as follows:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n f_n x_n + (1 - \alpha_n)(1 - x_n), \\ x_{n+1} = \frac{x_n + P_C[y_n]}{2} \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{4.1}$$

The projection point of y_n onto C can be expressed as

$$P_C[y_n] = \begin{cases} 0, & \text{if } y_n < 0; \\ y_n, & \text{if } y_n \in C; \\ 1, & \text{if } y_n > 0. \end{cases}$$

The iterates of algorithm (4.1) for initial guess $x_1 = 0, 0.2, 0.8, 1$ are shown in Table 1. From Table 1, we see that the iterations converge to 1/2 which is the unique fixed point of T. The convergence of each iteration is also shown in Figure 1 for comparison.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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