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Porosity of the free boundary for quasilinear parabolic variational problems

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Abstract
In this paper we consider a certain quasilinear parabolic variational problem with identically zero constraint. By using intrinsic scaling, the exact growth of the solution near the free boundary is established. A consequence of this is that the time level of the free boundary is porous (in N dimensions) and therefore its Hausdorff dimension is less than N. In particular, the N-dimensional Lebesgue measure of the free boundary is zero for each time level.

MSC: 35K86; 35K92; 35K65; 35K67

Keywords: parabolic equation; obstacle problem; free boundary; porosity

1 Introduction and main theorem

In this paper we consider a variational inequality for the quasilinear parabolic operator

$$\operatorname{div} a(x,\nabla u) - \partial_t u,$$

giving rise to a free boundary. Our purpose is to analyze the free boundary for a large class of obstacle problems associated with degenerate ($2 < p < \infty$) and non-degenerate ($1 < p \le 2$) parabolic equations. Therefore, let us start with the formulation of the problem in the weak sense. Let Ω be an open bounded domain of $\mathbb{R}^N$ ($N \ge 2$), $\Omega_T = \Omega \times (0,T)$. Denote the parabolic space by $V^{1,p}(\Omega_T)$, see [1],

$$V^{1,p}(\Omega_T) = L^\infty\big(0,T;L^1(\Omega)\big) \cap L^p\big(0,T;W^{1,p}(\Omega)\big) \quad (1 < p < \infty).$$

The Steklov average v_h of a function is defined by

$$v_h(x,t) = \frac{1}{h}\int_t^{t+h} v(x,\tau)\,\mathrm{d}\tau \quad \text{for } t \in (0,T-h],$$

and $v_h = 0$ for $t > T-h$. Let the function $a(x,\eta) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be Lipschitz continuous in $x \in \Omega$ and continuously differentiable in $\eta \in \mathbb{R}^N \setminus \{0\}$. Given bounded functions f, θ and the obstacle 0, the variational problems are to find a function

$$u \in \mathcal{K}_\theta := \mathcal{K}_\theta(p) = \big\{w : w \in V^{1,p}(\Omega_T), \forall t\ w = \theta \text{ on } \partial_p\Omega_T, w \ge 0 \text{ a.e. in } \Omega_T\big\},$$

where $\partial_p \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T])$, such that (for $h > 0$ and $0 < t < t+h < T$)

$$\int_\Omega \partial_t u_h (w-u)\,\mathrm{d}x + \int_\Omega \big(a(x,\nabla u)\big)_h \cdot \nabla(w-u)\,\mathrm{d}x + \int_\Omega f_h(w-u)\,\mathrm{d}x \geq 0, \tag{1}$$

a.e. in $t \in (0,T)$, and for all $w \in \mathcal{K}_\theta$.

Under certain conditions on f and θ, we will show that the free boundary of the solution to the variational problems (1) is porous for each t-level cut, which implies that the t-cuts of the free boundary has Lebesgue measure zero.

As is well known, in the obstacle problems associated with elliptic operators, to obtain the porosity of the free boundary one needs to prove that every solution has a certain growth rate near the free boundary; see [2–4] for instance. When focusing on p-parabolic variational problem ($1 < p < \infty$), we remark that due to the lack of the strong minimum principle or the Harnack inequality one cannot inherit each technique from the elliptic obstacle problems, and we need further arguments to establish the growth rate of solutions near the free boundary. In p-parabolic variational problems ($p \geq 2$), Shahgholian overcame this difficulty by using Hölder's estimates for solutions of parabolic equations. As a by-product, the author obtained the porosity of the free boundary for $p \geq 2$; see [5]. A fact that should be noticed is: although neither the technique of Hölder's estimates nor Harnack inequality can be applied to get the growth of solutions in the case of $1 < p < 2$ in [5], a 'minimum principle (in spatial variables)' for singular parabolic equations given in [6] (Lemma 2.3) may be used in our problem as a substitute tool at this step. Thus in this paper, using the main idea of [5] and techniques of compactness, we are interested in studying the porosity of the free boundary in a large class of variational problems governed by quasilinear parabolic operators. Our result contains not only the case of $p \geq 2$, but the singular case of $1 < p < 2$ as well, which is naturally an extension of [5].

Throughout this paper, unless specified, we always assume $1 < p < \infty$. We make the standard structural conditions on the function $a(x,\eta)$ for some positive constants γ_0, γ_1, namely,

(a_1) $a_i(x,0) = 0$,
(a_2) $\sum_{i,j=1}^N \frac{\partial a_i}{\partial \eta_j}(x,\eta)\xi_i\xi_j \geq \gamma_0 |\eta|^{p-2}|\xi|^2$,
(a_3) $\sum_{i,j=1}^N |\frac{\partial a_i}{\partial \eta_j}(x,\eta)| \leq \gamma_1 |\eta|^{p-2}$,
(a_4) $\sum_{i,j=1}^N |\frac{\partial a_i}{\partial x_j}(x,\eta)| \leq \gamma_1 |\eta|^{p-1}$,

for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, and all $\xi \in \mathbb{R}^N$.

Remark 1.1 Assumptions (a_1)-(a_4) imply that (see [7, 8] for instance)

$$a(x,\eta)\eta \geq \frac{\gamma_0}{p-1}|\eta|^p,$$

$$\big|a(x,\eta)\big| \leq \frac{\gamma_1}{p-1}|\eta|^{p-1},$$

$$\big\langle a(x,\eta_1) - a(x,\eta_2), \eta_1 - \eta_2\big\rangle \geq 0,$$

for a.e. $x \in \Omega$ and all $\eta, \eta_1, \eta_2 \in \mathbb{R}^N$. Thus the structural conditions for quasilinear operators in [1] are satisfied, which are needed in this paper.

Suppose that f and θ are bounded continuous functions on the closure of Ω_T. To establish the results obtained in this paper, further conditions on f and θ are imposed as follows.

(f) $0 < \lambda_0 \le f \le \Lambda$ in Ω_T, $f(x,t)$ is monotone non-increasing in t;
(θ) $\theta(x,0) = 0$, $\theta(x,t)$ is monotone non-decreasing in t.

Let us gather some properties (needed here) for the solution u to the variational inequality (1). The following theorem can be proven by classical techniques; we refer the reader to [2, 4, 5] for sketches of the proofs.

Classical theorem *There exists a unique solution u to the variational problem* (1) *in $\mathcal{K}_\theta$ with*

$$\begin{aligned} &0 \le u \le \|\theta\|_{\infty,\Omega_T} \quad \text{in } \Omega_T, \\ &\partial_t u \ge 0 \quad \text{in } \{u > 0\}. \end{aligned}$$

Moreover, u satisfies

$$\operatorname{div} a(x,\nabla u) - \partial_t u = g \quad \text{in } \{u > 0\}.$$

weakly in Ω_T with $g \in L^\infty(\Omega_T)$ satisfying

$$f\chi_{\{u>0\}} \le g \le f\chi_{\overline{\{u>0\}}} \quad \text{a.e. in } \Omega_T.$$

We recall the concept of porosity; see [2, 5].

Porosity *A set E in $\mathbb{R}^N$ is called porous with porosity constant δ if there is an $r_0 > 0$ such that for each $x \in E$ and $0 < r < r_0$ there is a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus E$.*

According to [9], a porous set has Hausdorff dimension not exceeding $N - C\delta^N$; thus, it is of Lebesgue measure zero.

Now we state the main theorem in this paper.

Theorem 1.1 *Let u be the solution to problem* (1) *in $\mathcal{K}_\theta$. Then for every compact set $K \subset \Omega_T$*

$$c_0 r^{\frac{p}{p-1}} \le \sup_{B_r(x_0)} u(\cdot,t_0) \le C_0 r^{\frac{p}{p-1}}, \quad \forall (x_0,t_0) \in \partial\{u>0\} \cap K.$$

Furthermore, the intersection $\partial\{u>0\} \cap K \cap \{t = t_0\}$ is porous (in $\mathbb{R}^N$) with the porosity constant

$$\delta = \delta\bigl(\|\theta\|_{\infty,\Omega_T}, \lambda_0, \Lambda_0, \operatorname{dist}(K, \partial_p\Omega_T), \gamma_0, \gamma_1, p\bigr).$$

Here c_0 depends on p, λ_0, γ_1, and C_0 depends on p, λ_0, Λ_0, γ_0, γ_1, $\|\theta\|_{\infty,\Omega_T}$.

2 A class of functions on the unit cylinder

We first let $q = \frac{p}{p-1}$ and $Q_r(z,s) = B_r(z) \times (-r^q + s, r^q + s)$ be the cylinder in $\mathbb{R}^{N+1}$. Write $Q_1 = Q_1(0,0)$, the unit cylinder. Due to the local character of the results obtained in this paper (Theorem 1.1), we may consider the following local formulation. We say that a function u is in $W^{1,p}(Q_1)$ belongs to the class $\mathcal{G}_a = \mathcal{G}_a(p, \gamma_0, \gamma_1)$ if

$$\left\| \operatorname{div} a(x, \nabla u) - \partial_t u \right\|_{\infty, Q_1} \le 1; \tag{2a}$$

$$0 \le u \le 1, \quad \text{a.e. in } Q_1; \tag{2b}$$

$$u(0,0) = 0; \tag{2c}$$

$$\partial_t u \ge 0 \quad \text{a.e. in } Q_1. \tag{2d}$$

Condition (2a) should be understood in the weak sense, *i.e.*, $\operatorname{div} a(x, \nabla u) - \partial_t u = h$ weakly for $h \in L^\infty(Q_1)$ with $\|h\|_{\infty, Q_1} \le 1$. Condition (2c) makes sense since (2a) and (2b) provide that $u \in C_x^{1,\alpha} \cap C_t^{0,\alpha}(Q_{\frac{1}{2}})$ and $u \in C^{1,\alpha}(Q_{\frac{1}{2}})$ for some $\alpha \in (0,1)$ in the case of $p \ge 2$ and $1 < p < 2$, respectively (see *e.g.* Chapter IX of [1]).

In this section, we discuss the behavior of solutions to problem (1) and functions in $\mathcal{G}_a$ near the free boundary.

2.1 Non-degeneracy of the solution near the free boundary

The following result gives a description of the solution u to problem (1) showing that it cannot grow too slowly near the free boundary. This property and the growth rate of the elements in $\mathcal{G}_a$ will pave the way to establish the porosity of the free boundary.

Lemma 2.1 *Let $u \in W^{1,p}(Q_1)$ be a non-negative continuous function in Q_1, satisfying*

$$\operatorname{div} a(x, \nabla u) - \partial_t u = f$$

weakly in $U^+ = \{u > 0\}$. Then for every $(z,s) \in \overline{U^+}$ and $r > 0$ with $Q_r(z,s) \subset Q_1$

$$\sup_{(x,t) \in \partial_p Q_r^-(z,s)} u(x,t) \ge c_0 r^{\frac{p}{p-1}} + u(z,s),$$

where $Q_r^-(z,s) = B_r(z) \times (s - r^q, s)$, c_0 is a positive constant depending only on p, λ_0, γ_1.

Proof First suppose that $(z,s) \in U^+$, and for small $\varepsilon > 0$ set

$$u_\varepsilon(x,t) = u(x,t) - (1-\varepsilon) u(z,s)$$

and

$$v(x,t) = C_1 |x - z|^{\frac{p}{p-1}} - C_2 (t - s),$$

where C_1, C_2 are positive constants, depending only on p, λ_0, γ_1, such that

$$\gamma_1 C_1^{p-1} \left(\frac{p}{p-1} \right)^p + C_2 \le \lambda_0.$$

We claim that for C_1, C_2

$$\operatorname{div} a(x,\nabla v) - \partial_t v \le \lambda_0, \quad \forall (x,t) \in U^+ \cap Q_r^-(z,s). \tag{3}$$

To prove (3), we need to calculate ∇v and the divergence of $a(x,\nabla v)$. Indeed,

$$\nabla v(x,t) = \frac{pC_1}{p-1}|x-z|^{\frac{2-p}{p-1}}(x-z), \qquad \left|D_{ij}v(x,t)\right| \le \frac{pC_1}{(p-1)^2}|x-z|^{\frac{2-p}{p-1}}.$$

One may verify that

$$\begin{aligned}
\operatorname{div} a(x,\nabla v) - \partial_t v &= \sum_{i=1}^N \frac{\partial a_i}{\partial x_i}(x,w) + \sum_{i,j=1}^N \frac{\partial a_i}{\partial \eta_j}(x,w)\frac{\partial w_j}{\partial x_i}(x) + C_2 \\
&\le \gamma_1|w|^{p-1} + \gamma_1|w|^{p-2}\frac{pC_1}{(p-1)^2}|x-z|^{\frac{2-p}{p-1}} + C_2 \\
&\le \gamma_1\left(\frac{pC_1}{p-1}\right)^{p-1}\left(|x-z| + \frac{1}{p-1}\right) + C_2 \\
&\le \gamma_1\left(\frac{pC_1}{p-1}\right)^{p-1}\left(1 + \frac{1}{p-1}\right) + C_2 \\
&\le \lambda_0,
\end{aligned}$$

where $w(x,t) = \nabla v(x,t) = \frac{pC_1}{p-1}|x-z|^{\frac{2-p}{p-1}}(x-z)$.

Notice that $\operatorname{div} a(x,\nabla u) - \partial_t u = \operatorname{div} a(x,\nabla u_\varepsilon) - \partial_t u_\varepsilon$ in $U^+ \cap Q_r^-(z,s)$. Recall condition (f), it follows that

$$\operatorname{div} a(x,\nabla v) - \partial_t v \le \operatorname{div} a(x,\nabla u_\varepsilon) - \partial_t u_\varepsilon \quad \text{in } \Omega_+ \cap Q_r^-(z,s).$$

It is easy to see $u_\varepsilon(x,t) = -(1-\varepsilon)u(z,s) \le 0$ on ∂U^+ and $v(x,t) \ge 0$ for any $t \le s$, thus $u_\varepsilon \le v$ on $\partial U^+ \cap Q_r^-(z,s)$. If also $u_\varepsilon \le v$ on $\partial Q_r^-(z,s) \cap U^+$, then we get by the comparison principle

$$u_\varepsilon \le v \quad \text{in } Q_r^-(z,s) \cap U^+.$$

But $u_\varepsilon(z,s) = \varepsilon u(z,s) > 0 = v(z,s)$, which is a contradiction. Therefore there exists some point $(y,\tau) \in \partial Q_r^-(z,s)$ such that

$$u_\varepsilon(y,\tau) \ge v(y,\tau) = c_0 r^{\frac{p}{p-1}},$$

where $c_0 = \min\{C_1, C_2\}$. Letting $\varepsilon \to 0$ we obtain the desired result for all $(z,s) \in U^+$, and by continuity for all $(z,s) \in \overline{U^+}$. □

2.2 Growth rate of the function u in $\mathcal{G}_a$

In this subsection we prove that every function u in $\mathcal{G}_a$ cannot grow too fast near the free boundary but has a growth rate of order $q = \frac{p}{p-1}$ (Theorem 2.1).

First we define the supremum norm of u over the cylinder $Q_r^-(z,s)$ as [5] by setting

$$S(r,u,z,s) = \sup_{x\in Q_r^-(z,s)} u(x,t), \quad \text{and} \quad S(r,u) = \sup_{x\in Q_r^-(0,0)} u(x,t).$$

For each $u \in \mathscr{G}_a$, define the set $\mathbb{M}_a(u,z,s)$ by setting

$$\mathbb{M}_a(u,z,s) = \{ j \in \mathbb{N}; AS(2^{-j-1},u,z,s) \geq S(2^{-j},u,z,s) \},$$

where $A = 2^q \max\{1, \frac{1}{c_0}\}$ with $q = \frac{p}{p-1}$, and c_0 as in Lemma 2.1. For simplicity, we write $\mathbb{M}_a(u) = \mathbb{M}_a(u,0,0)$.

It should be noticed that $\mathbb{M}_a(u) \neq \emptyset$ for all $u \in \mathscr{G}_a$ since $0 \in \mathbb{M}_a(u)$. Indeed, it follows from Lemma 2.1 that $S(1,u) \leq 1 = (\frac{1}{c_0 2^{-q}})c_0 2^{-q} \leq (\frac{1}{c_0 2^{-q}})S(2^{-1},u) = AS(2^{-1},u)$.

Now we state the growth property of the elements in the class $\mathscr{G}_a$.

Theorem 2.1 *There is a positive constant $M_0 = M_0(p,\gamma_0,\gamma_1)$ such that, for every $u \in \mathscr{G}_a$,*

$$|u(x,t)| \leq M_0 (d(x,t))^q \quad \forall (x,t) \in Q_{\frac{1}{2}},$$

where $d(x,t) = \sup\{r; Q_r(x,t) \subset U^+\}$ for $(x,t) \in U^+$, and $d(x,t) = 0$ otherwise.

To prove this theorem we need the following lemma.

Lemma 2.2 *There is a positive constant $M_1 = M_1(p,\gamma_0,\gamma_1)$ such that*

$$S(2^{-j-1},u) \leq M_1 (2^{-j})^q,$$

for all $u \in \mathscr{G}_a$ and $j \in \mathbb{M}_a(u)$.

Proof Arguing by contradiction, we assume that for every $k \in \mathbb{N}$, there exists $u_k \in \mathscr{G}_a$ and $j_k \in \mathbb{M}_a(u_k)$ such that

$$S(2^{-j_k-1},u_k) \geq k(2^{-j_k})^q. \tag{4}$$

Observe that by the uniform boundedness of u_k and (4) it follows that $j_k \to \infty$ as $k \to \infty$.

Consider the function

$$\widetilde{u}_k(x,t) = \frac{u_k(2^{-j_k}x, \alpha_k t)}{S(2^{-j_k-1},u_k)}$$

defined in the unit cylinder, where $\alpha_k = (2^{-j_k})^p (S(2^{-j_k-1},u_k))^{2-p}$. Note that by (4) we have

$$\begin{aligned} \alpha_k &\leq \frac{1}{k^{p-1}} (S(2^{-j_k-1},u_k))^{p-1} \cdot (S(2^{-j_k-1},u_k))^{2-p} \\ &= \frac{1}{k^{p-1}} S(2^{-j_k-1},u_k) \\ &\leq \frac{1}{k^{p-1}} \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

By the definition of $\mathbb{M}_a(u_k)$ and $\mathscr{G}_a$ it follows that

$$
\begin{aligned}
&0 \le \widetilde{u}_k \le A \quad \text{in } Q_1^-, \\
&\sup_{Q_{\frac{1}{2}}^-} \widetilde{u}_k \ge 1 \quad \bigl(\text{by (2d) and } \bigl(2^{-1}\bigr)^q \alpha_k \ge \bigl(2^{-j_k-1}\bigr)^q\bigr), \\
&\widetilde{u}_k(0,0) = 0, \\
&\partial_t \widetilde{u}_k \ge 0 \quad \text{in } Q_1^-.
\end{aligned}
$$

Now, define for $(x,\eta) \in B_1 \times \mathbb{R}^N$

$$
a^k(x,\eta) = \left(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}\right)^{p-1} \cdot a\left(2^{-j_k}x, \frac{S(2^{-j_k-1},u_k)}{2^{-j_k}}\eta\right).
$$

We claim that $a^k(x,\eta)$ satisfies the same structural conditions as $a(x,\eta)$ for large k. Indeed, letting $s_k = \frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}$, one may verify directly that

$$
\begin{aligned}
\sum_{i,j=1}^N \frac{\partial a_i^k}{\partial \eta_j}(x,\eta)\xi_i\xi_j &= \sum_{i,j=1}^N s_k^{p-2}\frac{\partial a_i}{\partial \eta_j}\bigl(2^{-j_k}x, s_k^{-1}\eta\bigr)\xi_i\xi_j \\
&\ge \gamma_0 s_k^{p-2}\bigl|s_k^{-1}\eta\bigr|^{p-2}|\xi|^2 \\
&= \gamma_0|\eta|^{p-2}|\xi|^2, \\
\sum_{i,j=1}^N \left|\frac{\partial a_i^k}{\partial \eta_j}(x,\eta)\right| &= \sum_{i,j=1}^N s_k^{p-2}\left|\frac{\partial a_i}{\partial \eta_j}\bigl(2^{-j_k}x, s_k^{-1}\eta\bigr)\right| \\
&\le \gamma_1 s_k^{p-2}\bigl|s_k^{-1}\eta\bigr|^{p-2} \\
&= \gamma_1|\eta|^{p-2}, \\
\sum_{i,j=1}^N \left|\frac{\partial a_i^k}{\partial x_j}(x,\eta)\right| &= \sum_{i,j=1}^N s_k^{p-1}2^{-j_k}\left|\frac{\partial a_i}{\partial x_j}\bigl(2^{-j_k}x, s_k^{-1}\eta\bigr)\right| \\
&\le 2^{-j_k}\gamma_1|\eta|^{p-1} \\
&\le \gamma_1|\eta|^{p-1}.
\end{aligned} \tag{5}
$$

Now by (2a) and (4) we obtain

$$
\begin{aligned}
\bigl\|\operatorname{div} a^k\bigl(x,\nabla\widetilde{u}_k(x,t)\bigr) - \partial_t\widetilde{u}_k(x,t)\bigr\|_\infty &= 2^{-j_k}s_k^{p-1}\bigl\|(Au_k - \partial_t u_k)\bigl(2^{-j_k}x,\alpha_k t\bigr)\bigr\|_\infty \\
&\le 2^{-j_k}\left(\frac{2^{-j_k}}{S(2^{-j_k-1},u_k)}\right)^{p-1} \\
&\le \frac{1}{k^{p-1}} \to 0 \quad \text{as } k\to\infty,
\end{aligned}
$$

where $(Au)(x,t)$ is defined by $(Au)(x,t) = \operatorname{div} a(x,\nabla u(x,t))$.

Observe that by (5), for any $M > 0$ we have

$$
\left|\frac{\partial a_i^k}{\partial x_j}(x,\eta)\right| \to 0 \quad \text{as } k\to\infty,
$$

uniformly in $(x,\eta) \in B_1 \times B_M$. Therefore the pointwise limit of $a^k(x,\eta)$ does not depend on x:

$$a^k(x,\eta) \to \widetilde{a}(\eta),$$

with $\widetilde{a}$ satisfying the same structural conditions as (a_1)-(a_4). Then invoking compactness arguments (see Lemma 14.1 on p.75 and 14-(iii) on p.115 of [1]), we deduce that, up to a subsequence, $\widetilde{u}_k$ converges locally uniformly in Q_1^- to a function u. Moreover, the limit function u satisfies

$$\begin{aligned} &\operatorname{div} \widetilde{a}(\nabla u) - \partial_t u = 0, \qquad u \geq 0, \qquad u(0,0) = 0, \\ &\sup_{Q^-_{\frac{1}{2}}} u \geq 1, \qquad \partial_t u \geq 0 \quad \text{in } Q_1^-. \end{aligned} \tag{6}$$

To get a contradiction, we divide our problem into two cases.

-Case 1 ($1 < p \leq 2$). In this case, we need the following lemma originating from [6], where the authors stated it for p-parabolic equations ($1 < p < 2$). One should pay attention to the fact that the proof of the following lemma can be repeated as in [6] with slight modifications. Moreover, the result is valid for $p = 2$ since the process is 'stable' as $p \nearrow 2$ so that one may recover the regularity results by letting $p \nearrow 2$ (see the proofs of Theorems 1 and 2, or the remarks in 1-(iii) on p.323 of [6]).

Lemma 2.3 (Theorem 2 [6]) *Let Ω be a region of $\mathbb{R}^N$, $\Omega_\infty = \Omega \times (0,\infty)$ and $u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,p}(\Omega))$ be any non-negative local solution of*

$$\operatorname{div} \widetilde{a}(\nabla u) - \partial_t u = 0 \quad \textit{in } \Omega_\infty.$$

Suppose $u(x_0,t_0) > 0$ for some $(x_0,t_0) \in \Omega_\infty$. Then, for any ball $B_\rho(x_0) \subset \Omega$,

$$u(x,t_0) > 0 \quad \forall x \in B_\rho(x_0).$$

Now notice that $\sup_{B_{\frac{1}{2}}} u(x,0) \geq 1$ by $\partial_t u \geq 0$ and (6). One may find $x_0 \in B_{\frac{1}{2}}(0)$ such that $u(x_0,0) \geq \frac{1}{2}$. On the other hand, since, for any $\overline{Q'} \subset Q_1$, $u \in C^{1,\alpha}(Q')$ for some $\alpha \in (0,1)$ (see Chapter IX of [1]), and then Lemma 2.3 gives $u(0,0) \geq \frac{1}{2}$, which is a contradiction. Indeed, in Lemma 2.3, one may let $\rho = \frac{1}{2} \in (|x_0|, 1 - |x_0|)$ and $\Omega = B_r \subset B_1$ with $r = \frac{\frac{1}{2} + |x_0| + 1}{2}$. Therefore $B_\rho(x_0) \subset \Omega$ and $0 \in B_\rho(x_0)$.

-Case 2 ($2 < p < \infty$). In this case, due to the lack of a strong minimum principle, we need further discussion to get a contradiction. At this point, we show that u is time independent, *i.e.*,

$$\partial_t u = 0 \quad \text{in } Q_1^-.$$

Then u is a nonzero, non-negative harmonic function in the unit ball and it vanishes at the origin. Indeed, this is a contradiction to the strong minimum principle; see [10] for instance. To this end, choosing $(x,t), (x',t') \in Q^-_{\frac{1}{2}}$ and using the definition of $\mathbb{M}_a(u_k)$ and

Hölder's estimates for solutions with $G_T = Q^-_{2^{-j_k}}$ and $K = Q^-_{2^{-j_k-1}}$ (see Theorem 1.1 on p.41 of [1]), we arrive at

$$\begin{aligned}
\left|\widetilde{u}_k(x,t) - \widetilde{u}_k(x,t')\right| &= \frac{|u_k(2^{-j_k}x,\alpha_k t) - u_k(2^{-j_k}x,\alpha_k t')|}{S(2^{-j_k-1},u_k)} \\
&\le A\frac{|u_k(2^{-j_k}x,\alpha_k t) - u_k(2^{-j_k}x,\alpha_k t')|}{S(2^{-j_k},u_k)} \\
&\le A\gamma\frac{\|u_k\|_{\infty,G_T}}{S(2^{-j_k},u_k)}\left(\frac{\|u_k\|_{\infty,G_T}^{\frac{p-2}{p}}\alpha_k^{\frac{1}{p}}|t-t'|^{\frac{1}{p}}}{\operatorname{dist}_p(K,\partial_p G_T;p)}\right)^{\alpha},
\end{aligned}$$

where $\operatorname{dist}_p(K,\partial_p G_T;p) = \inf_{(x,t)\in K,(y,s)\in\partial_p G_T}(|x-y| + \|u\|_{\infty,G_T}^{\frac{2-p}{p}}|t-s|^{\frac{1}{p}})$, $\alpha\in(0,1)$ is the Hölder exponent, the constant γ does not depend on $\|u_k\|_{\infty,G_T}$.

Now observe that $\operatorname{dist}_p(K,\partial_p G_T;p) \ge (2^{-j_k-1})^{\frac{q}{p}}\|u\|_{\infty,G_T}^{\frac{2-p}{p}}$. It follows that

$$\begin{aligned}
\left|\widetilde{u}_k(x,t) - \widetilde{u}_k(x,t')\right| &\le A\gamma\left|t-t'\right|^{\frac{\alpha}{p}}\left(\alpha_k\cdot 2^{(j_k+1)q}\right)^{\frac{\alpha}{p}} \\
&= A\gamma\left|t-t'\right|^{\frac{\alpha}{p}}\left[\left(2^{-j_k}\right)^p\cdot\left(S\left(2^{-j_k-1},u_k\right)\right)^{2-p}\cdot 2^{(j_k+1)q}\right]^{\frac{\alpha}{p}} \\
&\le A\gamma\left|t-t'\right|^{\frac{\alpha}{p}}\left[\left(2^{-j_k}\right)^{p-1}\cdot k^{2-p}\cdot\left(2^{-j_k}\right)^{q(2-p)}\cdot\left(2^{j_k}\right)^q\cdot 2^q\right]^{\frac{\alpha}{p}} \\
&= 2^{\frac{\alpha}{p-1}}A\gamma\left|t-t'\right|^{\frac{\alpha}{p}}k^{2-p}\to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Hence u is t-independent, and the proof is completed. □

Proof of Theorem 2.1 The proof of this theorem is standard (see [5]). For convenience, we recover the process. Let us take j for which

$$S\left(2^{-j},u\right) > 2^q M_1 2^{-qj}.$$

It follows that

$$S\left(2^{-j+1},u\right) \le 2^q M_1 2^{-q(j-1)} < 2^q S\left(2^{-j},u\right) \le AS\left(2^{-j},u\right), \tag{7}$$

i.e. $j-1\in\mathbb{M}_a(u)$, so Lemma 2.2 holds for $j-1$. Now we arrive at the following obvious contradiction to (7):

$$S\left(2^{-j},u\right) \le S\left(2^{-j+1},u\right) \le M_1 2^{-q(j-1)} = 2^q M_1 2^{-qj}.$$

Therefore

$$S\left(2^{-j},u\right) \le 2^q M_1 2^{-qj}, \quad \forall j,$$

which implies

$$\sup_{Q_r^-(0,0)} u \le 2^q M_1 2^{-qj}, \quad \forall r\le 1.$$

To obtain a similar estimate for u over the whole cylinder (and not only over the lower half part) we use an upper barrier. Define $w(x,t) = C_3|x|^q + C_4 t$ where $C_4 = 1 + \gamma_1(qC_3)^{p-1}(1 + \frac{1}{p-1})$ and $C_3 > 0$. Let now $Q_1^+ = B_1(0) \times (0,1)$. Then proceeding as Lemma 2.1, we deduce

$$\begin{aligned} \operatorname{div} a(x,\nabla w) - \partial_t w &\le \gamma_1(qC_3)^{p-1}\left(1 + \frac{1}{p-1}\right) - C_4 \\ &= -1 \le \operatorname{div} a(x,\nabla u) - \partial_t u \quad \text{in } Q_1^+. \end{aligned}$$

Since by choosing C_3 large, we will have $w \ge u$ on $\partial_p Q_1^+$, where for the estimate on $\{t = 0\}$ we have sued the previous discussion, *i.e.*, $S(r,u) \le Cr^q$. Hence by the comparison principle we have $w \ge u$ in Q_1^+. Therefore

$$\sup_{Q_r(0,0)} u \le M_2 r^q.$$

The proof is completed. □

3 Proof of the main theorem

Having the estimates from below and above for the function u, one can prove our main result as in [5]. For completeness we carry out the minor changes in the proof of [5].

Proof of the main theorem Without loss of generality, we assume that the compact set K in the main theorem is the closed unit cylinder $\overline{Q}_1$, and, moreover, that $\overline{Q}_2 \subset \Omega_T$.

For $(x,t) \in U^+ \cap \overline{Q}_1$, let $d(x,t)$ be defined as in Theorem 2.1 and take $(x^0,t^0) \in \partial U^+ \cap \overline{Q}_1$ which realizes this distance. Next define $\widetilde{u}(y,s) = u(x^0 + y, t^0 + s)$ in Q_1. Let $M = \max\{\|\theta\|_{\infty,\Omega_T}, \Lambda_0\}$, $\widetilde{a}(y,\eta) = \frac{a(x^0+y,M\eta)}{M}$ and $\widetilde{A}v(y,s) = \operatorname{div}\widetilde{a}(y,\nabla v(y,s))$. We claim that $\frac{\widetilde{u}}{M} \in \mathscr{G}_{\widetilde{a}}$. Indeed, one may verify directly that $\widetilde{a}$ satisfies all structural conditions (not necessarily with the same constants as a). Furthermore, we have

$$\begin{aligned} \widetilde{A}\left(\frac{\widetilde{u}}{M}\right) - \partial_s\left(\frac{\widetilde{u}}{M}\right) &= \frac{1}{M}\left[\operatorname{div} a\left(x^0 + y, \nabla u\left(x^0 + y, t^0 + s\right)\right) - \partial_s u\left(x^0 + y, t^0 + s\right)\right] \\ &= \frac{1}{M}\left[(Au) - \partial_s u\right]\left(x^0 + y, t^0 + s\right) \\ &\le \frac{\Lambda_0}{M} \\ &\le 1, \end{aligned}$$

and

$$0 \le \frac{\widetilde{u}}{M} \le \frac{\|\theta\|_{\infty,\Omega_T}}{M} \le 1 \quad \text{and} \quad \frac{\widetilde{u}(0,0)}{M} = 0.$$

Therefore we infer by Theorem 2.1 that

$$u(x,t) = \widetilde{u}\left(x - x^0, t - t^0\right) \le MM_0\left(d(x,t)\right)^{\frac{p}{p-1}}. \tag{8}$$

Let $(z,\tau) \in \partial U^+ \cap \overline{Q}_1$. Then, for $0 < r < 1$, by Lemma 2.1, there exists $x^1 \in \partial B_r(z)$ such that

$$u\left(x^1\right) \ge c_0 r^{\frac{p}{p-1}}.$$

It follows from (8) that

$$c_0 r^{\frac{p}{p-1}} \leq u(x^1, \tau) \leq MM_0 \big(d(x,\tau)\big)^{\frac{p}{p-1}}.$$

Let $\delta = (\frac{c_0}{MM_0})^{\frac{p-1}{p}}$. Then $d(x,\tau) \geq \delta r$, and $0 < \delta \leq 1$. Therefore

$$B_{\delta r}(x^1) \cap B_r(z) \subset U^+.$$

Now choose $y \in [z, x^1]$ such that $|y - x^1| = \frac{\delta r}{2}$. Then we have

$$B_{\frac{\delta r}{2}}(y) \subset B_{\delta r}(x^1) \cap B_r(z) \subset B_r(z) \setminus \partial U^+.$$

Indeed, for any $y_0 \in B_{\frac{\delta r}{2}}(y)$, we have

$$\left|y_0 - x^1\right| \leq |y_0 - y| + \left|y - x^1\right| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r.$$

Moreover, since $|y - z| = |z - x^1| - |y - x^1|$, we have

$$|y_0 - z| \leq |y_0 - y| + \left(\left|z - x^1\right| - \left|y - x^1\right|\right) \leq \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r.$$

This shows that $\partial U^+ \cap \{t = \tau\} \cap \overline{B}_1$ is porous with the porosity constant $\frac{\delta}{2}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. The three authors have contributed to the manuscript; they wrote, read, and approved the manuscript.

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