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Existence of periodic solutions for a class of second order Hamiltonian systems

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Abstract

By using the least action principle and the minimax methods, the existence of periodic solutions for a class of second order Hamiltonian systems is considered. The results obtained in this paper extend some previous results.

Keywords: periodic solutions; second order Hamiltonian systems; least action principle

1 Introduction and main results

Consider the second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$, continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \qquad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The corresponding functional $\varphi : H_T^1 \to \mathbb{R}$,

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt + \int_0^T F(t, u(t))\,dt,$$

is continuously differentiable and weakly lower semi-continuous on H_T^1, where H_T^1 is the usual Sobolev space with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot{u}(t)|^2\,dt\right)^{1/2}$$

for $u \in H_T^1$, and

$$\langle \varphi'(u), v\rangle = \int_0^T \left[(\dot{u}(t), \dot{v}) + (\nabla F(t, u(t)), v(t))\right] dt$$

for all $u, v \in H_T^1$, It is well known that the solutions of problem (1.1) correspond to the critical points of φ.

The existence of periodic solutions for problem (1.1) is obtained in [1–22] with many solvability conditions by using the least action principle and the minimax methods, such as the coercive type potential condition (see [2]), the convex type potential condition (see [5]), the periodic type potential conditions (see [16]), the even type potential condition (see [4]), the subquadratic potential condition in Rabinowitz's sense (see [9]), the bounded nonlinearity condition (see [6]), the subadditive condition (see [11]), the sublinear nonlinearity condition (see [3, 13]), and the linear nonlinearity condition (see [7, 15, 19, 20]).

In particular, when the nonlinearity $\nabla F(t,x)$ is bounded, that is, there exists $g(t) \in L^1([0,T], \mathbb{R}^+)$ such that $|\nabla F(t,x)| \le g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, and that

$$\int_0^T F(t,x)\,dt \to \pm\infty \quad \text{as } |x| \to \infty,$$

Mawhin and Willem [6] proved that problem (1.1) has at least one periodic solution.

In [3, 13], Han and Tang generalized these results to the sublinear case:

$$\bigl|\nabla F(t,x)\bigr| \le f(t)|x|^\alpha + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T] \tag{1.2}$$

and

$$|x|^{-2\alpha} \int_0^T F(t,x)\,dt \to \pm\infty \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $f(t), g(t) \in L^1([0,T], \mathbb{R}^+)$ and $\alpha \in [0,1)$.

Subsequently, when $\alpha = 1$ Zhao and Wu [19, 20] and Meng and Tang [7, 15] also proved the existence of periodic solutions for problem (1.1), *i.e.* $\nabla F(t,x)$ was linear:

$$\bigl|\nabla F(t,x)\bigr| \le f(t)|x| + g(t) \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T],$$

where $f(t), g(t) \in L^1([0,T], \mathbb{R}^+)$.

Recently, Wang and Zhang [21] used a control function $h(|x|)$ instead of $|x|^\alpha$ in (1.2) and (1.3) and got some new results, where h satisfied the following conditions:

(B) $h \in C([0,\infty), [0,\infty))$ and there exist constants $C_0 > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0,1)$ such that

(i) $h(s) \le h(t)$ $\forall s \le t$, $s, t \in [0,\infty)$,
(ii) $h(s+t) \le C_0(h(s) + h(t))$ $\forall s, t \in [0,\infty)$,
(iii) $0 \le h(s) \le K_1 s^\alpha + K_2$ $\forall s \in [0,\infty)$,
(iv) $h(s) \to \infty$ as $s \to \infty$.

Motivated by the results mentioned above, we will consider the periodic solutions for problem (1.1). The following are our main results.

Theorem 1.1 *Suppose that* $F(t,x) = F_1(t,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy assumption* (A) *and the following conditions*:

(1) *there exist* $f, g \in L^1([0,T]; \mathbb{R}^+)$ *such that*

$$\bigl|\nabla F_1(t,x)\bigr| \le f(t)h\bigl(|x|\bigr) + g(t),$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$, *here* h *satisfies* (B);

(2) *there exist constants* $r > 0$ *and* $\gamma \in [0, 2)$ *such that*

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \geq -r|x - y|^{\gamma},$$

for all $x, y \in \mathbb{R}^N$;

(3)

$$\liminf_{|x|\to\infty} h^{-2}\bigl(|x|\bigr) \int_0^T F(t,x)\,dt > \frac{T^2 C_0^2}{8\pi^2} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes* φ *on* H_T^1.

Theorem 1.2 *Suppose that* $F(t,x) = F_1(t,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy assumption* (A), (1), (2), *and the following conditions*:

(4) *there exist* $\delta \in [0, 2)$ *and* $\mu > 0$ *such that*

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \leq \mu|x - y|^{\delta},$$

for all $x, y \in R^N$;

(5)

$$\limsup_{|x|\to\infty} h^{-2}\bigl(|x|\bigr) \int_0^T F(t,x)\,dt < -\frac{3T^2 C_0^2}{8\pi^2} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes* φ *on* H_T^1.

Theorem 1.3 *Suppose that* $F(t,x) = F_1(t,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy assumption* (A), (1), *and the following conditions*:

(6) *there exists a constant* $0 < r < 4\pi^2/T^2$, *such that*

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \geq -r|x - y|^2,$$

for all $x, y \in R^N$;

(7)

$$\liminf_{|x|\to\infty} h^{-2}\bigl(|x|\bigr) \int_0^T F(t,x)\,dt > \frac{T^2}{2(4\pi^2 - rT^2)} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes* φ *on* H_T^1.

Theorem 1.4 *Suppose that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy assumption* (A), (1), *and the following conditions*:

(8) *there exist* $k \in L^1([0, T]; \mathbb{R}^+)$ *and* (λ, μ)-*subconvex potential* $G : \mathbb{R}^N \to \mathbb{R}$ *with* $\lambda > 1/2$ *and* $0 < \mu < 2\lambda^2$, *such that*

$$\bigl(\nabla F_2(t,x), y\bigr) \geq -k(t)G(x - y),$$

for all $x, y \in \mathbb{R}^N$;

(9)

$$\limsup_{|x|\to\infty} h^{-2}(|x|)\int_0^T F_1(t,x)\,dt < -\frac{3T^2C_0^2}{8\pi^2}\int_0^T f^2(t)\,dt,$$

$$\limsup_{|x|\to\infty} |x|^{-\beta}\int_0^T F_2(t,x)\,dt \le -8\mu\max_{|s|\le 1} G(s)\int_0^T k(t)\,dt,$$

where $\beta = \log_{2\lambda}(2\mu)$.

Then problem (1.1) *has at least one periodic solution which minimizes* φ *on* H_T^1.

Remark 1.5 Theorems 1.1-1.4 extend some existing results: (i) [22], Theorems 1.1-1.4, are special cases of Theorems 1.1-1.4 with control function $h(t) = t^\alpha, \alpha \in [0,1)$, $t \in [0,+\infty)$; (ii) if $F_2 = 0$, [15], Theorems 1 and 2, are special cases of Theorem 1.1 and Theorem 1.2, respectively; (iii) If $F_2 = 0$, Theorem 1.1 and Theorem 1.2 extend [21], Theorems 1.1 and 1.2, since we weaken the so-called Ahmad-Lazer-Paul type conditions with the control function $h(t)$.

2 Proof of theorems

For $u \in H_T^1$, let $\bar{u} = \frac{1}{T}\int_0^T |\dot{u}(t)|\,dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. Then one has

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2\,dt \quad \text{(Sobolev's inequality)},$$

$$\|\tilde{u}\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\,dt \quad \text{(Wirtinger's inequality)}.$$

For the sake of convenience, we denote $M_1 = (\int_0^T f^2(t)\,dt)^{1/2}$, $M_2 = \int_0^T f(t)\,dt$, $M_3 = \int_0^T g(t)\,dt$.

Proof of Theorem 1.1 Due to (3), we can choose an $a_1 > T^2/(4\pi^2)$ such that

$$\liminf_{|x|\to\infty} h^{-2}(|x|)\int_0^T F(t,x)\,dt > \frac{a_1C_0^2}{2}M_1^2. \tag{2.1}$$

For (B) and the Sobolev inequality, for any $u \in H_T^1$ we have

$$\begin{aligned}
&\left|\int_0^T \left[F_1(t,u(t)) - F_1(t,\bar{u})\right]dt\right| \\
&\quad = \left|\int_0^T\int_0^1 \left(\nabla F_1(t,\bar{u} + s\tilde{u}(t)),\tilde{u}(t)\right)ds\,dt\right| \\
&\quad \le \int_0^T\int_0^1 f(t)h(|\bar{u} + s\tilde{u}(t)|)|\tilde{u}(t)|\,ds\,dt + \int_0^T\int_0^1 g(t)|\tilde{u}(t)|\,ds\,dt \\
&\quad \le \int_0^T\int_0^1 C_0 f(t)\left(h(|\bar{u}|) + h(|\tilde{u}(t)|)\right)|\tilde{u}(t)|\,ds\,dt + M_3\|\tilde{u}\|_\infty \\
&\quad \le C_0h(|\bar{u}|)\left(\int_0^T f^2(t)\,dt\right)^{1/2}\left(\int_0^T |\tilde{u}(t)|^2\,dt\right)^{1/2} \\
&\qquad + C_0\int_0^T f(t)h(|\tilde{u}(t)|)|\tilde{u}(t)|\,dt + M_3\|\tilde{u}\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&\leq C_0M_1h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0\int_0^T f(t)\big(K_1|\tilde{u}(t)|^\alpha + K_2\big)|\tilde{u}(t)|\,dt + M_3\|\tilde{u}\|_\infty \\
&\leq C_0M_1h(|\bar{u}|)\|\tilde{u}\|_{L^2} + C_0M_2K_1\|\tilde{u}\|_\infty^{1+\alpha} + C_0M_2K_2\|\tilde{u}(t)\|_\infty + M_3\|\tilde{u}(t)\|_\infty \\
&\leq \frac{1}{2a_1}\|\widetilde{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) + C_0M_2K_1\|\tilde{u}\|_\infty^{1+\alpha} \\
&\quad + C_0M_2K_2\|\tilde{u}(t)\|_\infty + M_3\|\tilde{u}(t)\|_\infty \\
&\leq \frac{T^2}{8\pi^2a_1}\|\dot{u}\|_{L^2}^2 + \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) + \left(\frac{T}{12}\right)^{(1+\alpha)/2}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2}C_0M_2K_2\|\dot{u}\|_{L^2} + \left(\frac{T}{12}\right)^{1/2}M_3\|\dot{u}\|_{L^2}.
\end{aligned}
\tag{2.2}
$$

Similarly, from (2) and the Sobolev inequality, for any $u \in H_T^1$ we get

$$
\begin{aligned}
&\int_0^T \big[F_2(u(t)) - F_2(\bar{u})\big]\,dt \\
&\quad = \int_0^T\int_0^1 \frac{1}{s}\big(\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)\big)\,ds\,dt \\
&\quad \geq -\int_0^T\int_0^1 rs^{\gamma-1}|\tilde{u}(t)|^\gamma\,ds\,dt \\
&\quad \geq -\frac{rT}{\gamma}\|\widetilde{u}\|_\infty^\gamma \\
&\quad \geq -\frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma.
\end{aligned}
\tag{2.3}
$$

From (2.2) and (2.3) we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T \big[F_1(t,u(t)) - F_1(t,\bar{u})\big]\,dt \\
&\quad + \int_0^T \big[F_2(u(t)) - F_2(\bar{u})\big]\,dt + \int_0^T F(t,\bar{u})\,dt \\
&\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2a_1}\right)\|\dot{u}\|_{L^2}^2 - \frac{a_1(C_0M_1)^2}{2}h^2(|\bar{u}|) - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2}C_0M_2K_2\|\dot{u}\|_{L^2} - \left(\frac{T}{12}\right)^{1/2}M_3\|\dot{u}\|_{L^2} \\
&\quad - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\alpha/2}\|\dot{u}\|_{L^2}^\gamma + \int_0^T F(t,\bar{u})\,dt \\
&\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2a_1}\right)\|\dot{u}\|_{L^2}^2 + h^2(|\bar{u}|)\left(h^{-2}(|\bar{u}|)\int_0^T F(t,\bar{u})\,dt - \frac{a_1(C_0M_1)^2}{2}\right) \\
&\quad - \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}\|_{L^2} \\
&\quad - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}}C_0M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma,
\end{aligned}
$$

for all $u \in H_T^1$. So, by (2.1) we get $\varphi(u) \to \infty$ as $\|u\| \to \infty$.

Hence, applying the least action principle (see [6], Theorem 1.1 and Corollary 1.1), the proof is complete. □

Proof of Theorem 1.2 *Step* 1. First, we assert that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence, that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. For (5), we can choose an $a_2 > T^2/(4\pi^2)$ such that

$$\limsup_{|x|\to\infty} h^{-2}(|x|) \int_0^T F(t,x)\,dt < -\left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right) C_0^2 \int_0^T f^2(t)\,dt. \tag{2.4}$$

Similar to the proof of Theorem 1.1, we have

$$\begin{aligned} &\left|\int_0^T \big(\nabla F_1\big(t,u_n(t)\big),\tilde{u}_n(t)\big)\,dt\right| \\ &\quad\le \frac{T^2}{8\pi^2 a_2}\|\dot{u}_n\|_{L^2}^2 + \frac{a_2(C_0M_1)^2}{2}h^2\big(|\bar{u}_n|\big) + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} \\ &\qquad + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} \end{aligned} \tag{2.5}$$

and

$$\int_0^T \big(\nabla F_2\big(u_n(t)\big),\tilde{u}_n(t)\big)\,dt \ge -\frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^{\gamma},$$

for all n. Hence we have

$$\begin{aligned} \|\tilde{u}_n\| &\ge \big\langle\varphi'(u_n),\tilde{u}_n\big\rangle \\ &= \|\dot{u}_n\|_{L^2}^2 + \int_0^T \big(\nabla F\big(t,u_n(t)\big),\tilde{u}_n(t)\big)\,dt \\ &\ge \left(1 - \frac{T^2}{8\pi^2 a_2}\right)\|\dot{u}_n\|_{L^2}^2 - \frac{a_2(C_0M_1)^2}{2}h^2\big(|\bar{u}_n|\big) \\ &\quad - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} \\ &\quad - \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}_n\|_{L^2}^{\gamma}, \end{aligned} \tag{2.6}$$

for large n. So, by Wirtinger's inequality we get

$$\big\|(\tilde{u}_n)\big\| \le \frac{(T^2 + 4\pi^2)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2}. \tag{2.7}$$

From (2.6) and (2.7),

$$\begin{aligned} &\frac{a_2(C_0M_1)^2}{2}h^2\big(|\bar{u}_n|\big) \\ &\quad\ge \left(1 - \frac{T^2}{8\pi^2 a_2}\right)\|\dot{u}_n\|_{L^2}^2 - C_0M_2K_1\left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}}\|\dot{u}_n\|_{L^2}^{1+\alpha} \end{aligned}$$

$$- \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3) \|\dot{u}_n\|_{L^2}$$
$$- \|\tilde{u}_n\| - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2} \|\dot{u}_n\|_{L^2}^{\gamma} \geq \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + C_1, \tag{2.8}$$

where

$$C_1 = \min_{s\in[0,+\infty]} \left\{ \frac{4\pi^2 a_2 - T^2}{8\pi^2 a_2} s^2 - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} C_0 M_2 K_1 s^{1+\alpha} - \left[\frac{(T^2+4\pi^2)^{1/2}}{2\pi} + C_0 M_2 K_2 \left(\frac{T}{12}\right)^{1/2} + \left(\frac{T}{12}\right)^{1/2} M_3\right] s - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2} s^{\gamma} \right\}.$$

Note that $a_2 > T^2/(4\pi^2)$ implies $-\infty < C_1 < 0$. Hence, it follows from (2.8) that

$$\|\dot{u}_n\|_{L^2}^2 \leq a_2 C_0^2 M_1^2 h^2\big(|\overline{u}_n|\big) - 2C_1, \tag{2.9}$$

and then

$$\|\dot{u}_n\|_{L^2} \leq \sqrt{a_2} C_0 M_1 h\big(|\overline{u}_n|\big) + C_2, \tag{2.10}$$

where $0 < C_2 < +\infty$. Similar to the proof of Theorem 1.1, we have

$$\begin{aligned}
&\left|\int_0^T \big[F_1\big(t, u_n(t)\big) - F_1(t, \bar{u}_n)\big]\, dt\right| \\
&\quad \leq C_0 M_1 h\big(|\bar{u}_n|\big)\|\tilde{u}_n\|_{L^2} + C_0 M_2 K_1 \|\tilde{u}_n\|_\infty^{1+\alpha} + (C_0 M_2 K_2 + M_3)\|\tilde{u}_n\|_\infty \\
&\quad \leq \frac{\pi}{\sqrt{a_2}T}\|\tilde{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi} M_1^2 h^2\big(|\bar{u}_n|\big) \\
&\qquad + C_0 M_2 K_1 \|\tilde{u}_n\|_\infty^{1+\alpha} + (C_0 M_2 K_2 + M_3)\|\tilde{u}_n\|_\infty \\
&\quad \leq \frac{T}{4\pi\sqrt{a_2}}\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi} M_1^2 h^2\big(|\bar{u}_n|\big) + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\qquad + \left(\frac{T}{12}\right)^{1/2} (C_0 M_2 K_2 + M_3)\|\dot{u}_n\|_{L^2}.
\end{aligned} \tag{2.11}$$

By (4), we obtain

$$\begin{aligned}
&\int_0^T \big[F_2\big(u_n(t)\big) - F_2(\bar{u}_n)\big]\, dt \\
&\quad = \int_0^T \int_0^1 \frac{1}{s}\big(\nabla F_2\big(\bar{u}_n + s\tilde{u}_n(t)\big) - \nabla F_2(\bar{u}_n), s\tilde{u}_n(t)\big)\, ds\, dt \\
&\quad \leq \int_0^T \int_0^1 \mu s^{\delta-1}\big|\tilde{u}_n(t)\big|^{\delta}\, ds\, dt \leq \frac{\mu T}{\delta}\|\tilde{u}_n\|_\infty^{\delta} \\
&\quad \leq \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2} \|\dot{u}_n\|_{L^2}^{\delta}.
\end{aligned}$$

From the boundedness of $\varphi(u_n)$ and (2.9)-(2.11), we have

$$\begin{aligned}
C_3 &\le \varphi(u_n) \\
&= \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + \int_0^T \big[F_1\big(t,u_n(t)\big) - F_1(t,\bar{u}_n)\big]\,dt + \int_0^T \big[F_2\big(u_n(t)\big) - F_2(\bar{u}_n)\big]\,dt \\
&\quad + \int_0^T F\big(t,\bar{u}_n(t)\big)\,dt \\
&\le \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right)\|\dot{u}\|_{L^2}^2 + \frac{\sqrt{a_2}TC_0^2}{4\pi}M_1^2 h^2\big(|\bar{u}|\big) + C_0M_2K_1\left(\frac{T}{12}\right)^{(1+\alpha)/2}\|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} + \int_0^T F(t,\bar{u}_n)\,dt + \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2}\|\dot{u}_n\|_{L^2}^{\delta} \\
&\le \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\right)\big(a_2C_0^2M_1^2h^2\big(|\bar{u}_n|\big) - 2C_1\big) + \frac{\sqrt{a_2}TC_0^2}{4\pi}M_1^2h^2\big(|\bar{u}|\big) \\
&\quad + C_0M_2K_1\left(\frac{T}{12}\right)^{(1+\alpha)/2}\big(\sqrt{a_2}C_0M_1h\big(|\bar{u}_n|\big) + C_2\big)^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\big(\sqrt{a_2}C_0M_1h\big(|\bar{u}_n|\big) + C_2\big) \\
&\quad + \int_0^T F(t,\bar{u}_n)\,dt + \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2}\|\dot{u}_n\|_{L^2}^{\delta} \\
&\le \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right)C_0^2M_1^2h^2\big(|\bar{u}_n|\big) - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right)C_1 \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2}2^{\alpha}C_0M_2K_1\big[(\sqrt{a_2}C_0M_1)^{1+\alpha}h\big(|\bar{u}_n|\big)^{1+\alpha} + C_2^{1+\alpha}\big] \\
&\quad + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\big(\sqrt{a_2}C_0M_1h\big(|\bar{u}_n|\big) + C_2\big) \\
&\quad + \int_0^T F(t,\bar{u}_n)\,dt + \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2}2^{\delta-1}\big((\sqrt{a_2}M_1)^{\delta}h^{\delta}\big(|\bar{u}_n|\big) + C_2^{\delta}\big) \\
&= h^2\big(|\bar{u}_n|\big)\bigg[h^{-2}\big(|\bar{u}_n|\big)\int_0^T F(t,\bar{u}_n)\,dt + \left(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\right)C_0^2M_1^2 \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2}2^{\alpha}C_0^{2+\alpha}M_2M_1^{1+\alpha}K_1h^{\alpha-1}\big(|\bar{u}_n|\big) + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 \\
&\quad + M_3)\big(\sqrt{a_2}C_0M_1h^{-1}\big(|\bar{u}_n|\big)\big) + \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2}2^{\delta-1}(\sqrt{a_2}M_1)^{\delta}h^{\delta-2}\big(|\bar{u}_n|\big)\bigg] \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2}2^{\alpha}C_0M_2K_1C_2^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 \\
&\quad + M_3)C_2 - \left(1 + \frac{T}{2\pi\sqrt{a_2}}\right)C_1 + \frac{\mu T}{\delta}\left(\frac{T}{12}\right)^{\delta/2}2^{\delta-1}C_2^{\delta},
\end{aligned}$$

for large n. So, by (2.4) we see that $|\bar{u}|$ is bounded. Hence $\{u_n\}$ is bounded by (2.9). Arguing as in the proof of Proposition 4.1 of [6], we conclude that the (PS) condition is satisfied.

Step 2. Let $\tilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$. We assert that for $u \in \tilde{H}_T^1$,

$$\varphi(u) \to +\infty, \qquad \|u\| \to \infty. \tag{2.12}$$

In fact, from (1) and Sobolev's inequality, we get

$$\begin{aligned}
&\left|\int_0^T \bigl[F_1\bigl(t,u(t)\bigr) - F_1(t,0)\bigr]\,dt\right| \\
&\quad= \left|\int_0^T \int_0^1 \bigl(\nabla F\bigl(t,su(t)\bigr), u(t)\bigr)\,ds\,dt\right| \\
&\quad\le \int_0^T f(t)h\bigl(\bigl|u(t)\bigr|\bigr)\bigl|u(t)\bigr|\,dt + \int_0^T g(t)\bigl|u(t)\bigr|\,dt \\
&\quad\le \int_0^T f(t)\bigl(K_1\bigl|u(t)\bigr|^\alpha + K_2\bigr)\bigl|u(t)\bigr|\,dt + M_3\|u\|_\infty \\
&\quad\le M_2K_1\|u\|_\infty^{1+\alpha} + M_2K_2\|u\|_\infty + M_3\|u\|_\infty \\
&\quad\le \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{1/2}(M_2K_2 + M_3)\|\dot{u}\|_{L^2},
\end{aligned}$$

for all $u \in \tilde{H}_T^1$. It follows from (2) that

$$\begin{aligned}
&\int_0^T \bigl[F_2\bigl(u(t)\bigr) - F_2(0)\bigr]\,dt \\
&\quad= \int_0^T \int_0^1 \frac{1}{s}\bigl(\nabla F_2\bigl(s\tilde{u}(t)\bigr) - \nabla F_2(0), su(t)\bigr)\,ds\,dt \\
&\quad\ge -\int_0^T \int_0^1 rs^{\gamma-1}|u|^\gamma\,ds\,dt \\
&\quad\ge -\frac{rT}{\gamma}\|\dot{u}\|_\infty^\gamma \\
&\quad\ge -\frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma.
\end{aligned}$$

So, we get

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T \bigl[F\bigl(t,u(t)\bigr) - F(t,0)\bigr]\,dt + \int_0^T F(t,0)\,dt \\
&\ge \frac{1}{2}\|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{\frac{1+\alpha}{2}} M_2K_1\|\dot{u}\|_{L^2}^{1+\alpha} - \left(\frac{T}{12}\right)^{1/2}(M_2K_2 + M_3)\|\dot{u}\|_{L^2} \\
&\quad - \frac{rT}{\gamma}\left(\frac{T}{12}\right)^{\gamma/2}\|\dot{u}\|_{L^2}^\gamma + \int_0^T F(t,0)\,dt.
\end{aligned}$$

By Wirtinger's inequality, $\|u\| \to \infty$ if and only if $\|\dot{u}\|_{L^2} \to \infty$ in $\tilde{H}_T^1$. Hence (2.12) holds.

Step 3. By (5), we can easily see that $\int_0^T F(t,x)\,dt \to -\infty$ as $|x| \to \infty$ for all $x \in \mathbb{R}^N$. Thus, for all $u \in (\tilde{H}_T^1)^\perp = \mathbb{R}^N$,

$$\varphi(u) = \int_0^T F(t,u)\,dt \to -\infty \quad \text{as } |u| \to \infty.$$

Now, by saddle point theorem (see, [10], Theorem 4.6), the proof is completed. □

Proof of Theorem 1.3 By (7), we can choose an $a_3 > \frac{T^2}{4\pi^2 - rT^2}$ such that

$$\liminf_{|x|\to\infty} h^{-2}(|x|) \int_0^T F(t,x)\,dt > \frac{a_3}{2} M_1^2 C_0^2. \tag{2.13}$$

By (6) and the Sobolev inequality, we have

$$\begin{aligned}
&\int_0^T \bigl[F_2\bigl(u(t)\bigr) - F_2(\bar{u})\bigr]\,dt \\
&\quad = \int_0^T \int_0^1 \frac{1}{s}\bigl(\nabla F_2\bigl(\bar{u} + s\tilde{u}(t)\bigr) - \nabla F_2(\bar{u}), s\tilde{u}(t)\bigr)\,ds\,dt \\
&\quad \geq -\int_0^T \int_0^1 rs\bigl|\tilde{u}(t)\bigr|^2\,ds\,dt \geq -\frac{rT^2}{8\pi^2}\|\dot{u}\|_{L^2}^2.
\end{aligned}$$

By a similar method to that of the proof of Theorem 1.1, we get

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T F\bigl(t,u(t)\bigr)\,dt \\
&= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T \bigl[F_1\bigl(t,u(t)\bigr) - F_1(t,\overline{u})\bigr]\,dt \\
&\quad + \int_0^T \bigl[F_2\bigl(u(t)\bigr) - F_2(\overline{u})\bigr]\,dt + \int_0^T F(t,\overline{u})\,dt \\
&\geq \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{rT^2}{8\pi^2}\right)\|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}\|_{L_2}^{1+\alpha} \\
&\quad - \left(\frac{T}{12}\right)^{1/2}\left(M_3 + \frac{C_0 M_2 K_2}{2}\right)\|\dot{u}\|_{L^2} - \frac{a_3 C_0^2 M_1^2}{2} h^2\bigl(|\bar{u}|\bigr) + \int_0^T F(t,\overline{u})\,dt \\
&= \left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{rT^2}{8\pi^2}\right)\|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{1/2}\left(M_3 + \frac{C_0 M_2 K_2}{2}\right)\|\dot{u}\|_{L^2} \\
&\quad - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0 M_2 K_1 \|\dot{u}\|_{L_2}^{1+\alpha} + h^2\bigl(|\bar{u}|\bigr)\left(h^{-2}\bigl(|\bar{u}|\bigr)\int_0^T F(t,\bar{u})\,dt - \frac{a_3 C_0^2 M_1^2}{2}\right),
\end{aligned}$$

for all $u \in H_T^1$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by (2.13), due to the facts that $r < \frac{4\pi^2}{T^2}$ and $\|u\| \to \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2)^{1/2} \to \infty$. So, applying the least action principle, Theorem 1.3 holds. □

Proof of Theorem 1.4 First, we assert that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ satisfies $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (9), we can choose an $a_4 > \frac{T^2}{4\pi^2}$ such that

$$\limsup_{|x|\to\infty} h^{-2}\bigl(|x|\bigr) \int_0^T F_1(t,x)\,dt < -\left(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\right) C_0^2 M_1^2. \tag{2.14}$$

By the (λ,μ)-subconvexity of $G(x)$, we have

$$G(x) \leq \bigl(2\mu|x|^\beta + 1\bigr)G_0 \tag{2.15}$$

for all $x \in \mathbb{R}^N$, and a.e. $t \in [0, T]$, where $G_0 = \max_{|s|\leq 1} G(s)$, $\beta = \log_{2\lambda}(2\mu) < 2$ Then

$$\begin{aligned}
&\int_0^T \bigl(\nabla F_2\bigl(t, u_n(t)\bigr), \tilde{u}_n(t)\bigr)\,dt \\
&\quad\geq -\int_0^T k(t)G(\bar{u}_n)\,dt \\
&\quad\geq -\int_0^T k(t)\bigl(2\mu|\bar{u}_n|^\beta + 1\bigr)G_0\,dt \\
&\quad= -2\mu M_4|\bar{u}_n|^\beta - M_4,
\end{aligned} \tag{2.16}$$

where $M_4 = G_0 \int_0^T k(t)\,dt$. From (2.5) and (2.16), for large n, we have

$$\begin{aligned}
\|\bar{u}_n\| &\geq \bigl\langle \varphi(u_n), \tilde{u}_n\bigr\rangle \\
&= \|\dot{u}_n\|_{L^2}^2 + \int_0^T \bigl(\nabla F\bigl(t, u_n(t)\bigr), \tilde{u}_n(t)\bigr) \\
&\geq \left(1 - \frac{T^2}{8\pi^2 a_4}\right)\|\dot{u}_n\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} - \frac{(C_0M_1)^2a_4}{2}h^2\bigl(|\bar{u}_n|\bigr) \\
&\quad - \left(\frac{T}{12}\right)^{1/2}\left(M_3 + \frac{C_0M_2K_2}{2}\right)\|\dot{u}\|_{L^2} - 2\mu M_4|\bar{u}_n|^\beta - M_4.
\end{aligned} \tag{2.17}$$

So, from (2.7) and (2.17) we have

$$\begin{aligned}
&\frac{(C_0M_1)^2a_4}{2}h^2\bigl(|\bar{u}_n|\bigr) + 2\mu M_4|\bar{u}_n|^\beta \\
&\quad\geq \left(1 - \frac{T^2}{8\pi^2 a_4}\right)\|\dot{u}_n\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} \\
&\qquad - \left(\frac{T}{12}\right)^{1/2}\left(M_3 + \frac{C_0M_2K_2}{2}\right)\|\dot{u}\|_{L^2} - \frac{(T^2 + 4\pi^2)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2} - M_4 \\
&\quad\geq \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + C_4,
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
C_4 = \min\Biggl\{&\left(\frac{1}{2} - \frac{T^2}{8\pi^2 a_4}\right)s^2 - \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1 s^{1+\alpha} - \Biggl[\frac{(T^2 + 4\pi^2)^{1/2}}{2\pi} \\
&+ \left(\frac{T}{12}\right)^{1/2}\left(M_3 + \frac{C_0M_2K_2}{2}\right)\Biggr]s - M_4\Biggr\}.
\end{aligned}$$

Note that $-\infty < C_4 < 0$ due to $a_4 > \frac{T^2}{4\pi^2}$, by (2.18), one has

$$\|\dot{u}_n\|_{L^2}^2 \leq a_4(C_0M_1)^2h^2\bigl(|\bar{u}_n|\bigr) + 4\mu M_4|\bar{u}_n|^\beta - 2C_4, \tag{2.19}$$

and then

$$\|\dot{u}_n\|_{L^2} \leq \sqrt{a_4}C_0M_1h\bigl(|\bar{u}_n|\bigr) + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5, \tag{2.20}$$

where $C_5 > 0$. From (8) and (2.15), we have

$$\begin{aligned}
&\left|\int_0^T \bigl[F_2\bigl(t,u(t)\bigr) - F_2(t,\bar{u})\bigr]\,dt\right| \\
&\quad= \int_0^T \int_0^1 \bigl(\nabla F_2\bigl(t,\bar{u}_n + s\tilde{u}_n(t)\bigr),\tilde{u}_n(t)\bigr)\,ds\,dt \\
&\quad\le \int_0^T \int_0^1 k(t)G\bigl(\bar{u}_n + (s+1)\tilde{u}_n\bigr)\,ds\,dt \\
&\quad\le \int_0^T \int_0^1 k(t)\bigl(2\mu\bigl|\bar{u}_n + (s+1)\tilde{u}_n(t)\bigr|^\beta + 1\bigr) \\
&\quad\le 4\mu\int_0^T k(t)\bigl(|\bar{u}_n|^\beta + 2^\beta|\tilde{u}_n|^\beta\bigr)G_0\int_0^T k(t)\,dt \\
&\quad\le \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4\|\dot{u}_n\|_{L^2}^\beta + 4\mu M_4|\bar{u}_n|^\beta + M_4, \qquad (2.21)
\end{aligned}$$

for all $u \in H_T^1$. By the boundedness of $\{\varphi(u_n)\}$ and the inequalities (2.19)-(2.21), we get

$$\begin{aligned}
C_6 &\le \varphi(u_n) \\
&= \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + \int_0^T \bigl[F_1\bigl(t,u_n(t)\bigr) - F_1(t,\bar{u}_n)\bigr]\,dt \\
&\quad + \int_0^T \bigl[F_2\bigl(t,u_n(t)\bigr) - F_2(t,\bar{u}_n)\bigr]\,dt + \int_0^T F(t,\bar{u}_n)\,dt \\
&\le \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\right)\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_4}TC_0^2}{4\pi}M_1^2h^2\bigl(|\bar{u}_n|\bigr) + \left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2 + M_3)\|\dot{u}_n\|_{L^2} \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\|\dot{u}_n\|_{L^2}^{1+\alpha} + \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4\|\dot{u}\|_{L^2}^\beta \\
&\quad + 4\mu M_4|\bar{u}_n|^\beta + M_4 + \int_0^T F(t,\bar{u}_n)\,dt \\
&\le \left(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\right)\bigl(a_4(C_0M_1)^2h^2\bigl(|\bar{u}_n|\bigr) + 4\mu M_4|\bar{u}_n|^\beta - 2C_4\bigr) \\
&\quad + \frac{\sqrt{a_4}TC_0^2}{4\pi}M_1^2h^2\bigl(|\bar{u}_n|\bigr) \\
&\quad + \left(\frac{T}{12}\right)^{(1+\alpha)/2} C_0M_2K_1\bigl(\sqrt{a_4}C_0M_1h\bigl(|\bar{u}_n|\bigr) + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\bigr)^{1+\alpha} \\
&\quad + \left(\frac{T}{12}\right)^{1/2} (C_0M_2K_2 + M_3)\bigl(\sqrt{a_4}C_0M_1h\bigl(|\bar{u}_n|\bigr) + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\bigr) \\
&\quad + \left(\frac{T}{12}\right)^{\beta/2} 2^{\beta+2}\mu M_4\bigl(\sqrt{a_4}C_0M_1h\bigl(|\bar{u}_n|\bigr) + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\bigr)^\beta \\
&\quad + 4\mu M_4|\bar{u}_n|^\beta + M_4 + \int_0^T F(t,\bar{u}_n)\,dt \\
&\le \left(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\right)\bigl((C_0M_1)^2h^2\bigl(|\bar{u}_n|\bigr)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+\left(6+\frac{T}{\pi\sqrt{a_4}}\right)\mu M_4|\overline{u}_n|^{\beta}-\left(1+\frac{T}{2\pi\sqrt{a_4}}\right)C_4\\
&+\left(\frac{T}{12}\right)^{(1+\alpha)/2}C_0M_2K_1\bigl(2^{\alpha}a_4^{(1+\alpha)/2}(C_0M_1)^{1+\alpha}h^{1+\alpha}\bigl(|\overline{u}_n|\bigr)\\
&+2^{3\alpha+1}\mu^{\frac{1+\alpha}{2}}M_4^{\frac{1+\alpha}{2}}|\overline{u}_n|^{\beta(1+\alpha)}+2^{2\alpha}C_5^{1+\alpha}\bigr)\\
&+\left(\frac{T}{12}\right)^{\beta/2}2^{2+\beta}\mu M_4\bigl(2^{\beta-1}a_4^{\beta/2}\bigl(C_0M_1h\bigl(|\overline{u}_n|\bigr)\bigr)^{\beta}\\
&+2^{3\beta-2}\mu^{\beta/2}M_4^{\beta/2}|\overline{u}_n|^{\beta^2/2}+2^{2(\beta-1)}C_5^{\beta}\bigr)\\
&+\left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2+M_3)\bigl(\sqrt{a_4}C_0M_1h\bigl(|\overline{u}_n|\bigr)+2\sqrt{\mu M_4}|\overline{u}_n|^{\beta/2}+C_5\bigr)\\
&+M_4+\int_0^T F_1(t,\bar{u}_n)\,dt+\int_0^T F_2(\bar{u}_n)\,dt\\
=\;&h^2\bigl(|\bar{u}_n|\bigr)\biggl[h^{-2}\bigl(|\bar{u}_n|\bigr)\int_0^T F_1(t,\bar{u}_n)\,dt+\left(\frac{a_4}{2}+\frac{\sqrt{a_4}T}{2\pi}\right)(C_0M_1)^2\\
&+\left(\frac{T}{12}\right)^{(1+\alpha)/2}C_0^{2+\alpha}M_1^{1+\alpha}M_2K_12^{\alpha}a_4^{(1+\alpha)/2}h^{\alpha-1}\bigl(|\overline{u}_n|\bigr)\\
&+\left(\frac{T}{12}\right)^{\beta/2}2^{1+2\beta}\mu(C_0M_1)^{\beta}M_4a_4^{\beta/2}h^{\beta-2}\bigl(|\overline{u}_n|\bigr)\\
&+\left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2+M_3)\sqrt{a_4}C_0M_1h^{-1}\bigl(|\bar{u}_n|\bigr)\biggr]\\
&+|\overline{u}_n|^{\beta}\biggl[|\overline{u}_n|^{-\beta}\int_0^T F_2(\bar{u}_n)\,dt+\left(6+\frac{T}{\pi\sqrt{a_4}}\right)\mu M_4\\
&+\left(\frac{T}{12}\right)^{(1+\alpha)/2}2^{3\alpha+1}C_0M_2K_1\mu^{(1+\alpha)/2}M_4^{(1+\alpha)/2}|\overline{u}_n|^{\alpha\beta}\\
&+\left(\frac{T}{12}\right)^{\beta/2}2^{4\beta}\mu^{(\beta+2)/2}M_4^{(\beta+2)/2}|\overline{u}_n|^{\frac{\beta^2}{2}-\beta}\\
&+\left(\frac{T}{12}\right)^{1/2}2(C_0M_2K_2+M_3)\sqrt{\mu M_4}|\overline{u}_n|^{-\beta/2}\biggr]\\
&-\left(1+\frac{T}{2\pi\sqrt{a_4}}\right)C_4+\left(\frac{T}{12}\right)^{(1+\alpha)/2}2^{2\alpha}C_0M_2K_1C_5^{1+\alpha}\\
&+\left(\frac{T}{12}\right)^{1/2}(C_0M_2K_2+M_3)C_5+\left(\frac{T}{12}\right)^{\beta/2}2^{3\beta}\mu M_4C_5^{\beta}+M_4,
\end{aligned}
$$

for large n. The above inequality and (2.14) imply that $\{|\overline{u}|\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.19). By using the standard method, the (PS) condition holds.

Since the rest of the proof is similar to that of Theorem 1.2, we omit the details here. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by D-BW, D-BW prepared the manuscript initially, and KY performed a part of the steps of the proofs in this research. All authors read and approved the final manuscript.

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