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Implicit and explicit iterative algorithms for hierarchical variational inequality in uniformly smooth Banach spaces

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Abstract

The purpose of this paper is to solve the hierarchical variational inequality with the constraint of a general system of variational inequalities in a uniformly convex and 2-uniformly smooth Banach space. We introduce implicit and explicit iterative algorithms which converge strongly to a unique solution of the hierarchical variational inequality problem. Our results improve and extend the corresponding results announced by some authors.

MSC: 49J30; 47H09; 47J20; 49M05**Keywords:** system of variational inequalities; hierarchical variational inequality; nonexpansive mapping; fixed point; implicit and explicit iterative algorithms; global convergence

1 Introduction

Let X be a real Banach space with its topological dual X^* , and C be a nonempty closed convex subset of X . Let $T : C \rightarrow X$ be a nonlinear mapping on C . We denote by $\text{Fix}(T)$ the set of fixed points of T and by \mathbf{R} the set of all real numbers. A mapping $T : C \rightarrow X$ is called L -Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then T is called a nonexpansive mapping; if $L \in [0, 1)$ then T is called a contraction.

The normalized dual mapping $J : X \rightarrow 2^{X^*}$ is defined as

$$J(x) := \{\varphi \in X^* : \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}, \quad \forall x \in X, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing; see, e.g., [1] for further details.

Let $U := \{x \in X : \|x\| = 1\}$ be the unit sphere of X . Then the space X is said to

- have a Gâteaux differentiable norm if the limit $\lim_{t \rightarrow 0^+} (\|x + ty\| - \|x\|)/t$ exists for each $x, y \in U$.
- have a uniformly Gâteaux differentiable norm if the limit is attained uniformly for $x \in U$.

- be strictly convex if and only if for $x, y \in U$ with $x \neq y$, we have

$$\|(1 - \lambda)x + \lambda y\| < 1, \quad \forall \lambda \in (0, 1).$$

We note that if X is smooth, then the normalized duality mapping is single-valued; and if the norm of X is uniformly Gâteaux differentiable, then the normalized duality mapping is norm to weak star uniformly continuous on every bounded subset of X (see [1]). In the sequel, we shall denote by j the single-valued normalized duality mapping.

Let X be a smooth Banach space. Let $A, B : C \rightarrow X$ be two nonlinear mappings and λ, μ be two positive real numbers. The general system of variational inequalities (GSVI, for short) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C. \end{cases} \tag{2}$$

The equivalence between GSVI (2) and the fixed point problem of some nonexpansive mapping defined on a Banach space is established in Yao et al. [2]. The authors [2] introduced and analyzed implicit and explicit iterative algorithms for solving GSVI (2) by using this equivalence, and they proved the strong convergence of the sequences generated by the proposed algorithms. Subsequently, Ceng et al. [3] proposed and analyzed an implicit algorithm of Mann’s type and another explicit algorithm of Mann’s type for solving GSVI (2).

If X is a real Hilbert space, then GSVI (2) was introduced and studied by Ceng et al. [4]. In this case, for $A = B$, it was considered by Verma [5] (see also [6]). Further, in this case, when $x^* = y^*$, problem (2) reduces to the following classical variational inequality (in short, VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3}$$

This problem is a fundamental problem in the variational analysis, in particular, in the optimization theory and mechanics; see, e.g., [7–12] and the references therein. A large number of algorithms for solving this problem are essentially projection algorithms that employ projections onto the feasible set C of the VI, or onto some related set, so as to iteratively reach a solution. In particular, Korpelevich [13] proposed extragradient method for solving the VI in the Euclidean space. This method further has been improved by several researchers; see, e.g., [4, 14, 15] and the references therein.

In the case of a Banach space setting, that is, if $A = B$ and $x^* = y^*$, the VI is defined as

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{4}$$

Aoyama et al. [16] proposed an iterative scheme to find the approximate solution of (4) and proved the weak convergence of the sequences generated by the proposed scheme. It is also well known [16] that, in a smooth Banach space, this problem is equivalent to a fixed-point equation, containing a sunny nonexpansive retraction from any point of the space onto the feasible set, which is usually assumed to be closed and convex. For the complexity of the feasible set, the sunny nonexpansive retraction is difficult to compute.

To overcome this drawback in a Hilbert space, where the retraction is a metric projection, in Yamada [17], the feasible set is assumed to be the common fixed points of a finite family of nonexpansive mappings, and an explicit hybrid steepest-descent method is introduced. In this case, the variational inequality defined on such a feasible set is also called a hierarchical variational inequality (in short, HVI). Yamada’s method is then extended to solve more complex problems involving finite or infinite nonexpansive mappings (one can refer to, e.g., [18, 19] and the references therein). Zeng and Yao [19] introduced an implicit method that converges weakly to a solution of a variational inequality, containing a Lipschitz continuous and strongly monotone mapping in a Hilbert space H , where the feasible set is the common fixed points of a finite family of nonexpansive mappings on H . Ceng et al. [20] extended this result from nonexpansive mappings to Lipschitz pseudocontractive mappings and strictly pseudocontractive mappings on H . Recently, Buong and Anh [21] modified Yamada’s result and proposed a strongly convergent implicit method.

In this paper, we are going to solve the hierarchical variational inequality with the constraint of a general system of variational inequalities in a uniformly convex and 2-uniformly smooth Banach space. We introduce implicit and explicit iterative algorithms for finding a solution of the problem and derive the strong convergence of the proposed algorithms to a unique solution of the problem. Our results improve and extend the corresponding results announced by some others, e.g., Ceng et al. [3] and Buong and Phuong [18].

2 Preliminaries

Let X be a real Banach space with the dual space X^* . For simplicity, the norms of X and X^* are denoted by the symbol $\| \cdot \|$. Let X be a nonempty closed convex subset of a real Banach space X . We write $x_n \rightharpoonup x$ (respectively, $x_n \rightarrow x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x . A mapping $J : X \rightarrow 2^{X^*}$, defined by

$$J(x) = \{ \varphi \in X^* : \langle x, \varphi \rangle = \|\varphi\|^2 \text{ and } \|\varphi\| = \|x\| \},$$

is called the normalized duality mapping of X . We know that $J(tx) = tJ(x)$ for all $t > 0$ and $x \in X$, and $J(-x) = -J(x)$.

Let $U := \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|\frac{x+y}{2}\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$. It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space X is reflexive, then X is strictly convex if and only if X^* is smooth as well as X is smooth if and only if X^* is strictly convex.

Proposition 1 ([22]) *Let X be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \rightarrow \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x - y\|) \leq \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2, \quad \forall x, y \in B_r,$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

Here we define a function $\rho : [0, \infty) \rightarrow [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \rightarrow 0^+} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For further details on the geometry of Banach spaces, we refer to [1, 23] and the references therein. Takahashi et al. [24] reminded us of the fact that no Banach space is q -uniformly smooth for $q > 2$. So, in this paper, we focus on only a 2-uniformly smooth Banach space as in [2].

Lemma 1 ([25]) *Let q be a given real number with $1 < q \leq 2$, and let X be a q -uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + 2\|y\|^q, \quad \forall x, y \in X,$$

where κ is the q -uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \{ \varphi \in X^* : \langle x, \varphi \rangle = \|x\|^q, \|\varphi\| = \|x\|^{q-1} \}, \quad \forall x \in X.$$

Let D be a subset of C , and let Π be a mapping of C into D . Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π . A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D .

Lemma 2 ([26]) *Let C be a nonempty closed convex subset of a smooth Banach space X , D be a nonempty subset of C and Π be a retraction of C onto D . Then the following are equivalent:*

- (i) Π is sunny and nonexpansive;
- (ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, j(\Pi(x) - \Pi(y)) \rangle, \forall x, y \in C$;
- (iii) $\langle x - \Pi(x), j(y - \Pi(x)) \rangle \leq 0, \forall x \in C, y \in D$.

It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Π_C coincides with the metric projection from X onto C . Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X , and let T be a nonexpansive mapping of C into itself with the fixed point set $\text{Fix}(T) \neq \emptyset$. Then the set $\text{Fix}(T)$ is a sunny nonexpansive retract of C ; see, e.g., [2].

Lemma 3 ([2]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mappings $A, B : C \rightarrow X$ be α -inverse-strongly accretive and β -inverse-strongly accretive, respectively. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (2) if and only if $x^* \in \text{GSVI}(C, A, B)$,*

where $GSVI(C, A, B)$ is the set of fixed points of the mapping $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$ and $y^* = \Pi_C(x^* - \mu Bx^*)$.

Proposition 2 ([27]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let the mappings $A, B : C \rightarrow X$ be α -inverse-strongly accretive and β -inverse-strongly accretive, respectively. Then*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\kappa^2\lambda - \alpha)\|Ax - Ay\|^2$$

and

$$\|(I - \mu B)x - (I - \mu B)y\|^2 \leq \|x - y\|^2 + 2\mu(\kappa^2\mu - \beta)\|Bx - By\|^2.$$

In particular, if $0 \leq \lambda \leq \frac{\alpha}{\kappa^2}$ and $0 \leq \mu \leq \frac{\beta}{\kappa^2}$, then $I - \lambda A$ and $I - \mu B$ are nonexpansive.

Lemma 4 ([2]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mappings $A, B : C \rightarrow X$ be α -inverse-strongly accretive and β -inverse-strongly accretive, respectively. Let the mapping $G : C \rightarrow C$ be defined as $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$. If $0 \leq \lambda \leq \frac{\alpha}{\kappa^2}$ and $0 \leq \mu \leq \frac{\beta}{\kappa^2}$, then $G : C \rightarrow C$ is nonexpansive.*

Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mappings $A, B : C \rightarrow X$ be α -inverse-strongly accretive and β -inverse-strongly accretive, respectively. Let $F : C \rightarrow X$ be δ -strongly accretive and ζ -strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$, where κ is the 2-uniformly smooth constant of X (see Lemma 2). Very recently, in order to solve GSVI (2), Ceng et al. [3] introduced an implicit algorithm of Mann’s type.

Algorithm 1 ([3]) *For each $t \in (0, 1)$, choose a number $\theta_t \in (0, 1)$ arbitrarily. The net $\{x_t\}$ is generated by the implicit method*

$$x_t = t\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_t + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_t, \quad \forall t \in (0, 1),$$

where x_t is a unique fixed point of the contraction

$$W_t = t\Pi_C(I - \lambda A)\Pi_C(I - \mu B) + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B).$$

It was proven in [3] that the net $\{x_t\}$ converges in norm, as $t \rightarrow 0^+$, to the unique solution $x^* \in GSVI(C, A, B)$ to the following VI:

$$\langle F(x^*), j(x - x^*) \rangle \geq 0, \quad \forall x \in GSVI(C, A, B), \tag{5}$$

provided $\lim_{t \rightarrow 0^+} \theta_t = 0$. In the meantime, the authors [3] also proposed another explicit algorithm of Mann’s type.

Algorithm 2 ([3]) *For arbitrarily given $x_0 \in C$, let the sequence $\{x_k\}$ be generated iteratively by*

$$x_{k+1} = \beta_k x_k + \gamma_k \Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_k + (1 - \beta_k - \gamma_k)\Pi_C(I - \lambda_k F)\Pi_C(I - \lambda A)\Pi_C(I - \mu B)x_k,$$

where $\{\lambda_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ are three sequences in $[0, 1]$ such that $\beta_k + \gamma_k \leq 1, \forall k \geq 0$.

On the other hand, a mapping F with domain $D(F)$ and range $R(F)$ in X is called

(a) accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \geq 0,$$

where J is the normalized duality mapping;

(b) δ -strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \geq \delta \|x - y\|^2 \quad \text{for some } \delta \in (0, 1);$$

(c) α -inverse-strongly accretive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \geq \alpha \|Fx - Fy\|^2 \quad \text{for some } \alpha \in (0, 1);$$

(d) ζ -strictly pseudocontractive if for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \leq \|x - y\|^2 - \zeta \|x - y - (Fx - Fy)\|^2 \quad \text{for some } \zeta \in (0, 1).$$

It is easy to see that (5) can be rewritten as (see [28])

$$\langle (I - F)x - (I - F)y, j(x - y) \rangle \geq \zeta \|(I - F)x - (I - F)y\|^2, \tag{6}$$

where I denotes the identity mapping of X . Clearly, if F is ζ -strictly pseudocontractive with $\zeta = 0$, then it is said to be pseudocontractive. It is not hard to find that every nonexpansive mapping is pseudocontractive.

Let C be a nonempty closed convex subset of a smooth Banach space X and $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C . Then we set $\mathcal{F} := \bigcap_{i=1}^\infty \text{Fix}(T_i)$. In 2013, Buong and Phuong [18] considered the following HVI with $C = X$: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), j(x - x^*) \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{7}$$

In the case where $X = H$, a Hilbert space, we have $J = I$, and hence problem (7) reduces to the HVI: find $x^* \in \mathcal{F}$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{8}$$

Assume that $\mathcal{F} = \bigcap_{i=1}^N \text{Fix}(T_i)$ is the set of common fixed points of a family of N nonexpansive mappings T_i on H , and F is an L -Lipschitz continuous and η -strongly monotone mapping, i.e., $\|Fx - Fy\| \leq L\|x - y\|$ and $\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2$ for all $x, y \in H$. Zeng and Yao [19] introduced the following implicit iteration: for an arbitrarily initial point $x_0 \in H$, the sequence $\{x_k\}_{k=1}^\infty$ is generated as follows:

$$x_k = \beta_k x_{k-1} + (1 - \beta_k) [T_{[k]} x_k - \lambda_k \mu F(T_{[k]} x_k)], \quad \forall k \geq 1, \tag{9}$$

where $T_{[n]} = T_{n \bmod N}$, for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, \dots, N\}$. They proved the following result.

Theorem 3 ([19]) *Let H be a real Hilbert space, and let $F : H \rightarrow H$ be a mapping such that, for some positive constants L and η , F is L -Lipschitz continuous and η -strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive mappings on H such that $\mathcal{F} = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, $x_0 \in H$, $\{\lambda_k\}_{k=1}^\infty \subset [0, 1)$ and $\{\beta_k\}_{k=1}^\infty \subset (0, 1)$ satisfying the conditions $\sum_{k=1}^\infty \lambda_k < \infty$, and let $a \leq \beta_k \leq b, k \geq 1$, for some $a, b \in (0, 1)$. Then the sequence $\{x_k\}_{k=0}^\infty$, defined by (9), converges weakly to $x^* \in \mathcal{F}$.*

Recently, for deriving the strong convergence and weakening the condition on λ_k , Buong and Anh [21] proposed the following implicit iteration method:

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t \cdots T_1^t, \quad t \in (0, 1), \tag{10}$$

where the sequence $\{T_i^t\}_{i=0}^N$ is defined by

$$\begin{cases} T_i^t x := (1 - \beta_i^t)x + \beta_i^t T_i x, & i = 1, \dots, N, \\ T_0^t y := (I - \lambda_t \mu F)y, & x, y \in H, \end{cases} \tag{11}$$

and proved that the nets $\{x_t\}$, defined by (10)-(11), converge strongly to an element x^* in (8). Under the assumptions that $N = 1$, X is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and T is a continuous pseudocontractive mapping, Ceng et al. [29] proved the following result.

Theorem 4 ([29]) *Let F be a δ -strongly accretive and ζ -strictly pseudocontractive mapping with $\delta + \zeta > 1$, and let T be a continuous and pseudocontractive mapping on X , which is a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm such that $\mathcal{F} := \text{Fix}(T) \neq \emptyset$. For each $t \in (0, 1)$, choose a number $\mu_t \in (0, 1)$ arbitrarily, and let $\{z_t\}$ be defined by*

$$z_t = t(I - \mu_t F)z_t + (1 - t)Tz_t. \tag{12}$$

Then, as $t \rightarrow 0^+$, $\{z_t\}$ converges strongly to $x^ \in \mathcal{F}$.*

In [30], Takahashi introduced the following W -mapping which is generated by T_k, T_{k-1}, \dots, T_1 and real numbers $\alpha_k, \alpha_{k-1}, \dots, \alpha_1$ as follows:

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \alpha_k T_k U_{k,k+1} + (1 - \alpha_k)I, \\ U_{k,k-1} = \alpha_{k-1} T_{k-1} U_{k,k} + (1 - \alpha_{k-1})I, \\ \dots \\ U_{k,2} = \alpha_2 T_2 U_{k,3} + (1 - \alpha_2)I, \\ W_k = U_{k,1} = \alpha_1 T_1 U_{k,2} + (1 - \alpha_1)I. \end{cases} \tag{13}$$

Kikkawa and Takahashi [31] considered the following strongly convergent implicit method:

$$S_k x = \left(1 - \frac{1}{k}\right) Ux + \frac{1}{k} f(x), \quad \text{and} \quad Ux = \lim_{k \rightarrow \infty} W_k x = \lim_{k \rightarrow \infty} U_{k,1} x. \tag{14}$$

However, the method (14) is very difficult to be grasped due to the limit mapping U .

Motivated by methods (10) and (12), Buong and Phuong [18] considered two implicit methods by introducing a mapping V_k defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1} \dots T^k, \quad T^i = (1 - \alpha_i)I + \alpha_i T_i, \quad i = 1, 2, \dots, k, \tag{15}$$

where

$$\alpha_i \in (0, 1) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i < \infty. \tag{16}$$

In both methods, the iteration sequence $\{x_k\}_{k=1}^{\infty}$ is defined, respectively, by

$$x_k = V_k(I - \lambda_k F)x_k, \quad \forall k \geq 1, \tag{17}$$

and

$$x_k = \gamma_k(I - \lambda_k F)x_k + (I - \gamma_k)V_k x_k, \quad \forall k \geq 1, \tag{18}$$

where λ_k and γ_k are the positive parameters satisfying some additional conditions. The strong convergence theorems for methods (17) and (18) are also established.

We will make use of the following well-known results.

Lemma 5 *Let X be a real normed linear space. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall x, y \in X, \forall j(x + y) \in J(x + y).$$

Lemma 6 ([32]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $(I - T)x_n \rightarrow y$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \text{Fix}(T)$.*

Lemma 7 ([2]) *Let C be a nonempty closed convex subset of a real smooth Banach space X . Assume that the mapping $F : C \rightarrow X$ is accretive and weakly continuous along segments (that is, $F(x + ty) \rightarrow F(x)$ as $t \rightarrow 0$). Then the variational inequality*

$$x^* \in C, \quad \langle F(x^*), j(x - x^*) \rangle \geq 0, \quad \forall x \in C$$

is equivalent to the following Minty type variational inequality:

$$x^* \in C, \quad \langle F(x), j(x - x^*) \rangle \geq 0, \quad \forall x \in C.$$

Lemma 8 ([3]) *Let X be a real smooth Banach space and $F : C \rightarrow X$ be a mapping.*

- (a) *If F is ζ -strictly pseudocontractive, then F is Lipschitz continuous with constant $1 + \frac{1}{\zeta}$.*
- (b) *If F is δ -strongly accretive and ζ -strictly pseudocontractive with $\delta + \zeta > 1$, then $I - F$ is contractive with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$.*
- (c) *If F is δ -strongly accretive and ζ -strictly pseudocontractive with $\delta + \zeta > 1$, then for any fixed number $\lambda \in (0, 1)$, $I - \lambda F$ is contractive with constant $1 - \lambda(1 - \sqrt{\frac{1-\delta}{\zeta}}) \in (0, 1)$.*

3 Iterative algorithms and convergence criteria

In this section, we study iterative methods for computing approximate solutions of the HVI (for an infinite family of nonexpansive mappings) with a GSVI constraint. We introduce implicit and explicit iterative algorithms for solving such a problem. We show the strong convergence theorems for the sequences generated by the proposed algorithms.

The following lemmas will be used to prove our main results in the sequel.

Lemma 9 ([18]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X , and let $\{T_i\}_{i=1}^k, k \geq 1$, be k nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^k \text{Fix}(T_i) \neq \emptyset$. Let a, b and $\alpha_i, i = 1, 2, \dots, k$, be real numbers such that $0 < a \leq \alpha_i \leq b < 1$, and let V_k be a mapping defined by (15) for all $k \geq 1$. Then $\text{Fix}(V_k) = \mathcal{F}$.*

Lemma 10 ([18]) *Let C be a nonempty closed convex subset of a Banach space X , and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^\infty \text{Fix}(T_i) \neq \emptyset$. Let V_k be a mapping defined by (15), and let α_i satisfy (16). Then, for each $x \in C$ and $i \geq 1$, $\lim_{k \rightarrow \infty} V_k^i x$ exists.*

Remark 1

- (i) We can define the mappings

$$V_\infty^i x := \lim_{k \rightarrow \infty} V_k^i x \quad \text{and} \quad Vx := V_\infty^1 x = \lim_{k \rightarrow \infty} V_k x, \quad \forall x \in C.$$

- (ii) It can be readily seen from the proof of Lemma 10 that if D is a nonempty and bounded subset of C , then the following holds:

$$\lim_{k \rightarrow \infty} \sup_{x \in D} \|V_k^i x - V_\infty^i x\| = 0, \quad \forall i \geq 1.$$

In particular, whenever $i = 1$, we have

$$\lim_{k \rightarrow \infty} \sup_{x \in D} \|V_k x - Vx\| = 0.$$

Lemma 11 ([18]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X , and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^\infty \text{Fix}(T_i) \neq \emptyset$. Let α_i satisfy the first condition in (16). Then $\text{Fix}(V) = \mathcal{F}$.*

Lemma 12 ([33]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X , and let $\{\alpha_k\}$ be a sequence in $[0, 1]$ such that*

$$0 < \liminf_{k \rightarrow \infty} \alpha_k \leq \limsup_{k \rightarrow \infty} \alpha_k < 1.$$

Suppose that $x_{k+1} = \alpha_k x_k + (1 - \alpha_k)z_k, \forall k \geq 1$, and

$$\limsup_{k \rightarrow \infty} (\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|) \leq 0.$$

Then $\lim_{k \rightarrow \infty} \|z_k - x_k\| = 0$.

Lemma 13 ([34]) *Assume that $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \leq (1 - \gamma_k)a_k + \gamma_k \delta_k, \quad \forall k \geq 1,$$

where $\{\gamma_k\}$ is a sequence in $[0, 1]$ and $\{\delta_k\}$ is a sequence in \mathbf{R} such that

- (i) $\sum_{k=1}^\infty \gamma_k = \infty$;
- (ii) $\limsup_{k \rightarrow \infty} \delta_k \leq 0$ or $\sum_{k=1}^\infty |\gamma_k \delta_k| < \infty$.

Then $\lim_{k \rightarrow \infty} a_k = 0$.

Now, we are in a position to prove the following main results.

Theorem 5 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X . Let Π_C be a sunny nonexpansive retraction from X onto C . Let the mappings $A, B : C \rightarrow X$ be α -inverse-strongly accretive and β -inverse-strongly accretive, respectively. Let $F : C \rightarrow X$ be δ -strongly accretive and ζ -strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\lambda \in (0, \frac{\alpha}{\kappa^2})$ and $\mu \in (0, \frac{\beta}{\kappa^2})$ where κ is the 2-uniformly smooth constant of X . Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^\infty \text{Fix}(T_i) \cap \text{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (15) and (16). Let $\{x_k\}_{k=1}^\infty$ be defined by*

$$x_k = \gamma_k \Pi_C(I - \lambda_k F) \Pi_C(I - \lambda A) \Pi_C(I - \mu B) V_k x_k + (1 - \gamma_k) \Pi_C(I - \lambda A) \Pi_C(I - \mu B) V_k x_k, \quad \forall k \geq 1,$$

where $\{\gamma_k\}$ and $\{\lambda_k\}$ are sequences in $(0, 1]$ such that $\gamma_k \rightarrow 0$ and $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$. Then $\{x_k\}_{k=1}^\infty$ converges strongly to a unique solution $x^ \in \mathcal{F}$ to the following VI:*

$$\langle F(x^*), j(x - x^*) \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{19}$$

Proof Let the mapping $G : C \rightarrow C$ be defined as $G := \Pi_C(I - \lambda A) \Pi_C(I - \mu B)$, where $0 < \lambda < \frac{\alpha}{\kappa^2}$ and $0 < \mu < \frac{\beta}{\kappa^2}$. In terms of Lemma 4 we know that $G : C \rightarrow C$ is nonexpansive. Then

the implicit iterative scheme can be rewritten as

$$x_k = \gamma_k \Pi_C(I - \lambda_k F)GV_k x_k + (1 - \gamma_k)GV_k x_k, \quad \forall k \geq 1. \tag{20}$$

Consider the mapping $U_k x = \gamma_k \Pi_C(I - \lambda_k F)GV_k x + (1 - \gamma_k)GV_k x, \forall x \in C$. From Lemmas 4 and 8(c), it follows that for each $x, y \in C$,

$$\begin{aligned} \|U_k x - U_k y\| &= \|\gamma_k (\Pi_C(I - \lambda_k F)GV_k x - \Pi_C(I - \lambda_k F)GV_k y) \\ &\quad + (1 - \gamma_k)(GV_k x - GV_k y)\| \\ &\leq \gamma_k \|(I - \lambda_k F)GV_k x - (I - \lambda_k F)GV_k y\| + (1 - \gamma_k)\|GV_k x - GV_k y\| \\ &\leq \gamma_k(1 - \lambda_k \tau)\|GV_k x - GV_k y\| + (1 - \gamma_k)\|GV_k x - GV_k y\| \\ &= (1 - \gamma_k \lambda_k \tau)\|GV_k x - GV_k y\| \\ &\leq (1 - \gamma_k \lambda_k \tau)\|x - y\|, \end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$ (by using $\delta + \zeta > 1$). Due to $\gamma_k \lambda_k \tau \in (0, 1)$, U_k is a contraction of C into itself. Hence, by Banach’s contraction principle, there exists a unique element $x_k \in C$ satisfying (20).

Next, we divide the rest of the proof into several steps.

Step 1. We show that $\{x_k\}_{k=1}^\infty$ is bounded. Indeed, take an arbitrarily given $p \in \mathcal{F}$. Then we have $V_k p = p$ and $Gp = p$. Hence, by Lemma 8(c) we get

$$\begin{aligned} \|x_k - p\|^2 &= \|\gamma_k \Pi_C(I - \lambda_k F)GV_k x_k + (1 - \gamma_k)GV_k x_k - p\|^2 \\ &\leq \gamma_k \|\Pi_C(I - \lambda_k F)GV_k x_k - p\|^2 + (1 - \gamma_k)\|GV_k x_k - p\|^2 \\ &\leq \gamma_k \|(I - \lambda_k F)GV_k x_k - p\|^2 + (1 - \gamma_k)\|GV_k x_k - p\|^2 \\ &= \gamma_k \|(I - \lambda_k F)GV_k x_k - (I - \lambda_k F)p - \lambda_k F(p)\|^2 + (1 - \gamma_k)\|GV_k x_k - p\|^2 \\ &\leq \gamma_k [(1 - \lambda_k \tau)\|GV_k x_k - p\| + \lambda_k \|F(p)\|]^2 + (1 - \gamma_k)\|GV_k x_k - p\|^2 \\ &\leq \gamma_k [(1 - \lambda_k \tau)\|GV_k x_k - p\|^2 + \lambda_k \tau^{-1}\|F(p)\|^2] + (1 - \gamma_k)\|GV_k x_k - p\|^2 \\ &= (1 - \gamma_k \lambda_k \tau)\|GV_k x_k - p\|^2 + \gamma_k \lambda_k \tau^{-1}\|F(p)\|^2 \\ &\leq (1 - \gamma_k \lambda_k \tau)\|x_k - p\|^2 + \gamma_k \lambda_k \tau^{-1}\|F(p)\|^2. \end{aligned} \tag{21}$$

Therefore, $\|x_k - p\| \leq \|F(p)\|/\tau$, which also leads to the boundedness of $\{x_k\}_{k=1}^\infty$. So, the sequences $\{V_k x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty$ and $\{F(y_k)\}_{k=1}^\infty$, where $y_k = GV_k x_k$, are also bounded. Since $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$, and the following relation holds

$$\begin{aligned} \|x_k - GV_k x_k\| &= \gamma_k \|\Pi_C(I - \lambda_k F)GV_k x_k - GV_k x_k\| \\ &\leq \gamma_k \|(I - \lambda_k F)GV_k x_k - GV_k x_k\| \\ &= \gamma_k \lambda_k \|F(y_k)\| \leq \gamma_k \|F(y_k)\|, \end{aligned}$$

we obtain from the boundedness of $\{F(y_k)\}$ that $\|x_k - GV_k x_k\| \rightarrow 0$ as $k \rightarrow \infty$.

Step 2. We show that $\|z_k - Gz_k\| \rightarrow 0$ as $k \rightarrow \infty$, where $z_k = V_k x_k$ for all $k \geq 1$. Indeed, for simplicity, put $q = \Pi_C(p - \mu Bp)$ and $u_k = \Pi_C(z_k - \mu Bz_k)$. Then $y_k = Gz_k = \Pi_C(u_k - \lambda Au_k)$ for all $k \geq 1$. From Lemma 2, we have

$$\begin{aligned} \|u_k - q\|^2 &= \|\Pi_C(z_k - \mu Bz_k) - \Pi_C(p - \mu Bp)\|^2 \\ &\leq \|z_k - p - \mu(Bz_k - Bp)\|^2 \\ &\leq \|z_k - p\|^2 - 2\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2 \\ &\leq \|x_k - p\|^2 - 2\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \|y_k - p\|^2 &= \|\Pi_C(u_k - \lambda Au_k) - \Pi_C(q - \lambda Aq)\|^2 \\ &\leq \|u_k - q - \lambda(Au_k - Aq)\|^2 \\ &\leq \|u_k - q\|^2 - 2\lambda(\alpha - \kappa^2\lambda)\|Au_k - Aq\|^2. \end{aligned} \tag{23}$$

Substituting (22) for (23), we obtain

$$\begin{aligned} \|y_k - p\|^2 &\leq \|x_k - p\|^2 - 2\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2 \\ &\quad - 2\lambda(\alpha - \kappa^2\lambda)\|Au_k - Aq\|^2. \end{aligned} \tag{24}$$

From (21) and (24), we have

$$\begin{aligned} \|x_k - p\|^2 &\leq (1 - \gamma_k \lambda_k \tau)\|Gz_k - p\|^2 + \gamma_k \lambda_k \tau^{-1}\|F(p)\|^2 \\ &\leq \|Gz_k - p\|^2 + \gamma_k \tau^{-1}\|F(p)\|^2 \\ &\leq \|x_k - p\|^2 - 2\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2 \\ &\quad - 2\lambda(\alpha - \kappa^2\lambda)\|Au_k - Aq\|^2 + \gamma_k \frac{\|F(p)\|^2}{\tau}, \end{aligned}$$

which immediately yields

$$2\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2 + 2\lambda(\alpha - \kappa^2\lambda)\|Au_k - Aq\|^2 \leq \gamma_k \frac{\|F(p)\|^2}{\tau}.$$

So, from $\lambda \in (0, \frac{\alpha}{\kappa^2})$, $\mu \in (0, \frac{\beta}{\kappa^2})$ and $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$, we deduce that

$$\lim_{k \rightarrow \infty} \|Bz_k - Bp\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|Au_k - Aq\| = 0. \tag{25}$$

Utilizing Proposition 1 and Lemma 2, we have

$$\begin{aligned} \|u_k - q\|^2 &= \|\Pi_C(z_k - \mu Bz_k) - \Pi_C(p - \mu Bp)\|^2 \\ &\leq \langle (z_k - \mu Bz_k) - (p - \mu Bp), j(u_k - q) \rangle \\ &= \langle (z_k - p, j(u_k - q)) \rangle + \mu \langle Bp - Bz_k, j(u_k - q) \rangle \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} [\|z_k - p\|^2 + \|u_k - q\|^2 - g_1(\|z_k - u_k - (p - q)\|)] \\ &\quad + \mu \|Bp - Bz_k\| \|u_k - q\| \\ &\leq \frac{1}{2} [\|x_k - p\|^2 + \|u_k - q\|^2 - g_1(\|z_k - u_k - (p - q)\|)] \\ &\quad + \mu \|Bp - Bz_k\| \|u_k - q\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|u_k - q\|^2 &\leq \|x_k - p\|^2 - g_1(\|z_k - u_k - (p - q)\|) \\ &\quad + 2\mu \|Bp - Bz_k\| \|u_k - q\|. \end{aligned} \tag{26}$$

In the same way, we derive

$$\begin{aligned} \|y_k - p\|^2 &= \|\Pi_C(u_k - \lambda Au_k) - \Pi_C(q - \lambda Aq)\|^2 \\ &\leq \langle u_k - \lambda Au_k - (q - \lambda Aq), j(y_k - p) \rangle \\ &= \langle u_k - q, j(y_k - p) \rangle + \lambda \langle Aq - Au_k, j(y_k - p) \rangle \\ &\leq \frac{1}{2} [\|u_k - q\|^2 + \|y_k - p\|^2 - g_2(\|u_k - y_k + (p - q)\|)] \\ &\quad + \lambda \|Aq - Au_k\| \|y_k - p\|, \end{aligned}$$

which implies that

$$\|y_k - p\|^2 \leq \|u_k - q\|^2 - g_2(\|u_k - y_k + (p - q)\|) + 2\lambda \|Aq - Au_k\| \|y_k - p\|. \tag{27}$$

Substituting (26) for (27), we get

$$\begin{aligned} \|y_k - p\|^2 &\leq \|x_k - p\|^2 - g_1(\|z_k - u_k - (p - q)\|) - g_2(\|u_k - y_k + (p - q)\|) \\ &\quad + 2\mu \|Bp - Bz_k\| \|u_k - q\| + 2\lambda \|Aq - Au_k\| \|y_k - p\|. \end{aligned} \tag{28}$$

So, from (21) and (28) it follows that

$$\begin{aligned} \|x_k - p\|^2 &\leq (1 - \gamma_k \lambda_k \tau) \|Gz_k - p\|^2 + \gamma_k \lambda_k \tau^{-1} \|F(p)\|^2 \\ &\leq \|y_k - p\|^2 + \gamma_k \frac{\|F(p)\|^2}{\tau} \\ &\leq \|x_k - p\|^2 - g_1(\|z_k - u_k - (p - q)\|) - g_2(\|u_k - y_k + (p - q)\|) \\ &\quad + 2\mu \|Bp - Bz_k\| \|u_k - q\| + 2\lambda \|Aq - Au_k\| \|y_k - p\| + \gamma_k \frac{\|F(p)\|^2}{\tau}, \end{aligned}$$

which hence leads to

$$\begin{aligned} &g_1(\|z_k - u_k - (p - q)\|) + g_2(\|u_k - y_k + (p - q)\|) \\ &\leq 2\mu \|Bp - Bz_k\| \|u_k - q\| + 2\lambda \|Aq - Au_k\| \|y_k - p\| + \gamma_k \frac{\|F(p)\|^2}{\tau}. \end{aligned} \tag{29}$$

From (25), $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$, and the boundedness of $\{u_k\}$ and $\{y_k\}$, we deduce that

$$\lim_{k \rightarrow \infty} g_1(\|z_k - u_k - (p - q)\|) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} g_2(\|u_k - y_k + (p - q)\|) = 0.$$

Utilizing the properties of g_1 and g_2 , we conclude that

$$\lim_{k \rightarrow \infty} \|z_k - u_k - (p - q)\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u_k - y_k + (p - q)\| = 0. \tag{30}$$

From (30), we get

$$\|z_k - y_k\| \leq \|z_k - u_k - (p - q)\| + \|u_k - y_k + (p - q)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

That is,

$$\lim_{k \rightarrow \infty} \|z_k - Gz_k\| = \lim_{k \rightarrow \infty} \|z_k - y_k\| = 0. \tag{31}$$

This together with $\|x_k - GV_k x_k\| \rightarrow 0$ implies that

$$\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|x_k - V_k x_k\| = \lim_{k \rightarrow \infty} \|x_k - z_k\| = 0. \tag{32}$$

Step 3. We show that $\omega_w(x_k) \subset \mathcal{F}$, where

$$\omega_w(x_k) = \{x \in C : x_{k_i} \rightarrow x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Indeed, we first claim that $\|x_k - V_k x_k\| \rightarrow 0$ as $k \rightarrow \infty$. It is easy to see from Remark 1(ii) that if D is a nonempty and bounded subset of C , then, for $\varepsilon > 0$, there exists $k_0 > i$ such that for all $k > k_0$

$$\sup_{x \in D} \|V_k^i x - V_\infty^i x\| \leq \varepsilon.$$

Taking $D = \{x_k : k \geq 1\}$ and $i = 1$, we have

$$\|V_k x_k - V x_k\| \leq \sup_{x \in D} \|V_k x - V x\| \leq \varepsilon.$$

So, it follows that

$$\lim_{k \rightarrow \infty} \|V_k x_k - V x_k\| = 0. \tag{33}$$

Noting that

$$\begin{aligned} \|GV_k x_k - V_k x_k\| &\leq \|GV_k x_k - GV_k x_k\| + \|GV_k x_k - V_k x_k\| + \|V_k x_k - V_k x_k\| \\ &\leq \|V_k x_k - V_k x_k\| + \|Gz_k - z_k\| + \|V_k x_k - V_k x_k\| \\ &= 2\|V_k x_k - V_k x_k\| + \|Gz_k - z_k\|, \end{aligned} \tag{34}$$

from (31), (33) and (34) we obtain that

$$\lim_{k \rightarrow \infty} \|GVx_k - Vx_k\| = 0. \tag{35}$$

Also, noting that $\|x_k - Vx_k\| \leq \|x_k - V_kx_k\| + \|V_kx_k - Vx_k\|$, from (32) and (33) we get

$$\lim_{k \rightarrow \infty} \|x_k - Vx_k\| = 0. \tag{36}$$

In addition, observing that

$$\begin{aligned} \|x_k - Gx_k\| &\leq \|x_k - z_k\| + \|z_k - Gz_k\| + \|Gz_k - Gx_k\| \\ &\leq \|x_k - z_k\| + \|z_k - Gz_k\| + \|z_k - x_k\| \\ &= 2\|x_k - z_k\| + \|z_k - Gz_k\|, \end{aligned}$$

from (31) and (32) we get

$$\lim_{k \rightarrow \infty} \|x_k - Gx_k\| = 0. \tag{37}$$

Since X is reflexive, there exists at least a weak convergence subsequence of $\{x_k\}$, and hence $\omega_w(x_k) \neq \emptyset$. Take an arbitrary $p \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup p$. Since V_k is nonexpansive for all $k \geq 1$, V is a nonexpansive self-mapping on C . Also, since X is uniformly convex and V and G are two nonexpansive self-mappings on C , utilizing Lemma 6 we know from (36) and (37) that $p \in \text{GSVI}(C, A, B)$ and $p \in \text{Fix}(V) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ (due to Lemma 11). Consequently, $p \in \text{Fix}(V) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \cap \text{GSVI}(C, A, B) =: \mathcal{F}$. This shows that $\omega_w(x_k) \subset \mathcal{F}$.

Step 4. We show that $\omega_w(x_k) = \omega_s(x_k)$, where

$$\omega_s(x_k) = \{x \in C : x_{k_i} \rightarrow x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Indeed, by Lemma 9, we have $\|GV_kx_k - z\| \leq \|x_k - z\|$ for any fixed $z \in \mathcal{F}$, and hence

$$\begin{aligned} \|x_k - z\|^2 &= \|\gamma_k \Pi_C(I - \lambda_k F)GV_kx_k + (1 - \gamma_k)GV_kx_k - z\|^2 \\ &= \gamma_k \left[\langle \Pi_C(I - \lambda_k F)GV_kx_k - (I - \lambda_k F)GV_kx_k, j(x_k - z) \rangle \right. \\ &\quad \left. + \langle \lambda_k(I - F)GV_kx_k + (1 - \lambda_k)GV_kx_k - z, j(x_k - z) \rangle \right] \\ &\quad + (1 - \gamma_k) \langle GV_kx_k - z, j(x_k - z) \rangle \\ &= \gamma_k \left[\langle \Pi_C(I - \lambda_k F)GV_kx_k - (I - \lambda_k F)GV_kx_k, j(\Pi_C(I - \lambda_k F)GV_kx_k - z) \rangle \right. \\ &\quad \left. + \langle \Pi_C(I - \lambda_k F)GV_kx_k - (I - \lambda_k F)GV_kx_k, j(x_k - z) \rangle \right. \\ &\quad \left. - j(\Pi_C(I - \lambda_k F)GV_kx_k - z) \right] \\ &\quad + \langle \lambda_k(I - F)GV_kx_k + (1 - \lambda_k)GV_kx_k - z, j(x_k - z) \rangle \\ &\quad + (1 - \gamma_k) \langle GV_kx_k - z, j(x_k - z) \rangle \end{aligned}$$

$$\begin{aligned}
 &\leq \gamma_k \left[\left\| \Pi_C(I - \lambda_k F)G V_k x_k - (I - \lambda_k F)G V_k x_k \right\| \left\| j(x_k - z) \right. \right. \\
 &\quad \left. \left. - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| \right. \\
 &\quad \left. + \langle \lambda_k(I - F)G V_k x_k + (1 - \lambda_k)G V_k x_k - z, j(x_k - z) \rangle \right] \\
 &\quad + (1 - \gamma_k) \langle G V_k x_k - z, j(x_k - z) \rangle \\
 &\leq \gamma_k \left[\left\| \Pi_C(I - \lambda_k F)G V_k x_k - (I - \lambda_k F)G V_k x_k \right\| \left\| j(x_k - z) \right. \right. \\
 &\quad \left. \left. - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| \right. \\
 &\quad \left. + \lambda_k \langle (I - F)G V_k x_k - z, j(x_k - z) \rangle + (1 - \lambda_k) \|x_k - z\|^2 \right] \\
 &\quad + (1 - \gamma_k) \|x_k - z\|^2 \\
 &\leq \gamma_k \left(\left\| \Pi_C(I - \lambda_k F)G V_k x_k - G V_k x_k \right\| \right. \\
 &\quad \left. + \lambda_k \left\| F(G V_k x_k) \right\| \right) \left\| j(x_k - z) - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| \\
 &\quad + \gamma_k \lambda_k \langle (I - F)G V_k x_k - z, j(x_k - z) \rangle + (1 - \gamma_k \lambda_k) \|x_k - z\|^2 \\
 &\leq 2\gamma_k \lambda_k \left\| F(G V_k x_k) \right\| \left\| j(x_k - z) - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| \\
 &\quad + \gamma_k \lambda_k \langle (I - F)G V_k x_k - (I - F)z - F(z), j(x_k - z) \rangle + (1 - \gamma_k \lambda_k) \|x_k - z\|^2.
 \end{aligned}$$

Therefore, by Lemma 8(b) we get

$$\begin{aligned}
 \|x_k - z\|^2 &\leq (1 - \tau) \|x_k - z\|^2 - \langle F(z), j(x_k - z) \rangle \\
 &\quad + 2 \left\| F(x_k) \right\| \left\| j(x_k - z) - j(\Pi_C(I - \lambda_k F)x_k - z) \right\|,
 \end{aligned}$$

which immediately leads to

$$\begin{aligned}
 \|x_k - z\|^2 &\leq \frac{1}{\tau} \left(\langle F(z), j(z - x_k) \rangle \right. \\
 &\quad \left. + 2 \left\| F(G V_k x_k) \right\| \left\| j(x_k - z) - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| \right), \quad \forall z \in \mathcal{F}, \quad (38)
 \end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$. Note that

$$\begin{aligned}
 \left\| (x_k - z) - (\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| &\leq \left\| x_k - (I - \lambda_k F)y_k \right\| \\
 &\leq \|x_k - y_k\| + \lambda_k \left\| F(y_k) \right\|.
 \end{aligned}$$

Since the uniform smoothness of X guarantees the uniform continuity of j on every nonempty bounded subset of X , we deduce from (32), $\lambda_k \rightarrow 0$ and the boundedness of $\{y_k\}$ that

$$\lim_{k \rightarrow \infty} \left\| j(x_k - z) - j(\Pi_C(I - \lambda_k F)G V_k x_k - z) \right\| = 0.$$

Now, take an arbitrary $p \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightarrow p$. In terms of Step 3, we know that $p \in \omega_w(x_k) \subset \mathcal{F}$. Thus, we can substitute x_{k_i}

for x_k and p for z in (38) to get

$$\begin{aligned} \|x_{k_i} - p\|^2 &\leq \frac{1}{\tau} \langle (F(p), j(p - x_{k_i})) \rangle \\ &\quad + 2 \|F(y_{k_i})\| \|j(x_{k_i} - p) - j(\Pi_C(I - \lambda_{k_i}F)y_{k_i} - p)\|. \end{aligned} \tag{39}$$

Consequently, the weak convergence of $\{x_{k_i}\}$ to p together with (39) actually implies that $x_{k_i} \rightarrow p$ as $i \rightarrow \infty$, and hence $p \in \omega_s(x_k)$. This shows that $\omega_w(x_k) = \omega_s(x_k)$.

Step 5. We show that each $p \in \omega_s(x_k)$ solves the variational inequality (19). Indeed, take an arbitrary $p \in \omega_s(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightarrow p$ as $i \rightarrow \infty$. According to Steps 3 and 4, we know that $p \in \omega_s(x_k) (= \omega_w(x_k) \subset \mathcal{F})$. Replacing x_k in (39) with x_{k_i} , and noticing that $x_{k_i} \rightarrow p$, we have the Minty type variational inequality

$$\langle F(z), j(z - p) \rangle \leq 0, \quad \forall z \in \mathcal{F},$$

which is equivalent to the variational inequality (see Lemma 7)

$$\langle F(p), j(p - z) \rangle \leq 0, \quad \forall z \in \mathcal{F}. \tag{40}$$

That is, $p \in \mathcal{F}$ is a solution of (19).

Step 6. We show that $\{x_k\}$ converges strongly to a unique solution in \mathcal{F} to VI (19). Indeed, we first claim that the solution set of (19) is a singleton. As a matter of fact, assume that $\bar{p} \in \mathcal{F}$ is also a solution of (19). Then we have

$$\langle F(\bar{p}), j(\bar{p} - p) \rangle \leq 0.$$

From (40), we have

$$\langle F(p), j(p - \bar{p}) \rangle \leq 0.$$

So, by the δ -strong accretiveness of F , we have

$$\begin{aligned} \langle F(\bar{p}), j(\bar{p} - p) \rangle + \langle F(p), j(p - \bar{p}) \rangle &\leq 0 \\ \Rightarrow \langle F(\bar{p}) - F(p), j(\bar{p} - p) \rangle &\leq 0 \\ \Rightarrow \delta \|\bar{p} - p\|^2 &\leq 0. \end{aligned}$$

Therefore, $\bar{p} = p$. In summary, we have shown that each cluster point of $\{x_k\}$ (as $k \rightarrow \infty$) equals p . Consequently, $x_k \rightarrow p$ as $k \rightarrow \infty$. □

Theorem 6 *Let $C, X, \Pi_C, A, B, F, \{T_i\}_{i=1}^\infty, \mathcal{F}, \delta, \zeta, \lambda$ and μ be as in Theorem 5. Let $\{V_k\}_{k=1}^\infty$ be defined by (15) and (16). For arbitrarily given $x_1 \in C$, let $\{x_k\}_{k=1}^\infty$ be defined by*

$$\begin{aligned} x_{k+1} &= \beta_k x_k + \gamma_k \Pi_C(I - \lambda A) \Pi_C(I - \mu B) V_k x_k \\ &\quad + (1 - \beta_k - \gamma_k) \Pi_C(I - \lambda_k F) \Pi_C(I - \lambda A) \Pi_C(I - \mu B) V_k x_k, \quad \forall k \geq 1, \end{aligned}$$

where $\{\lambda_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ are three sequences in $[0, 1]$ with $\beta_k + \gamma_k \leq 1, \forall k \geq 1$, and satisfy the following conditions:

- (i) $\lim_{k \rightarrow \infty} \lambda_k = 0$ and $\sum_{k=1}^{\infty} \lambda_k = \infty$;
- (ii) $\lim_{k \rightarrow \infty} \left(\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k} \right) = 0$;
- (iii) $0 < \liminf_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} (\beta_k + \gamma_k) < 1$.

Then $\{x_k\}_{k=1}^{\infty}$ converges strongly to a unique solution $x^* \in \mathcal{F}$.

Proof Let the mapping $G : C \rightarrow C$ be defined as $G := \Pi_C(I - \lambda A)\Pi_C(I - \mu B)$, where $0 < \lambda < \frac{\alpha}{\kappa^2}$ and $0 < \mu < \frac{\beta}{\kappa^2}$. In terms of Lemma 4 we know that $G : C \rightarrow C$ is nonexpansive. Then the explicit iterative scheme can be rewritten as

$$x_{k+1} = \beta_k x_k + \gamma_k G V_k x_k + (1 - \beta_k - \gamma_k) \Pi_C(I - \lambda_k F) G V_k x_k, \quad \forall k \geq 1. \tag{41}$$

Next, we divide the rest of the proof into several steps.

Step 1. We show that $\{x_k\}_{k=1}^{\infty}$ is bounded. Indeed, take an arbitrarily given $p \in \mathcal{F}$. Then we have $V_k p = p$ and $G p = p$. Hence, by Lemma 8(c) we get

$$\begin{aligned} \|x_{k+1} - p\| &= \|\beta_k x_k + \gamma_k G V_k x_k + (1 - \beta_k - \gamma_k) \Pi_C(I - \lambda_k F) G V_k x_k - p\| \\ &\leq \beta_k \|x_k - p\| + \gamma_k \|G V_k x_k - p\| + (1 - \beta_k - \gamma_k) \|\Pi_C(I - \lambda_k F) G V_k x_k - p\| \\ &\leq \beta_k \|x_k - p\| + \gamma_k \|x_k - p\| \\ &\quad + (1 - \beta_k - \gamma_k) \|(I - \lambda_k F) G V_k x_k - (I - \lambda_k F) p - \lambda_k F(p)\| \\ &\leq (\beta_k + \gamma_k) \|x_k - p\| + (1 - \beta_k - \gamma_k) \|(I - \lambda_k F) G V_k x_k - (I - \lambda_k F) p\| \\ &\quad + \lambda_k (1 - \beta_k - \gamma_k) \|F(p)\| \\ &\leq (\beta_k + \gamma_k) \|x_k - p\| + (1 - \beta_k - \gamma_k) (1 - \lambda_k \tau) \|G V_k x_k - p\| \\ &\quad + \lambda_k (1 - \beta_k - \gamma_k) \|F(p)\| \\ &\leq [1 - (1 - \beta_k - \gamma_k) \lambda_k \tau] \|x_k - p\| + (1 - \beta_k - \gamma_k) \lambda_k \tau \frac{\|F(p)\|}{\tau}. \end{aligned}$$

By induction, we conclude that

$$\|x_{k+1} - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|F(p)\|}{\tau} \right\}.$$

Therefore, $\{x_k\}_{k=1}^{\infty}$ is bounded. So, the sequences $\{z_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty}$ and $\{F(y_k)\}_{k=1}^{\infty}$, where $z_k = V_k x_k$ and $y_k = G V_k x_k$, are also bounded.

Step 2. We show that $\|x_{k+1} - x_k\| \rightarrow 0$ and $\|x_k - y_k\| \rightarrow 0$ as $k \rightarrow \infty$. Indeed, observe that

$$\begin{aligned} \|y_{k+1} - y_k\| &= \|G V_{k+1} x_{k+1} - G V_k x_k\| \\ &\leq \|V_{k+1} x_{k+1} - V_k x_k\| \\ &\leq \|V_{k+1} x_{k+1} - V_{k+1} x_k\| + \|V_{k+1} x_k - V_k x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|V_k T^{k+1} x_k - V_k x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|T^{k+1} x_k - x_k\| \\ &= \|x_{k+1} - x_k\| + \alpha_{k+1} \|x_k - T_{k+1} x_k\|. \end{aligned}$$

Set $x_{k+1} = \beta_k x_k + (1 - \beta_k)w_k$ for all $k \geq 1$. Then $w_k = \frac{\gamma_k y_k + (1 - \beta_k - \gamma_k)\Pi_C(I - \lambda_k F)y_k}{1 - \beta_k}$. Note that

$$\begin{aligned} & \left\| \Pi_C(I - \lambda_{k+1}F)y_{k+1} - \Pi_C(I - \lambda_k F)y_k \right\| \\ & \leq \left\| (I - \lambda_{k+1}F)y_{k+1} - (I - \lambda_k F)y_k \right\| \\ & = \left\| y_{k+1} - y_k - \lambda_{k+1}F(y_{k+1}) + \lambda_k F(y_k) \right\| \\ & \leq \|y_{k+1} - y_k\| + \lambda_{k+1} \|F(y_{k+1})\| + \lambda_k \|F(y_k)\| \\ & \leq \|x_{k+1} - x_k\| + \lambda_{k+1} \|F(y_{k+1})\| + \lambda_k \|F(y_k)\| + \alpha_{k+1} \|x_k - T_{k+1}x_k\|. \end{aligned}$$

Hence

$$\begin{aligned} & \|w_{k+1} - w_k\| \\ & = \left\| \frac{\gamma_{k+1}y_{k+1} + (1 - \beta_{k+1} - \gamma_{k+1})\Pi_C(I - \lambda_{k+1}F)y_{k+1}}{1 - \beta_{k+1}} \right. \\ & \quad \left. - \frac{\gamma_k y_k + (1 - \beta_k - \gamma_k)\Pi_C(I - \lambda_k F)y_k}{1 - \beta_k} \right\| \\ & \leq \left\| \frac{\gamma_{k+1}}{1 - \beta_{k+1}}y_{k+1} - \frac{\gamma_k}{1 - \beta_k}y_k \right\| \\ & \quad + \left\| \frac{(1 - \beta_{k+1} - \gamma_{k+1})}{1 - \beta_{k+1}}\Pi_C(I - \lambda_{k+1}F)y_{k+1} - \frac{(1 - \beta_k - \gamma_k)}{1 - \beta_k}\Pi_C(I - \lambda_k F)y_k \right\| \\ & \leq \left| \frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k} \right| \|y_{k+1}\| + \frac{\gamma_k}{1 - \beta_k} \|y_{k+1} - y_k\| \\ & \quad + \left| \frac{(1 - \beta_{k+1} - \gamma_{k+1})}{1 - \beta_{k+1}} - \frac{(1 - \beta_k - \gamma_k)}{1 - \beta_k} \right| \left\| \Pi_C(I - \lambda_{k+1}F)y_{k+1} \right\| \\ & \quad + \frac{(1 - \beta_k - \gamma_k)}{1 - \beta_k} \left\| \Pi_C(I - \lambda_{k+1}F)y_{k+1} - \Pi_C(I - \lambda_k F)y_k \right\| \\ & \leq \left| \frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k} \right| (\|y_{k+1}\| + \left\| \Pi_C(I - \lambda_{k+1}F)y_{k+1} \right\|) \\ & \quad + \frac{\gamma_k}{1 - \beta_k} (\|x_{k+1} - x_k\| + \alpha_{k+1} \|x_k - T_{k+1}x_k\|) \\ & \quad + \frac{(1 - \beta_k - \gamma_k)}{1 - \beta_k} (\|x_{k+1} - x_k\| + \lambda_{k+1} \|F(y_{k+1})\| + \lambda_k \|F(y_k)\| + \alpha_{k+1} \|x_k - T_{k+1}x_k\|) \\ & \leq \left| \frac{\gamma_{k+1}}{1 - \beta_{k+1}} - \frac{\gamma_k}{1 - \beta_k} \right| (\|y_{k+1}\| + \left\| \Pi_C(I - \lambda_{k+1}F)y_{k+1} \right\|) \\ & \quad + \|x_{k+1} - x_k\| + \lambda_{k+1} \|F(y_{k+1})\| + \lambda_k \|F(y_k)\| + \alpha_{k+1} (\|x_k\| + \|T_{k+1}x_k\|). \end{aligned}$$

Since $\{x_k\}$, $\{y_k\}$ and $\{F(y_k)\}$ are bounded, we have that $\{\|y_{k+1}\| + \|\Pi_C(I - \lambda_{k+1}F)y_{k+1}\|\}$ and $\{\|x_k\| + \|T_{k+1}x_k\|\}$ are bounded. So it follows from $\alpha_k \rightarrow 0$ and conditions (i) and (ii) that

$$\limsup_{k \rightarrow \infty} (\|w_{k+1} - w_k\| - \|x_{k+1} - x_k\|) \leq 0.$$

Hence, by Lemma 12, we get $\|w_k - x_k\| \rightarrow 0$ as $k \rightarrow \infty$. Consequently,

$$\lim_{k \rightarrow \infty} \|x_{k+1} - x_k\| = \lim_{k \rightarrow \infty} (1 - \beta_k) \|w_k - x_k\| = 0. \tag{42}$$

We also note that

$$\begin{aligned} \|w_k - y_k\| &= \left\| \frac{\gamma_k y_k + (1 - \beta_k - \gamma_k) \Pi_C(I - \lambda_k F)y_k}{1 - \beta_k} - y_k \right\| \\ &= \frac{1 - \beta_k - \gamma_k}{1 - \beta_k} \|\Pi_C(I - \lambda_k F)y_k - y_k\| \\ &\leq \|\Pi_C(I - \lambda_k F)y_k - \Pi_C y_k\| \\ &\leq \lambda_k \|F(y_k)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

It follows that

$$\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0. \tag{43}$$

Step 3. We show that $\|z_k - Gz_k\| \rightarrow 0$ and $\|x_k - V_k x_k\| \rightarrow 0$ as $k \rightarrow \infty$. Indeed, for simplicity, put $q = \Pi_C(p - \mu Bp)$ and $u_k = \Pi_C(z_k - \mu Bz_k)$. Then $y_k = Gz_k = \Pi_C(u_k - \lambda A u_k)$ for all $k \geq 1$. Repeating the same arguments as those of (24), we obtain

$$\|y_k - p\|^2 \leq \|x_k - p\|^2 - 2\mu(\beta - \kappa^2 \mu) \|Bz_k - Bp\|^2 - 2\lambda(\alpha - \kappa^2 \lambda) \|A u_k - Aq\|^2. \tag{44}$$

Observe that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \beta_k \|x_k - p\|^2 + \gamma_k \|G V_k x_k - p\|^2 + (1 - \beta_k - \gamma_k) \|\Pi_C(I - \lambda_k F)G V_k x_k - p\|^2 \\ &\leq \beta_k \|x_k - p\|^2 + \gamma_k \|G V_k x_k - p\|^2 \\ &\quad + (1 - \beta_k - \gamma_k) \|(I - \lambda_k F)G V_k x_k - (I - \lambda_k F)p - \lambda_k F(p)\|^2 \\ &\leq \beta_k \|x_k - p\|^2 + \gamma_k \|G V_k x_k - p\|^2 \\ &\quad + (1 - \beta_k - \gamma_k) [(1 - \lambda_k \tau) \|G V_k x_k - p\| + \lambda_k \|F(p)\|]^2 \\ &\leq \beta_k \|x_k - p\|^2 + \gamma_k \|G V_k x_k - p\|^2 \\ &\quad + (1 - \beta_k - \gamma_k) [(1 - \lambda_k \tau) \|G V_k x_k - p\|^2 + \lambda_k \tau^{-1} \|F(p)\|^2] \\ &\leq \beta_k \|x_k - p\|^2 + \gamma_k \|G V_k x_k - p\|^2 \\ &\quad + (1 - \beta_k - \gamma_k) \|G V_k x_k - p\|^2 + \lambda_k \tau^{-1} \|F(p)\|^2 \\ &= \beta_k \|x_k - p\|^2 + (1 - \beta_k) \|G V_k x_k - p\|^2 + \lambda_k \tau^{-1} \|F(p)\|^2, \end{aligned} \tag{45}$$

which together with (44) implies that

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \beta_k \|x_k - p\|^2 + (1 - \beta_k) [\|x_k - p\|^2 - 2\mu(\beta - \kappa^2 \mu) \|Bz_k - Bp\|^2 \\ &\quad - 2\lambda(\alpha - \kappa^2 \lambda) \|A u_k - Aq\|^2] + \lambda_k \tau^{-1} \|F(p)\|^2 \\ &= \|x_k - p\|^2 - 2(1 - \beta_k) [\mu(\beta - \kappa^2 \mu) \|Bz_k - Bp\|^2 \\ &\quad + \lambda(\alpha - \kappa^2 \lambda) \|A u_k - Aq\|^2] + \lambda_k \tau^{-1} \|F(p)\|^2. \end{aligned}$$

So, it follows that

$$\begin{aligned}
 & 2(1 - \beta_k)[\mu(\beta - \kappa^2\mu)\|Bz_k - Bp\|^2 + \lambda(\alpha - \kappa^2\lambda)\|Au_k - Aq\|^2] \\
 & \leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \lambda_k\tau^{-1}\|F(p)\|^2 \\
 & \leq \|x_k - x_{k+1}\|(\|x_k - p\| + \|x_{k+1} - p\|) + \lambda_k\tau^{-1}\|F(p)\|^2.
 \end{aligned} \tag{46}$$

Since $\lambda \in (0, \frac{\alpha}{\kappa^2})$, $\mu \in (0, \frac{\beta}{\kappa^2})$ and $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$, we deduce from (42) and condition (iii) that

$$\lim_{k \rightarrow \infty} \|Bz_k - Bp\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|Au_k - Aq\| = 0. \tag{47}$$

Repeating the same arguments as those of (28), we get

$$\begin{aligned}
 \|y_k - p\|^2 & \leq \|x_k - p\|^2 - g_1(\|z_k - u_k - (p - q)\|) - g_2(\|u_k - y_k + (p - q)\|) \\
 & \quad + 2\mu\|Bp - Bz_k\|\|u_k - q\| + 2\lambda\|Aq - Au_k\|\|y_k - p\|.
 \end{aligned} \tag{48}$$

Combining (45) and (48), we have

$$\begin{aligned}
 \|x_{k+1} - p\|^2 & \leq \beta_k\|x_k - p\|^2 + (1 - \beta_k)[\|x_k - p\|^2 - g_1(\|z_k - u_k - (p - q)\|) \\
 & \quad - g_2(\|u_k - y_k + (p - q)\|) + 2\mu\|Bp - Bz_k\|\|u_k - q\| \\
 & \quad + 2\lambda\|Aq - Au_k\|\|y_k - p\|] + \lambda_k\tau^{-1}\|F(p)\|^2 \\
 & \leq \|x_k - p\|^2 - (1 - \beta_k)[g_1(\|z_k - u_k - (p - q)\|) + g_2(\|u_k - y_k + (p - q)\|)] \\
 & \quad + 2\mu\|Bp - Bz_k\|\|u_k - q\| + 2\lambda\|Aq - Au_k\|\|y_k - p\| + \lambda_k\tau^{-1}\|F(p)\|^2,
 \end{aligned}$$

which immediately leads to

$$\begin{aligned}
 & (1 - \beta_k)[g_1(\|z_k - u_k - (p - q)\|) + g_2(\|u_k - y_k + (p - q)\|)] \\
 & \leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2 \\
 & \quad + 2\mu\|Bp - Bz_k\|\|u_k - q\| + 2\lambda\|Aq - Au_k\|\|y_k - p\| + \lambda_k\tau^{-1}\|F(p)\|^2 \\
 & \leq \|x_k - x_{k+1}\|(\|x_k - p\| + \|x_{k+1} - p\|) \\
 & \quad + 2\mu\|Bp - Bz_k\|\|u_k - q\| + 2\lambda\|Aq - Au_k\|\|y_k - p\| + \lambda_k\tau^{-1}\|F(p)\|^2.
 \end{aligned}$$

Since $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$, and $\{u_k\}$ and $\{y_k\}$ are bounded, we deduce from (42), (47) and condition (iii) that

$$\lim_{k \rightarrow \infty} g_1(\|z_k - u_k - (p - q)\|) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} g_2(\|u_k - y_k + (p - q)\|) = 0.$$

Utilizing the properties of g_1 and g_2 , we conclude that

$$\lim_{k \rightarrow \infty} \|z_k - u_k - (p - q)\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u_k - y_k + (p - q)\| = 0. \tag{49}$$

From (49), we get

$$\|z_k - y_k\| \leq \|z_k - u_k - (p - q)\| + \|u_k - y_k + (p - q)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

That is,

$$\lim_{k \rightarrow \infty} \|z_k - Gz_k\| = \lim_{k \rightarrow \infty} \|z_k - y_k\| = 0. \tag{50}$$

This together with (43) implies that

$$\lim_{k \rightarrow \infty} \|x_k - V_k x_k\| = \lim_{k \rightarrow \infty} \|x_k - z_k\| = 0. \tag{51}$$

Step 4. We show that $\omega_w(x_k) \subset \mathcal{F}$, where

$$\omega_w(x_k) = \{x \in C : x_{k_i} \rightharpoonup x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Indeed, repeating the same arguments as those of (36) and (37) in the proof of Theorem 5, we obtain that

$$\lim_{k \rightarrow \infty} \|x_k - Vx_k\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|x_k - Gx_k\| = 0. \tag{52}$$

Utilizing Lemma 6 and Lemma 11 and the nonexpansivity of V and G , we conclude that $\omega_w(x_k) \subset \mathcal{F}$.

Step 5. We show that $\limsup_{k \rightarrow \infty} \langle F(x^*), j(x^* - v_k) \rangle \leq 0$, where $v_k = \Pi_C(I - \lambda_k F)y_k$ for all $k \geq 1$ and $x^* \in \mathcal{F}$ is the unique solution of VI (19). Indeed, we first take a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that

$$\limsup_{k \rightarrow \infty} \langle F(x^*), j(x^* - v_k) \rangle = \lim_{i \rightarrow \infty} \langle F(x^*), j(x^* - v_{k_i}) \rangle.$$

We may also assume that $v_{k_i} \rightharpoonup z$. Note that

$$\begin{aligned} \|v_k - x_k\| &= \|\Pi_C(I - \lambda_k F)y_k - x_k\| \\ &\leq \|(I - \lambda_k F)y_k - x_k\| \leq \|y_k - x_k\| + \lambda_k \|F(y_k)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \tag{53}$$

Combining (53) with $\omega_w(x_k) \subset \mathcal{F}$ (due to Step 4), we get $z \in \mathcal{F}$. So, it follows from VI (19) that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle F(x^*), j(x^* - v_k) \rangle &= \lim_{i \rightarrow \infty} \langle F(x^*), j(x^* - v_{k_i}) \rangle \\ &= \langle F(x^*), j(x^* - z) \rangle \leq 0. \end{aligned}$$

Since $v_k = \Pi_C(I - \lambda_k F)y_k$ for all $k \geq 1$, according to Lemma 2(iii), we have

$$\langle (I - \lambda_k F)y_k - \Pi_C(I - \lambda_k F)y_k, j(x^* - v_k) \rangle \leq 0. \tag{54}$$

From (54), we have

$$\begin{aligned}
 \|v_k - x^*\|^2 &= \langle \Pi_C(I - \lambda_k F)y_k - x^*, j(v_k - x^*) \rangle \\
 &= \langle \Pi_C(I - \lambda_k F)y_k - (I - \lambda_k F)y_k, j(v_k - x^*) \rangle \\
 &\quad + \langle (I - \lambda_k F)y_k - x^*, j(v_k - x^*) \rangle \\
 &\leq \langle (I - \lambda_k F)y_k - x^*, j(v_k - x^*) \rangle \\
 &= \langle (I - \lambda_k F)y_k - (I - \lambda_k F)x^*, j(v_k - x^*) \rangle + \lambda_k \langle F(x^*), j(x^* - v_k) \rangle \\
 &\leq (1 - \lambda_k \tau) \|y_k - x^*\| \|v_k - x^*\| + \lambda_k \langle F(x^*), j(x^* - v_k) \rangle \\
 &\leq \frac{1}{2} (1 - \lambda_k \tau)^2 \|y_k - x^*\|^2 + \frac{1}{2} \|v_k - x^*\|^2 + \lambda_k \langle F(x^*), j(x^* - v_k) \rangle,
 \end{aligned}$$

which immediately yields

$$\begin{aligned}
 \|v_k - x^*\|^2 &\leq (1 - \lambda_k \tau) \|y_k - x^*\|^2 + 2\lambda_k \langle F(x^*), j(x^* - v_k) \rangle \\
 &\leq (1 - \lambda_k \tau) \|x_k - x^*\|^2 + 2\lambda_k \langle F(x^*), j(x^* - v_k) \rangle.
 \end{aligned} \tag{55}$$

Step 6. We show that $x_k \rightarrow x^*$ as $k \rightarrow \infty$. Indeed, from (45) and (55), we have

$$\begin{aligned}
 \|x_{k+1} - x^*\|^2 &\leq \beta_k \|x_k - x^*\|^2 + \gamma_k \|y_k - x^*\|^2 + (1 - \beta_k - \gamma_k) \|v_k - x^*\|^2 \\
 &\leq \beta_k \|x_k - x^*\|^2 + \gamma_k \|x_k - x^*\|^2 \\
 &\quad + (1 - \beta_k - \gamma_k) [(1 - \lambda_k \tau) \|x_k - x^*\|^2 + 2\lambda_k \langle F(x^*), j(x^* - v_k) \rangle] \\
 &= [1 - \lambda_k (1 - \beta_k - \gamma_k) \tau] \|x_k - x^*\|^2 \\
 &\quad + \lambda_k (1 - \beta_k - \gamma_k) \tau \cdot \frac{2}{\tau} \langle F(x^*), j(x^* - v_k) \rangle.
 \end{aligned} \tag{56}$$

Since $\sum_{k=1}^\infty \lambda_k = \infty$, $\limsup_{k \rightarrow \infty} (\beta_k + \gamma_k) < 1$ and $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in (0, 1)$, we get

$$\sum_{k=1}^\infty \lambda_k (1 - \beta_k - \gamma_k) \tau = \infty.$$

Taking into account $\limsup_{k \rightarrow \infty} \langle F(x^*), j(x^* - v_k) \rangle \leq 0$, we can apply Lemma 13 to relation (56) and conclude that $x_k \rightarrow x^*$ as $k \rightarrow \infty$. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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