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Fundamental tone of minimal hypersurfaces with finite index in hyperbolic space

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Abstract

Let M be a complete minimal hypersurface in hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant sectional curvature -1. We prove that if M has a finite index and finite L^2 norm of the second fundamental form, then the fundamental tone $\lambda_1(M)$ is bounded above by n^2.

MSC: 53C40; 53C42

Keywords: minimal hypersurface; finite index; hyperbolic space; fundamental tone; eigenvalue

1 Introduction

McKean [1] proved that the fundamental tone of an n-dimensional complete simply connected Riemannian manifold M with sectional curvature bounded above by $-\kappa^2 < 0$ is bigger than or equal to $\frac{(n-1)^2\kappa^2}{4}$, where κ is a real number. Moreover, his result is sharp since the equality is attained by the hyperbolic space $\mathbb{H}^n(-\kappa^2)$ with constant sectional curvature $-\kappa^2$. We recall that the *fundamental tone* $\lambda_1(M)$ is defined by

$$\lambda_1(M) = \inf\left\{\frac{\int_M |\nabla f|^2}{\int_M f^2} : 0 \neq f \in W_0^{1,2}(M)\right\}.$$

Interestingly, Cheung and Leung [2] obtained the same lower bound for the fundamental tone of complete submanifold in $\mathbb{H}^m(-\kappa^2)$ with bounded mean curvature as follows (see also [3, 4]).

Theorem [2] *Let M be an n-dimensional complete noncompact submanifold in $\mathbb{H}^m(-\kappa^2)$ with the mean curvature vector H. If $|H| \leq \alpha < n-1$, then*

$$\lambda_1(M) \geq \frac{(n-1-\alpha)^2\kappa^2}{4}.$$

There have been extensive investigations to obtain an upper bound for the fundamental tone of complete minimal submanifolds in hyperbolic space. Castillon [5] proved that the spectrum of the Laplacian on a complete minimal hypersurface with finite L^n norm of the second fundamental form in $\mathbb{H}^{n+1}$, denoted by Spec(Δ), is given by Spec(Δ) = $[\frac{(n-1)^2}{4}, +\infty)$. Candel [6] was able to prove that the fundamental tone of complete simply connected stable minimal surfaces in $\mathbb{H}^3(-1)$ is at most $\frac{4}{3}$. In [7], the author proved that if M is a complete stable minimal hypersurface in $\mathbb{H}^{n+1}(-1)$ with finite L^2 norm of the second fundamental form, then $\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2$. Later, Bérard *et al.* [8] improved the upper bound for complete stable minimal surfaces in $\mathbb{H}^3(-1)$. Indeed, they proved that the fundamental tone of complete stable minimal surfaces in $\mathbb{H}^3(-1)$ is at most $\frac{4}{7}$. Fu and Tao [9] showed that if M is an n-dimensional complete submanifold in $\mathbb{H}^m(-1)$ with parallel mean curvature vector H and with finite L^p norm of the traceless second fundamental form for $p \geq n$, then $\lambda_1(M)$ is less than or equal to $\frac{(n-1)^2(1-|H|^2)}{4}$. Recently, Gimeno [10] proved that if M^2 is a complete minimal surface in $\mathbb{H}^m(-1)$ with finite L^2 norm of the second fundamental form, then $\lambda_1(M) = \frac{1}{4}$.

The aim of this paper is to obtain an upper bound for the fundamental tone of complete minimal hypersurfaces in $\mathbb{H}^{n+1}(-1)$ with finite index and finite L^2 norm of the second fundamental form. More precisely, we prove the following.

Theorem 1.1 *Let M be a complete orientable minimal hypersurface in $\mathbb{H}^{n+1}(-1)$ with $\int_M |A|^2 < \infty$. Suppose M has finite index. Then we have*

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2.$$

It is obvious that a complete stable minimal hypersurface in $\mathbb{H}^{n+1}(-1)$ has index 0. Hence our theorem can be regarded as an extension of the results in [6–8]. When $n = 2$, we remark that the finite index condition can be omitted, since the finiteness of the L^2 norm of the second fundamental form implies that M has finite index, which was proved by Bérard *et al.* [11]. However, in this case, our theorem is weaker than Theorem 4.1 in [5] or Theorem A in [10].

2 Proof of Theorem 1.1

In this section, we prove our main theorem.

Proof of Theorem 1.1 The lower bound of $\lambda_1(M)$ is given by $\frac{(n-1)^2}{4}$, which was done by Cheung and Leung [2] as mentioned in the Introduction. Thus it suffices to prove that the upper bound of $\lambda_1(M)$ is n^2.

Since M has a finite index, there exists a compact subset $K \subset M$ such that $M \setminus K$ is stable (see [12] for example), *i.e.*, for any compactly supported Lipschitz function f on $M \setminus K$,

$$\int_{M\setminus K} |\nabla f|^2 - \bigl(|A|^2 - n\bigr) f^2 \, dv \geq 0, \tag{1}$$

where $|A|^2$ denotes the squared length of the second fundamental form on M and dv denotes the volume form for the induced metric on M. Note that, for some geodesic ball $B(R_0) \subset M$ centered at $p \in M$ of radius R_0 containing the compact set K, the region $M \setminus B(R_0)$ is still stable. Thus, without loss of generality, we may assume that $K = B(R_0)$.

Choose a geodesic ball $B(R) \subset M$ centered at $p \in M$ of radius $R > R_0$ and take a cut-off function $0 \le \phi \le 1$ on M satisfying

$$\phi = \begin{cases} 0 & \text{on } B(R_0), \\ 1 & \text{on } B(2R+R_0) \setminus B(R+R_0), \\ 0 & \text{on } M \setminus B(3R+R_0), \end{cases}$$

and $|\nabla\phi| \le \frac{1}{R}$ on M. By the definition of the fundamental tone and the domain monotonicity of the eigenvalue, we see that

$$\lambda_1(M) \le \lambda_1\big(M \setminus B(R_0)\big) \le \frac{\int_{M\setminus B(R_0)} |\nabla f|^2}{\int_{M\setminus B(R_0)} f^2}$$

for any $f \in W_0^{1,2}(M \setminus B(R_0))$. Substituting f with $|A|\phi$ gives

$$\begin{aligned} &\lambda_1(M) \int_{M\setminus B(R_0)} |A|^2\phi^2 \\ &\quad\le \int_{M\setminus B(R_0)} \big|\nabla\big(|A|\phi\big)\big|^2 \\ &\quad= \int_{M\setminus B(R_0)} \phi^2\big|\nabla|A|\big|^2 + \int_{M\setminus B(R_0)} |A|^2|\nabla\phi|^2 + 2\int_{M\setminus B(R_0)} |A|\phi\big\langle\nabla|A|, \nabla\phi\big\rangle. \end{aligned}$$

Using the Schwarz inequality and the geometric-arithmetic mean inequality, we get

$$2\int_{M\setminus B(R_0)} |A|\phi\big\langle\nabla|A|, \nabla\phi\big\rangle \le \varepsilon\int_{M\setminus B(R_0)} |A|^2|\nabla\phi|^2 + \frac{1}{\varepsilon}\int_{M\setminus B(R_0)} \phi^2\big|\nabla|A|\big|^2$$

for any $\varepsilon > 0$. Therefore

$$\begin{aligned} \lambda_1(M) \int_{M\setminus B(R_0)} |A|^2\phi^2 \le{}& (1+\varepsilon)\int_{M\setminus B(R_0)} |A|^2|\nabla\phi|^2 \\ &+ \left(1+\frac{1}{\varepsilon}\right)\int_{M\setminus B(R_0)} \phi^2\big|\nabla|A|\big|^2. \end{aligned} \tag{2}$$

On the other hand, a Simons-type inequality [13, 14] for minimal hypersurfaces in $\mathbb{H}^{n+1}$ asserts that

$$|A|\Delta|A| + |A|^4 + n|A|^2 = |\nabla A|^2 - \big|\nabla|A|\big|^2.$$

Applying the Kato inequality [15],

$$|\nabla A|^2 - \big|\nabla|A|\big|^2 \ge \frac{2}{n}\big|\nabla|A|\big|^2,$$

we have

$$|A|\Delta|A| + |A|^4 + n|A|^2 \ge \frac{2}{n}\big|\nabla|A|\big|^2.$$

Multiplying both sides by the function ϕ^2 and integrating over $B(3R+R_0)\setminus B(R_0)$, we get

$$\begin{aligned}\frac{2}{n}\int_{M\setminus B(R_0)}\phi^2\big|\nabla|A|\big|^2 &\le \int_{M\setminus B(R_0)}\phi^2|A|^4 + n\int_{M\setminus B(R_0)}\phi^2|A|^2\\ &\quad - \int_{M\setminus B(R_0)}\phi^2\big|\nabla|A|\big|^2 - 2\int_{M\setminus B(R_0)}|A|\phi\big\langle\nabla|A|,\nabla\phi\big\rangle, \end{aligned}\tag{3}$$

where we used the divergence theorem.

Replacing f with $\phi|A|$ in the stability inequality (1) on $M\setminus B(R_0)$ gives

$$\int_{M\setminus B(R_0)}\big|\nabla\big(\phi|A|\big)\big|^2 \ge \int_{M\setminus B(R_0)}\big(|A|^2-n\big)|A|^2\phi^2,$$

which implies

$$\begin{aligned}&\int_{M\setminus B(R_0)}|A|^2|\nabla\phi|^2 + \int_{M\setminus B(R_0)}\phi^2\big|\nabla|A|\big|^2 + 2\int_{M\setminus B(R_0)}|A|\phi\big\langle\nabla|A|,\nabla\phi\big\rangle\\ &\quad\ge \int_{M\setminus B(R_0)}|A|^4\phi^2 - n\int_{M\setminus B(R_0)}|A|^2\phi^2. \end{aligned}\tag{4}$$

Combining (3) with (4), we obtain

$$\frac{2}{n}\int_{M\setminus B(R_0)}\phi^2\big|\nabla|A|\big|^2 \le \int_{M\setminus B(R_0)}|A|^2|\nabla\phi|^2 + 2n\int_{M\setminus B(R_0)}|A|^2\phi^2. \tag{5}$$

Hence, using (2) and (5), we have

$$2\left\{\frac{1}{n} - \frac{n(1+\frac{1}{\varepsilon})}{\lambda_1(M)}\right\}\int_{M\setminus B(R_0)}\phi^2\big|\nabla|A|\big|^2 \le \left\{1+\frac{2n(1+\varepsilon)}{\lambda_1(M)}\right\}\int_{M\setminus B(R_0)}|A|^2|\nabla\phi|^2. \tag{6}$$

We now suppose that $\lambda_1(M) > n^2$. For a sufficiently large $\varepsilon > 0$, letting $R\to\infty$ in (6) shows that $|\nabla|A||\equiv 0$ on $M\setminus B(R_0)$, which implies that $|A|$ is constant on $M\setminus B(R_0)$. Since the volume of any complete minimal hypersurface in hyperbolic space is infinite and L^2 norm of $|A|$ is finite by our assumption, we see that $|A|\equiv 0$ outside the compact subset $B(R_0)$. It follows from the maximum principle for minimal hypersurfaces in $\mathbb{H}^{n+1}$ that M must be totally geodesic. However, due to McKean [1], the fundamental tone of totally geodesic hyperplanes in $\mathbb{H}^{n+1}$ is equal to $\frac{(n-1)^2}{4}$, which gives a contradiction. Therefore we get the conclusion. □

Remark 2.1 The proof of Theorem 1.1 relies on the inequality (6), which is called a Caccioppoli-type inequality. In [16], Ilias *et al.* intensively studied a Caccioppoli-type inequality on constant mean curvature hypersurfaces in Riemannian manifolds.

Competing interests
The author declares that he has no competing interests.

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