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General note on the theorem of Stampfli

Cheniti Bensalloua and Mostefa Nadir*

*Correspondence:
mostefanadir@yahoo.fr
Department of Mathematics,
University of Msila, Msila, Algeria

Abstract

It is well known that for every bounded operator A in L(H), there exists a compact operator K in K(H) such that the Weyl spectrum $\sigma_W(A)$ of the operator A coincides with the spectrum $\sigma(A + K)$ of the perturbed operator A. In this work, we show the extension of this relation by the use of Kato's decomposition to the set of semi-Fredholm operators.

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1 Introduction

In 1965 Schechter demonstrated in [1] this result that for every bounded operator A in $L(H)$, there exists a compact operator K in $K(H)$ such that the Weyl spectrum $\sigma_W(A)$ of the operator A coincides with the spectrum $\sigma(A + K)$ of the perturbed operator A for a compact perturbation of Fredholm operators of index zero. In 1973 Stampfli showed in [2] that the minimum of this perturbation is achieved for a certain compact operator. This work is intended to extend this result to semi-Fredholm operators of any index using the decomposition of Kato, an extension already done by Apostol in [3] and Herrero in [4].

Let H be a complex, separable, infinite dimensional Hilbert space, and let $L(H)$ denote the algebra of all linear bounded operators on H, $C(H)$ the set of linear operators A with domain $D(A)$ dense in H and range $R(A)$ contained in H and a graph $G(A)$ closed in $H \times H$. $K(H)$ is the set of compact elements of $L(H)$.

For $A \in C(H)$, we let $\sigma(A)$, $\rho(A)$, and $N(A)$ denote the spectrum, the resolvent set, and the null space of A, respectively. The nullity $\alpha(A)$ of A is defined as the dimension of $N(A)$ and the deficiency $\beta(A)$ of A is defined as the codimension of $R(A)$ in H.

$\sigma_W(A)$ will denote the Weyl spectrum. Recall that the Weyl spectrum is the union of the essential spectrum $\sigma_e(A)$ and all bounded components of $\mathbb{C}/\sigma_e(A)$ associated with a nonzero Fredholm index. In other words, the Weyl spectrum of A is the set

$$\sigma_W(A) = \sigma(A) \backslash \Phi_0(A),$$

where $\Phi_0(A)$ designates the set of all scalars $\mu \in \mathbb{C}$ such that $R(A - \mu I)$ is closed and both $\dim N(A - \mu I)$ and $\dim N(A^* - \overline{\mu} I)$ are finite such that the index $\mathrm{ind}(A - \mu I)$ of $(A - \mu I)$ is null where

$$\mathrm{ind}(A - \mu I) = \dim N(A - \mu I) - \dim N(A^* - \overline{\mu} I).$$

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Note that it was shown in [1] that for every bounded operator A in $L(H)$, there exists a compact operator K in $K(H)$ such that

$$\sigma_W(A) = \sigma(A + K).$$

Proposition 1 (See [5]) *For $A \in C(H)$ we get*
  (1) *$R(A)$ is a closed subspace in H if and only if $c(A) > 0$,*
  (2) *$c(A) = c(A^*)$,*
where A^* is the adjoint of A and $c(A)$ is given by

$$c(A) = \inf_{u \in D(A) \cap N(A)^\perp} \frac{\|Au\|}{\|u\|}. \tag{1.1}$$

Definition 1 The operator $A \in C(H)$ is called a semi-Fredholm operator, denoted $A \in SF(H)$, if the following conditions are satisfied:
  (1) $R(A)$ is a closed subspace in H,
  (2) $\min \operatorname{ind}(A) = \{\dim N(A), \dim N(A^*)\} < \infty$.

Theorem 1 (See [6]) *Let $A \in SF(H)$, there is a direct decomposition $H = M \oplus N$ such that*
  (1) *M and N are invariant by A,*
  (2) *$A/_M$ is regular,*
  (3) *$N \subseteq D(A)$, $\dim N < \infty$, and $A/_N$ is nilpotent.*

This decomposition is known as the decomposition of Kato [6]. Operators admitting such decomposition were characterized in 1976 by Labrousse in [7] and are called quasi-Fredholm operators, also generalized by Mbekhta to the operators called pseudo-Fredholm [8].

Definition 2 Let $A \in C(H)$; the complex set $\rho_e(A)$ of the operator A given by

$$\rho_e(A) = \big\{\mu \in \mathbb{C}, (A - \mu I) \in SF(H)\big\},$$

will be called the essential resolvent (resp. $\sigma_e(A) = \mathbb{C} \backslash \rho_e(A)$) and it will stand for the essential spectrum of the operator A.

For $A \in C(H)$, we define the set

$$M_e(A) = \big\{\mu \in \mathbb{C} \text{ such that } (A - \mu I) \text{ is invertible on the left or on the right}\big\}.$$

Lemma 1 *If $(A - \mu I) \in SF(A)$ such that*

$$\min\big\{\dim N(A - \mu I), \dim\big(A^* - \mu I\big)\big\} = 0,$$

then $\mu \in M_e(A)$.

Definition 3 Let $A \in C(H)$, the complex point $\mu \in \mathbb{C}$ is called a regular point of A, denoted reg(A), if the following conditions are satisfied:

(1) $R(A - \mu I)$ is a closed subspace in H,

(2) $N[(A - \mu I)^n] \subseteq R(A - \mu I)$, $\forall n \in \mathbb{N}$.

Remark 1 The set reg(A) of regular points of A is open in $\mathbb{C}$.

Theorem 2 (See [9]) $A \in SF(H)$ *if and only if* $A^* \in SF(H)$ *and* ind(A) $= -$ ind(A^*).

Theorem 3 (See [10]) *For any* $A \in L(H) \cap SF(H)$ *and* $B \in L(H)$ *such that* $(A - B) \in K(H)$ *we get* $B \in SF(H)$ *and* ind(A) $=$ ind(B).

Theorem 4 (See [11]) ind($A - \mu I$) *is constant on each connected component of* $\rho_e(A)$.

Definition 4 Let $A, B \in C(H)$; we denote by $P_{G(A)}$ the orthogonal projection of $H \times H$ on $G(A)$. We set

$$\delta(A, B) = \left\| (I - P_A) P_B \right\| \tag{1.2}$$

and

$$g(A, B) = \| P_{G(A)} - P_{G(B)} \|. \tag{1.3}$$

Remark 2 $g(A, B)$ defines a metric on $C(H)$.

Proposition 2 (See [5]) *Let* M, N *be two closed subspaces of* H *with* dim$N < \infty$. *Then if* $\delta(M, N) < 1$ *we have* dim$M \leq$ dim$N$.

Definition 5 Let $A \in L(H)$. A is called quasi-nilpotent and will be noted $A \in QN(H)$ if and only if $\lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} = 0$.

Proposition 3 (See [12]) *Let* $A \in L(H)$, *then* $A \in QN(H)$ *if and only if* $\sigma(A) = 0$.

Definition 6 $A \in L(H)$ is called a Riesz operator if and only if $\sigma_e(A) = 0$.

Theorem 5 (See [1]) *Let* $A \in L(H)$, *that is,* $K \in K(H)$ *such that*

$$\sigma_W(A) = \bigcap_{K \in K(H)} \sigma(A + K).$$

In 1973 Stampfli demonstrated in [2] that there is an operator $K \in K(H)$ such that

$$\sigma_W(A) = \sigma(A + K).$$

Passing to the complement in the expression $\sigma_W(A) = \sigma(A + K) = \sigma(A) \cap [\Phi_0(A)]^c$ we obtain $\rho(A) \cup \Phi_0(A) = \rho(A + K)$. Hence $\rho(A) \subseteq \Phi_0(A)$; then the result of Stampfli is equivalent to $\Phi_0(A) = \rho(A + K)$.

2 Main results

Theorem 6 *Let* $A \in L(H)$, *there exists a compact operator* $K \in K(H)$ *such that*

$$\rho_e(A) = M_e(A + K).$$

2.1 Correction of the operator *A*

2.1.1 First correction

This correction involves the construction of a compact perturbation K_1 such that if $A \in L(H)$ and $B = (A + K_1) \in L(H)$ then

$$\rho_e(A) \cap \mathrm{reg}(A + K_1) \subseteq M_e(A + K_1).$$

Lemma 2 *Let $A \in L(H)$ such that A is invertible on the left or on the right, K_1 an operator of $K(H)$ and $B = A + K_1$. If $\|K_1\| < c(A)$ then*

(1) $\delta(N(B), N(A)) \leq \frac{\|K_1\|}{c(A)}$,

(2) *B is invertible on the left or on the right.*

Proof We have by definition $\delta(N(B), N(A)) = \|(I - P_{N(B)})P_{N(A)}\|$. Assume that $u \in H$ and let $v = (I - P_{N(A)})P_{N(B)}u$. Then v is orthogonal to $N(A)$ and we have

$$\|Av\| \geq c(A)\|v\|,$$

or still

$$\|v\| \leq \frac{\|Av\|}{c(A)},$$

$$\frac{\|Av\|}{c(A)} = \frac{\|A((I - P_{N(A)})P_{N(B)}u)\|}{c(A)}$$

$$= \frac{\|A(P_{N(B)}u)\|}{c(A)}$$

$$= \frac{\|(A - B)(P_{N(B)}u)\|}{c(A)}$$

$$\leq \frac{\|(A - B)\|\|P_{N(B)}u\|}{c(A)}.$$

Using Proposition 2, we have

$$\delta\big(N(B), N(A)\big) < 1 \quad \Rightarrow \quad \dim N(B) \leq \dim N(A).$$

If A is one-to-one, then $\dim N(A) = 0 \Rightarrow \dim N(B) = 0$. Therefore $N(B) = \{0\}$, hence B is one-to-one. As $R(B)$ is closed (compact perturbation of semi-Fredholm operator), we deduce that B is left invertible.

If $N(A) \neq \{0\}$, then A is not left invertible. Hence A is right invertible. Then $N(A^*) = \{0\}$. $A, B \in L(H) \Rightarrow A^*, B^* \in L(H)$, $A^* - B^* = A - B$, and $c(A) = c(A^*)$.

By a similar reasoning we will have

$$\delta\big(N(B^*), N(A^*)\big) \leq \frac{\|A^* - B^*\|}{c(A^*)} < 1 \quad \Rightarrow \quad \dim N\big(B^*\big) = 0.$$

Then $N(B^*) = \{0\}$ and B^* is one-to-one. $B = (A + K_1) \in SF(H)$, in particular $R(B)$ is a closed subspace of H, $R(B) = N(B^*)^\perp = \{0\}^\perp = H$, then B is onto. Thus it is right invertible. $\qquad\square$

Lemma 3 *Let M, N be two closed subspace of H such that* $\dim N \geq \dim M > 0$ *and* $\dim M <$ ∞. *Then there is U, a partial isometry of H such that*

$$R(U) \subseteq N \quad and \quad R(U^*) = M.$$

Proof Let $e_1, e_2, \ldots, e_m$ be an orthonormal bases of M and $f_1, f_2, \ldots, f_m$ an orthonormal family of vectors of N. Let P the orthogonal projection of H onto M and Q the orthogonal projection onto the space spanned by $f_1, f_2, \ldots, f_m$. If $u \in H$, there are m complex numbers $a_1, a_2, \ldots, a_m$ such that $Pu = \sum_{i=1}^{m} a_i e_i$. We put $Uu = \sum_{i=1}^{m} a_i f_i$. Then $R(U) \subseteq N$ and if $v \in H$, there are m complex numbers $b_1, b_2, \ldots, b_m$ such that $Qv = \sum_{i=1}^{m} b_i f_i$. Then $\langle Uu, v \rangle = \sum_{i=1}^{m} a_i \overline{b}_i = \langle u, \sum_{i=1}^{m} b_i e_i \rangle = \langle u, U^* v \rangle$, hence $U^* v = \sum_{i=1}^{m} b_i e_i$. It is easy to see that $R(U*) = M$, U is a partial isometry of H, and

$$U^* U = P, \qquad UU^* = Q. \qquad \qquad \square$$

Lemma 4 *Let $A \in L(H)$, $\beta \in \rho_e(A)$, and $\alpha \in \mathbb{R}_+^*$. Then there is $T(A, \beta) \in L(H)$ such that*
  (1) $T(A, \beta)$ *is of finite rank,*
  (2) $\|T(A, \beta)\| \leq 1$,
  (3) $\beta \in M_e(A + \alpha T(A, \beta))$.

Proof We suppose at first $\operatorname{ind}(A - \beta I) \geq 0$. Hence $\dim N(A^* - \beta I) < \infty$. By Lemma 3, there is U such that $R(U) \subseteq N(A - \beta I)$, $R(U^*) = N(A^* - \beta I)$. We take $U^* = T(A, \beta)$. Then $T(A, \beta)$ is of finite rank, thus compact, and we have $\beta \in \rho_e(A + \alpha T(A, \beta))$. Furthermore $A + \alpha T(A, \beta) - \beta I$ is onto. Indeed $R(A - \beta I)$ is a closed subspace of H. Hence $H = R(A - \beta I) \oplus N(A^* - \beta I)$. Let $u \in H$, then $u = u_1 + u_2$ with $u_1 \in R(A - \beta I)$ and $u_2 \in N(A* - \beta I)$. Let Q be a projection onto $N(A - \beta I)$. There is v_1 and v_2 such that $(A - \beta I)v_1 = u_1$ where $Qv_1 = 0$ and $u_2 = \alpha T v_2$, where $v_2 = Qv_2$, that is, $v_2 \in N(A - \beta I)$. If we take $v = v_1 + v_2$ it follows that

$$\bigl(A + \alpha T(A, \beta) - \beta I\bigr)v = (A - \beta I)(I - Q)v_1 + \alpha T(A, \beta)v_2$$

$$= (A - \beta I)v_1 + \alpha T(A, \beta)v_2 = u_1 + u_2 = u.$$

Hence the operator $(A + \alpha T(A, \beta) - \beta I)$ is onto and $\beta \in M_e(A + \alpha(T, \beta))$. If $\operatorname{ind}(A - \beta I) < 0$ and $\dim N(A - \beta I) > 0$, we proceed as above by replacing $A - \beta I$ by $A^* - \beta I$ and T by T^* as $\operatorname{ind}(A - \beta I) = -\operatorname{ind}(A^* - \beta I)$. The operator $(A + \alpha T(A, \beta) - \beta I)$ is one-to-one, it follows that

$$\beta \in M_e\bigl(A + \alpha T(A, \beta)\bigr). \qquad \qquad \square$$

Remark 3 If $\operatorname{ind}(A - \beta I) = 0$, then $A + \alpha T(A, \beta) - \beta I$ is onto and one-to-one.

Remark 4 If $\min \operatorname{ind}(A - \beta I) = 0$, we take $T = 0$. Indeed in this case $A + \alpha T(A, \beta) - \beta I = A - \beta I$ is invertible on the left or right.

Lemma 5 *Let $A \in L(H)$. Then for all $\varepsilon > 0$, there is $K_1 \in K(H)$ such that*
  (1) $\|K_1\| \leq \varepsilon$,
  (2) $\rho_e(A + K_1) \cap \operatorname{reg}(A + K_1) \subseteq M_e(A + K_1)$.

Proof Suppose that $\rho_e(A) = \rho_e(B) = \bigcup_{1 \leq i \leq n} C_i$ where C_i, $i = 1, 2, 3, \ldots, n$, is a connected component. For each i, let $\beta_i \in C_i$. We inductively define a sequence of positive real numbers and a sequence of operators in $L(H)$ as follows:

(1) $\alpha_1 = \alpha$, $A_0 = A$,

(2) $A_i = A_{i-1} + \alpha_i T_i$, $i = 1, 2, 3, \ldots, n$, where $T_i = T(A_{i-1}, \beta_i)$,

(2) $\alpha_{i+1} = \frac{1}{3} \min\{\alpha_i, c(A_i - \beta_i I), i = 1, 2, 3, \ldots, n - 1\}$.

Note that (1) and (3) give $\alpha_{i+1} \leq \frac{1}{3}\alpha_i \Rightarrow \alpha_{i+1} \leq \frac{\alpha}{3^i}$. We set $S_i = \sum_{1 \leq j \leq i} \alpha_j T_j$; then

$$\|S_i\| \leq \sum_{1 \leq j \leq i} \alpha_j \|T_j\| \leq \sum_{1 \leq j \leq i} \alpha_j \leq \sum_{1 \leq j \leq \infty} \alpha_j \leq \alpha \sum_{1 \leq j \leq \infty} \frac{1}{3^{j-1}} = \frac{3}{2}\alpha.$$

Therefore $\|S_i\| \leq \frac{3}{2}\alpha$. For each i, $A_i = A + S_i$ where S_i is a finite rank operator and therefore compact. We must have $\rho_e(A) = \rho_e(A_i)$ and $c(A_i - \beta_i I) > 0$. This shows that all α_i are strictly positive. In addition, by the previous lemma $A_i - \beta_i I$ is left or right invertible, from which we deduce that $C_i \cap \text{reg}(A_i) \subseteq M_e(A_i)$. Finally

$$\|S_n - S_i\| \leq \sum_{i+1 \leq j \leq n} \alpha_j \leq \frac{3}{2}\alpha_{i+1} \leq \frac{1}{2}c(A_i - \beta_i I) < c(A_i - \beta_i I).$$

Hence $A_n - \beta_i I - (A_i - \beta_i I) = S_n - S_i$; using Lemma 2, we deduce that for $i = 1, 2, 3, \ldots, n$, $A_n - \beta_i I$ is left or right invertible and therefore $\rho_e(A + S_n) \cap \text{reg}(A + S_n) \subseteq M_e(A + S_n)$. Accordingly we get $K_1 = S_n$ and $\alpha = \frac{2}{3}\varepsilon$, and the theorem is proved. If $\rho_e(A)$ has a countable infinite numbers of connected components, the only other possible case, we observe that the sequence (S_n) converges normally, and taking $K_1 = \lim_{n \to \infty} S_n$, we proceed exactly as in the previous case. As, for every n, S_n is an operator of finite rank, K_1 is a compact operator. We set $B = A + K_1$. □

2.1.2 Second correction

Let $B \in L(H)$ such that $\rho_e(B) \cap \text{reg}(B) \subseteq M_e(B)$. We will build a second compact perturbation K_2 such that $\rho_e(B) \subseteq M_e(B + K_2)$. If $B \in L(H)$ such that $\rho_e(B) \cap \text{reg}(B) \subseteq M_e(B)$, then $\rho_e(B) \backslash M_e(B)$ is a finite or countable set denoted μ_j. Note that $\sigma_e(B) = \emptyset$, because in [6], there exist $n \in \mathbb{N}$ and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n \in \mathbb{C}$ such that $\sigma(B) = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}$. Hence $\dim H < \infty$, this is contrary to our hypotheses. As $\sigma_e(B)$ is a closed set of $\sigma(B)$ (which is compact), $\sigma_e(B)$ is also compact. Since the distance function is continuous on $\sigma_e(B)$, there is a sequence of points $(\mu'_l)_{j \in \mathbb{N}^*} \in \sigma_e(B)$ such that $d(\mu_j, \sigma_e(B)) = |\mu_j - \mu'_j|$. Finally if the sequence (μ_n) is infinite, then $\lim_{j \to \infty} |\mu_j - \mu'_j| = 0$, because otherwise μ_n would have an accumulation point in $\rho_e(B)$, which is contradictory. For every μ_j, let M_j, N_j be the two subspaces of H corresponding to the decomposition of Kato of the operator $B - \mu_j I$ and let P_j be the associated projection of the kernel M_j and the range N_j.

Construction of subspaces M'_j and N'_j

For $j \in \mathbb{N}$, let $M'_0 = H$, $M'_{j+1} = M'_j \cap M_{j+1}$, and $N'_1 = M'^{\perp}_1$, $N'_{j+1} = M'_j \cap M'^{\perp}_{j+1}$. Finally, we denote by P'_j the orthogonal projection onto N'_j.

Lemma 6

(1) *If $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$ then $N'_i \perp N'_j$ if and only if $P'_i P'_j = 0$.*

(2) *If $1 \leq j \leq n$, then $N'_j \oplus M'_j = N_j \oplus M'_j = M'_{j-1}$ and $\dim N'_j = \dim N_j < \infty$.*

(3) *If $1 \leq j \leq n$, then $\bigcap_{1 \leq i \leq j} N'_i = M'_j$ and $\operatorname{co}\dim M'_j = \sum_{1 \leq i \leq j} \dim N_i$.*

Proof For any $i > j$, we have $N'_i \perp N'_j$. Indeed $N'_i \subseteq M'_{i-1} \subseteq M'_j$; and $N'_j \subseteq M'^{\perp}_j$, then $M'_j \subseteq N'^{\perp}_j$ and we deduce $N'_i \subseteq N'^{\perp}_j$. Suppose $P'_i P'_j = 0$ and let $x \in N'_i$. Then $\langle x, y \rangle = \langle P'_i x, P'_j y \rangle = \langle x, P'_i P'_j y \rangle = 0$. N'_i is orthogonal to N'_j. Reciprocally if N'_i is orthogonal to N'_j and $u \in H$, then $P'_j u \in N'_j$ and $P'_i u \in N'^{\perp}_j$ then $P'_i P'_j u = 0$ and $P'_i P'_j = 0$.

$N'_j \cap M'_j \subseteq M'^{\perp}_j \cap M'_j = \{0\}$ and $N'_j \cap M'_j \subseteq N'_j \cap M'_j = \{0\}$. $M'_j + N'_j = M'_{j-1}$ (because $N'_j = M'_{j-1} \cap M'^{\perp}_j$).

For any $i > j$, $N'_i \subseteq M'_{j-1} = N'_j \oplus M'_j$, then $M'_j \oplus N'_i \subseteq M'_j \oplus N'_j$; $N'_j \subseteq M'_{j-1} = M'_j \oplus N_j \Rightarrow M'_j \oplus N'_i \subseteq M'_j \oplus N_j$.

If $i = j$, the result is obvious. Suppose that it is true for j. Then $\operatorname{co}\dim M'_j < \infty$ and therefore $M'^{\perp}_j \oplus M'_{j+1}$ is closed. Hence $M'^{\perp}_j \oplus M'_{j+1} = N'^{\perp}_{j+1}$ and we have

$$\bigcap_{1 \leq i \leq j+1} N'^{\perp}_i = M'_j \cap N'^{\perp}_{j+1} = M'_j \cap \left(M'^{\perp}_j \oplus M'_{j+1} \right) = M'_{j+1}.$$

It follows that for any $j \in \mathbb{N}$, $1 \leq j \leq n$; $M'^{\perp}_{j+1} = M'^{\perp}_j \oplus N'_j$ It is easy to see that $\operatorname{co}\dim M'_j = \sum_{1 \leq i \leq j} \dim N'_i < \infty$. □

Remark 5 If (P_i) is a sequence of mutually orthogonal projections in H, then $\sum_{1 \leq i \leq n} P_i$ converges to P, which is an orthogonal projection (see [13]).

Suppose first that the sequence (μ_n) is finite, $1 \leq j \leq n$. We put $H_1 = \bigcap_{1 \leq j \leq n} M_j$ and $K_2 = \sum_{1 \leq j \leq n} (\mu'_j - \mu_j) P_j$. It is obvious that K_2 is normal.

Lemma 7 *If $\mu \in \rho_e(B)$, then $N(B + K_2 - \mu I) \subseteq N(B - \mu I) \cap H_1$.*

Proof First show by induction that if $u \in N(B + K_2 - \mu I)$ then $u \in H_1$. Let $u \in N(B + K_2 - \mu I)$, then $P'_1 = 0$. Indeed

$$(B + K_2 - \mu I)(I - P'_1)u + (B + K_2 - \mu I)P'_1 u = (B + K_2 - \mu I) = 0.$$

But

$$K_2 P'_1 u = \left[\sum_{1 \leq k \leq n} (\mu'_k - \mu_k) P'_k \right] P'_1 u = (\mu'_1 - \mu_1) P'_1 u.$$

Hence

$$0 = (B + K_2 - \mu I)(I - P'_1)u + B - \mu_1 I + (\mu'_1 - \mu_1)IP'_1 u$$

$$= (B + K_2 - \mu I)(I - P'_1)u + B - \mu_1 I + (\mu'_1 - \mu_1)(I - P_1)P'_1 u$$

$$+ B - \mu_1 I + (\mu'_1 - \mu_1)IP_1 P'_1 u.$$

Now we account for the last two terms of the amount in M_1 (because $K_2(I - P'_1) = (I - P'_1)K_2$ is invariant under B) and the last part of N_1 which is invariant under B. So as $H = M_1 \oplus N_1$ then $B - \mu_1 I + (\mu'_1 - \mu)IP_1 P'_1 u = 0$. The operator $(B - \mu_1 I)|N_1$ is nilpotent; and as

$\mu_1 = \mu$, then $B - \mu_1 I + (\mu_1' - \mu)I$ is one-to-one. It follows that $P_1 P_1' u = 0$. Hence $P_1' u = 0$. Suppose that always $u \in N(B + K_2 - \mu I)$; $P_j' = 0$ with $j = 1, 2, 3, \ldots, m$. According to Lemma 5, $u \in M_m' u$, we show $P_{m+1}' u = 0$. Indeed

$$(B + K_2 - \mu I)\big(I - P_{m+1}'\big)u + (B + K_2 - \mu I)P_{m+1}' u = (B + K_2 - \mu I) = 0.$$

But $K_2 P_{m+1}' u = (\mu_{m+1}' - \mu_{m+1})P_{m+1}' u$, and $(I - P_{m+1}') \in M_{m+1}' \subseteq M_{m+1}$. Hence

$$(B + K_2 - \mu I)\big(I - P_{m+1}'\big)u + B - \mu_{m+1}I + \big(\mu_{m+1}' - \mu\big)I P_{m+1}' u = 0. \tag{2.1}$$

But

$$(B + K_2 - \mu I)\big(I - P_{m+1}'\big)u = (B - \mu I)\big(I - P_{m+1}'\big)u + K_2\big(I - P_{m+1}' u\big) \in M_{m+1}.$$

Because M_{m+1} is invariant under $B - \mu I$ and M_{m+1}' is invariant under K_2,

$$\begin{aligned}
&B - \mu_{m+1}I + \big(\mu_{m+1}' - \mu\big)I P_{m+1}' u \\
&\quad = B - \mu_{m+1}I + \big(\mu_{m+1}' - \mu\big)I(I - P_{m+1})P_{m+1}u \\
&\quad\quad + B - \mu_{m+1}I + \big(\mu_{m+1}' - \mu\big)I P_{m+1} P_{m+1}' u.
\end{aligned}$$

The first term is in M_{m+1} while the second is in N_{m+1} (same reasons as before). So as $H = M_{m+1} \oplus N_{m+1}$, and as above, we deduce from (2.1) that $B - \mu_{m+1}I + (\mu_{m+1}' - \mu)I P_{m+1} P_{m+1}' u = 0$ then $P_{m+1} P_{m+1}' u = 0$. Hence $P_{m+1} u \in M_{m+1}$. So by Lemma 6 and using the induction hypothesis $u \in M_{m+1}$ and therefore $P_{m+1}' u = 0$, which states that $u \in H_1$ and $K_2 u = 0$. If the sequence μ_n is infinite, we set $H_1 = \bigcap_{1 \le j \le \infty} M_j$ and $K_2 \sum_{1 \le j \le \infty} (\mu_j' - \mu_j)P_j'$, the previous proposal remains valid because the series defining K_2 is normally convergent. Indeed $\lim_{j \to \infty}(\mu_j' - \mu_j) = 0 \Leftrightarrow (\forall \varepsilon > 0)\ (\exists N(\varepsilon) > 0)\ (\forall j \ge N(\varepsilon) \Rightarrow |\mu_j' - \mu_j| < \varepsilon)$. Let $u \in H$ and $n \ge N(\varepsilon)$ then

$$\begin{aligned}
\left\| \sum_{k \ge n}(\mu_k' - \mu_k)P_k' u \right\|^2 &= \left\langle \sum_{k \ge n}(\mu_k' - \mu_k)P_k' u, \sum_{j \ge n}(\mu_j' - \mu_j)P_j' u \right\rangle \\
&= \left\langle \sum_{k \ge n}\sum_{j \ge n}(\mu_k' - \mu_k)\overline{(\mu_j' - \mu_j)}P_j' P_k' u, u \right\rangle \\
&= \sum_{k \ge n}\big|\mu_k' - \mu_k\big|^2 \langle P_k' u, P_k' u \rangle \le \varepsilon^2 \left\| \sum_{k \ge n} P_k' u \right\|^2 \\
&\le \varepsilon^2 \big\| P' u \big\|^2 \le \varepsilon^2 \|u\|^2.
\end{aligned}$$

Therefore $\| \sum_{k \ge n}(\mu_k' - \mu_k)P_k' u \| \le \varepsilon$, then $\sum_{1 \le j \le \infty}(\mu_j' - \mu_j)P_j'$ is normally convergent. The rest of the proof is similar to the case of finite μ_j. □

Lemma 8 *If $\mu \in \rho_e(B)$, then $H_1^\perp \subseteq R(B + K_2 - \mu I) + H_1$.*

Proof (A) First prove that for any $j \in \mathbb{N}$, if $\mu_j \in N_j'$ then there exists one and only one $w_j \in N_j'$ such that

$$P_j'(B + K_2 - \mu I)w_j = u_j. \tag{2.2}$$

Uniqueness. Suppose that w_j and w_j' satisfy the requirement. Then $w_j - w_j' \in N(B + K_2 - \mu I) \subseteq H_1$ (see Lemma 6), hence $w_j - w_j' \in N_j' \cap H_1 = \{0\}$.

Existence. If $u_j \in N_j'$ then $u_j = x_j + y_j$ with $x_j \in N_j$, $y_j \in M_j'$ and $u_j = P_j' x_j$. We have $B - \mu I + (\mu_j' - \mu_j)I = B - \mu_j I + (\mu_j' - \mu)I$ and as $\mu_j' \neq \mu$, there is $v_j \in N_j$ such that $x_j = [B - \mu I + (\mu_j' - \mu_j)I]v_j$ then

$$
\begin{aligned}
u_j &= P_j'\big[B - \mu I + (\mu_j' - \mu_j)I\big]\big[P_j'v_j + (I - P_j')v_j\big] \\
&= P_j'[B - \mu I + K_2]P_j'v_j + P_j'[B - \mu I + (\mu_j' - \mu_j)I](I - P_j')v_j.
\end{aligned}
$$

But $(I - P_j')vj \in M_j'$. Hence $[B - \mu I + (\mu_j' - \mu_j)I](I - P_j')v_j \in M_j'$ and therefore $P_j'[B - \mu I + (\mu_j' - \mu_j)I](I - P_j')v_j = 0$. We take $w_j = P_j'v_j \in N_j'$.

(B) Let $Q = \sum_j P_j'$, $u_j = P_j'u$, and $u \in H_1^{\perp}$. We show that there is $w \in H_1^{\perp}$ such that

$$
u = Q(B + K_2 - \mu I)w. \tag{2.3}
$$

It follows from the existence of $w_1 \in N_1'$ that $P_1'(B + K_2 - \mu I)w_1 = u_1$. Suppose we know $w_1, w_2, w_3, \ldots, w_n$, $u_{n+1} - \sum_{1 \leq j \leq n} P_{n+1}'(B + K_2 - \mu I)w_j \in N_{n+1}'$, then there is $w_{n+1} \in N_{n+1}'$ such that

$$
P_{n+1}'(B + K_2 - \mu I)w_{n+1} = u_{n+1} - \sum_{1 \leq j \leq n} P_{n+1}'(B + K_2 - \mu I)w_j,
$$

and therefore

$$
u_{n+1} = \sum_{1 \leq j \leq n} P_{n+1}'(B + K_2 - \mu I)w_j. \tag{2.4}
$$

Let $w = \sum_j w_j$. Then

$$
\begin{aligned}
Q(B + K_2 - \mu I)w &= \sum_j P_j'(B + K_2 - \mu I)\sum_i w_i \\
&= \sum_j \sum_i P_j'(B + K_2 - \mu I)w_i \\
&= \sum_j \sum_i P_j'[B - \mu I + (\mu_i' - \mu_i)I]w_i.
\end{aligned}
$$

But $w_i \in M_{i-1}'$, $[B - \mu I + (\mu_i' - \mu_i)I]w_i \in M_{i-1}'$ and if $j < i$ we have $P_j'(M_{i-1}') = 0$. Then

$$
Q(B + K_2 - \mu I)w = \sum_j \sum_{1 \leq i \leq j} P_j'(B + K_2 - \mu I)w_i.
$$

Using this we find $Q(B + K_2 - \mu I)w = \sum_j u_j = u$; then $u = (B + K_2 - \mu I)w - (I - Q)(B + K_2 - \mu I)w$. Hence $u \in R(B + K_2 - \mu I) + H_1$. This completes the proof of the lemma. □

Lemma 9 *Let $B \in L(H)$ such that $\rho_e(B) \cap \mathrm{reg}(B) \subseteq M_e(B)$. Then there is a compact operator $K_2 \in K(H)$ such that*

$$
\rho_e(B) = \rho_e(B + K_2) \subseteq M_e(B + K_2).
$$

Proof We show that

$$\rho e(B) \subseteq \text{reg}(B + K_2). \tag{2.5}$$

Let $\mu \in \rho_e(B)$ and $u \in N(B + K_2 - \mu I)$. Then $u \in N(B - \mu I) \cap H_1$. But $(B - \mu I)|H_1$ is regular because for any $j \in \mathbb{N}$, $(B - \mu I)|M_j$ is also regular. Hence for any $j \in \mathbb{N}$, $u \in R[(B - \mu I)^j |H_1] \subseteq R[(B + K_2 - \mu I)^j]$. As further $B + K_2 - \mu I$ is a compact perturbation of the semi-Fredholm operator $B - \mu I$, it is itself semi-Fredholm and hence $R(B + K_2 - \mu I)$ is a closed subspace, which completes the demonstration We now show that $\text{reg}(B + K_2) \subseteq M_e(B + K_2)$. Let $\rho e(B) = \bigcup_k C_k = \text{reg}(B + K_2)$ where C_k is the nth k-connected component of $\rho_e(B)$.

For any $k \in \mathbb{N}^*$, $C_k \cap \text{reg}(B) = \emptyset$. Let $\mu \in C_k \cap \text{reg}(B) = C_k \cap M_e(B)$.

Two cases are possible.

First case. $B - \mu I$ is one-to-one and therefore $N(B + K_2 - \mu I) \subseteq N(B - \mu I) = 0 \Rightarrow B + K_2 - \mu I$ is one-to-one, hence $C_k \subseteq M_e(B + K_2)$.

Second case. $B - \mu I$ is onto and therefore $H_1 \subseteq R(B - \mu I)$. For any $u \in H_1$, there is v such that $(B - \mu I)v = u$;

$$(B - \mu I)Qv = (B - \mu I)v - (B - \mu I)(I - Q)v,$$

$(B - \mu I)v \in H_1$, $(I - Q)v \in H_1 \Rightarrow (B - \mu I)(I - Q)v \in H_1$, $Qv = \sum_j P'_j v = v_j$ where $v_j = \sum_j P'_j v$. We have

$$(B - \mu I)Qv = \sum_j (B - \mu I)v_j = \sum_j (B - \mu I)P_j v_j + \sum_j (B - \mu I)(I - P_j)v_j,$$

where

$$\sum_j (B - \mu I)P_j v_j \in N_j \quad \text{and} \quad \sum_j (B - \mu I)(I - P_j)v_j \in M'_j$$

because $v_j \in N'_j \subseteq M'_{j-1} \Rightarrow (I - P_j)v_j \in M'_j$. We have

$$(B - \mu I)P_1 v_1 = (B - \mu I)Qv - \sum_{j>1}(B - \mu I)P_j v_j - \sum_j (B - \mu I)(I - P_j)v_j \in M_1.$$

Hence $(B - \mu I)P_1 v_1 = 0$ because

$$(B - \mu I)P_1 v_1 \in M_1 \cap N_1 = 0.$$

As $\mu \in \text{reg}(B) \Rightarrow \mu_1 = \mu$ and $B - \mu I = B - \mu_1 + (\mu_1 - \mu)I$. We can see that $P_1 v_1 = 0$, hence $v_1 \in M_1 \cap N_1 = 0 \Rightarrow v_1 = 0$. Suppose we have shown that $v_1 = v_2 = v_3 = \cdots = v_n = 0$ and we demonstrate that $v_{n+1} = 0$. As previously $(B - \mu I)P_{n+1}v_{n+1} \in M'_{n+1}$ hence $(B - \mu I)P_{n+1}v_{n+1} = 0$, and as $\mu \in \text{reg}(B) \Rightarrow \mu = \mu_{n+1} \Rightarrow P_{n+1}v_{n+1} = 0$, then $v_{n+1} \in M_{n+1} \cap N'_{n+1} = 0 \Rightarrow v_{n+1} = 0$. It follows that $(B - \mu I)Qv = 0$ hence $u = (B - \mu I)w$ with $w = (I - Q)v \in H_1$; therefore $u = (B + K_2 - \mu I)w \Rightarrow u \in R(B + K_2 - \mu I)$. Hence $H_1 \subseteq R(B + K_2 - \mu I)$ and (by Lemma 7) we have $H_1^\perp \subseteq R(B + K_2 - \mu I)$; hence $H \subseteq R(B + K_2 - \mu I)$, which implies that $B + K_2 - \mu I$ is onto and $C_k \subseteq M_e(B + K)_2$ and therefore $\rho_e(B) = \sum_k C_k \subseteq M_e(B + K_2)$, which is equivalent to $\rho_e(B) = M_e(B + K_2)$. $\square$

Theorem 7 *Let $A \in L(H)$. There is a compact operator $K \in K(H)$ such that*

$$\rho_e(A) = M_e(A + K).$$

Proof This is an immediate consequence of Lemma 5 and Lemma 9; simply take $K = K_1 + K_2$. □

Corollary 1 *Let A be a Riesz operator. Then there exists a normal compact operator K such that $(A - K) \in QN(H)$.*

Proof A satisfies the hypothesis of Lemma 9 and also $\rho_e(A) = \mathbb{C}\backslash 0$ and $\mathrm{reg}(A) = \mathbb{C}\backslash \sigma(A) = \rho(A)$.

$\rho_e(A) \cap \mathrm{reg}(A) = \rho(A) = M_e(A)$. Hence there is a compact operator K such that $\rho(A - K) = M_e(A - K) = \rho_e(A) = \mathbb{C}\backslash 0 \Rightarrow \sigma(A - K) = 0 \Rightarrow (A - K) \in QN(H)$. We find the result of [14]. □

Remark 6 The result of Stampfli is a special case of Theorem 7 because if $\mu \in \Phi_0(A)$, then $\mu \in \rho_e(A) \subseteq M_e(A + K)$, hence $A + K - \mu I$ is one-to-one or onto. If $A + K - \mu I$ is one-to-one (resp. onto), then $N(A + K - \mu I) = 0$ (resp. $N(A^* + K^* - \mu I) = 0$) and $\mathrm{ind}(A + K - \mu I) = \mathrm{ind}(A - \mu I) = 0N(A^* + K^* - \mu I) = 0$ (resp. $N(A + K - \mu I) = 0$) and therefore $A + K - \mu I$ is one-to-one and onto. Hence $\mu \in \rho(A + K) \Rightarrow \Phi_0(A) = \rho(A + K)$.

Remark 7 In 1966, West has shown in [14] that for any Riesz operator $A \in L(H)$ such that $\rho_e(A) \subseteq \mathbb{C}\backslash 0$ there is a compact operator $K \in K(H)$ and $Q \in QN(H)$ such that $A = K + Q$. We find this result as a special case of our result (see Corollary 1).

3 Conclusion
After the famous result of Stampfli, for all bounded operator A the Weyl spectrum is $\sigma_W(A) = \bigcap \sigma(A + K)$ where the intersection is taken over all closed ideal compact operators. In this work, we prove the possibility to extend this result by the use of Kato's decomposition to the set of semi-Fredholm operators.

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