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Weak convergence theorems of a hybrid algorithm in Hilbert spaces

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Abstract
In this paper, a hybrid algorithm is investigated for solving common solutions of a generalized equilibrium problem, a variational inequality, and fixed point problems of an asymptotically strict pseudocontraction. Weak convergence theorems are established in the framework of real Hilbert spaces.

Keywords: equilibrium problem; variational inequality; nonexpansive mapping; common solution

1 Introduction

Monotone variational inequalities recently have been investigated as an effective and powerful tool for studying a wide class of real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1–9] and the references therein. Monotone variational inequalities, which include many important problems in nonlinear analysis and optimization, such as the Nash equilibrium problem, complementarity problems, fixed point problems, saddle point problems, and game theory recently have been extensively studied based on projection methods. Many well-known problems can be studied by using methods which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set in which the required solution lies. The problem of finding a point in the intersection of these convex subsets is then of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such a point. Krasnoselskii-Mann iteration, which is also known as a one-step iteration, is a classic algorithm to study fixed points of nonlinear operators. However, Krasnoselskii-Mann iteration only enjoys weak convergence for nonexpansive mappings; see [10] and the references therein.

The purposes of this paper is to study common solutions of a generalized equilibrium problem, a variational inequality, and fixed point problems of an asymptotically strict pseudocontraction based on a hybrid algorithm. Weak convergence theorems are established in the framework of real Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a hybrid algorithm is introduced and the convergence analysis is given. Weak convergence theorems are established in a real Hilbert space.

2 Preliminaries

From now on, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C is a nonempty closed convex subset of H and P_C denotes the metric projection from H onto C.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also call it an α-inverse-strongly monotone mapping.

A set-valued mapping $T : H \to 2^H$ is said to be monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle > 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{ w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C \}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v \rangle \geq 0$ for all $u \in C$; see [6] and the references therein.

Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

It is known that $x \in C$ is a solution to (2.1) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping. Projection methods recently have been studied for variational inequality (2.1); see [11–22] and the references therein.

Let $S : C \to C$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. Recall that S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|Sx - Sy\| \leq k_n \|x - y\|, \quad \forall x, y \in C.$$

S is said to be κ-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa \big\|(x - Sx) - (y - Sy)\big\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [23]. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction.

T is said to be an asymptotically κ-strict pseudocontraction if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and a constant $\kappa \in [0,1)$ such that

$$\big\|T^n x - T^n y\big\|^2 \le k_n\|x - y\|^2 + \kappa \big\|\big(I - T^n\big)x - \big(I - T^n\big)y\big\|^2, \quad \forall x, y \in C, n \ge 1.$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [24]. It is clear that every asymptotically nonexpansive mapping is an asymptotically 0-strict pseudocontraction.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $A : C \to H$ is an inverse-strongly monotone mapping. In this paper, we consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2.2}$$

In this paper, the set of such $x \in C$ is denoted by $\mathrm{EP}(F,A)$, *i.e.*,

$$\mathrm{EP}(F,A) = \big\{x \in C : F(x,y) + \langle Ax, y - x\rangle \ge 0, \forall y \in C\big\}.$$

To study the generalized equilibrium problem (2.2), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F\big(tz + (1 - t)x, y\big) \le F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

If $A \equiv 0$, then the generalized equilibrium problem (2.2) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \ge 0, \quad \forall y \in C. \tag{2.3}$$

In this paper, the set of such $x \in C$ is denoted by $\mathrm{EP}(F)$, *i.e.*,

$$\mathrm{EP}(F) = \big\{x \in C : F(x,y) \ge 0, \forall y \in C\big\}.$$

If $F \equiv 0$, then the generalized equilibrium problem (2.2) is reduced to the classical variational inequality (2.1).

Recently, equilibrium problems (2.2) and (2.3) have been investigated by many authors; see [25–31] and the references therein. Motivated by the research going on in this direction, we study a hybrid algorithm for solving common solutions of variational inequality

(2.1), generalized equilibrium problem (2.2), and fixed points of an asymptotically strict pseudocontraction. Possible computation errors are taken into account. Weak convergence theorems are established in the framework of real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [32] *Let C be a nonempty closed convex subset of H, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued*;

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = \mathrm{EP}(F)$;

(d) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.2 [24] *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be an asymptotically strict pseudocontraction. Then $I - S$ is demi-closed, that is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 2.3 [33] *Let H be a Hilbert space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \qquad \limsup_{n\to\infty} \|y_n\| \leq d$$

and

$$\lim_{n\to\infty} \big\| t_n x_n + (1 - t_n) y_n \big\| = d$$

hold for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4 [34] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping, and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $S : C \to C$ be an asymptotically κ-strict pseudocontraction with the sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $\Omega = F(S) \cap \mathrm{VI}(C,B) \cap \mathrm{EP}(F,A)$ is not empty. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, and $\{\beta_n\}$ be real number sequences in $(0,1)$. Let $\{r_n\}$ and $\{s_n\}$ be two positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following process*:

$$\begin{cases} x_1 \in C, \\ F(z_n, z) + \langle Ax_n, z - z_n\rangle + \frac{1}{r_n}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = P_C(z_n - s_n Bz_n), \\ x_{n+1} = \alpha_n x_n + \alpha'_n(\beta_n y_n + (1-\beta_n)S^n y_n) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;
(b) $0 < p \leq \alpha_n \leq q < 1$ *and* $\sum_{n=1}^{\infty} \alpha''_n < \infty$;
(c) $0 < \kappa < \beta_n \leq b < 1$;
(d) $0 < s \leq s_n \leq s' < 2\beta$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where p, q, b, s, s', r, r' are real constants. Then $\{x_n\}$ converges weakly to some point in Ω.

Proof First, we show that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. Let $p \in \Omega$ be fixed arbitrarily. For any $x, y \in C$, we see that

$$\begin{aligned} &\|(I - r_nA)x - (I - r_nA)y\|^2 \\ &\quad = \|(x-y) - r_n(Ax - Ay)\|^2 \\ &\quad = \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\quad \leq \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax-Ay\|^2. \end{aligned} \tag{3.1}$$

Using the restriction (d), we see that $\|(I - r_nA)x - (I - r_nA)y\| \leq \|x - y\|$. This implies that $I - r_nA$ is nonexpansive. In the same way, we find that $I - s_nB$ is also nonexpansive. Using the restriction (c), we obtain that

$$\begin{aligned} &\|\beta_n y_n + (1-\beta_n)S^n y_n - p\|^2 \\ &\quad = \beta_n\|y_n - p\|^2 + (1-\beta_n)\|S^n y_n - S^n p\|^2 \\ &\qquad - \beta_n(1-\beta_n)\|(y_n - p) - (S^n y_n - S^n p)\|^2 \\ &\quad \leq \beta_n\|y_n - p\|^2 + (1-\beta_n)\big(k_n\|y_n - p\|^2 + \kappa\|(y_n - p) - (S^n y_n - S^n p)\|^2\big) \\ &\qquad - \beta_n(1-\beta_n)\|(y_n - p) - (S^n y_n - S^n p)\|^2 \\ &\quad = k_n\|y_n - p\|^2 - (1-\beta_n)(\beta_n - \kappa)\|(y_n - p) - (S^n y_n - S^n p)\|^2 \\ &\quad \leq k_n\|y_n - p\|^2. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\alpha_n'\big\|\beta_n y_n+(1-\beta_n)S^n y_n-p\big\|^2+\alpha_n''\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\|y_n-p\|^2+\alpha_n''\|e_n-p\|^2\\ &= \alpha_n\|x_n-p\|^2+\alpha_n'k_n\big\|P_C(I-s_nB)z_n-p\big\|^2+\alpha_n''\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\big\|T_{r_n}(I-r_nA)x_n-p\big\|^2+\alpha_n''\|e_n-p\|^2\\ &\le k_n\|x_n-p\|^2+\alpha_n''\|e_n-p\|^2.\end{aligned}$$

This implies from Lemma 2.4 that $\lim_{n\to\infty}\|x_n-p\|$ exists. This shows that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. From (3.2), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\|y_n-p\|^2+\alpha_n''\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\big\|(I-s_nB)z_n-p\big\|^2+\alpha_n''\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\big(\|z_n-p\|^2-s_n(2\beta-s_n)\|Bz_n-Bp\|^2\big)+\alpha_n''\|e_n-p\|^2\\ &\le k_n\|x_n-p\|^2-s_nk_n\alpha_n'(2\beta-s_n)\|Bz_n-Bp\|^2+\alpha_n''\|e_n-p\|^2.\end{aligned}$$

It follows that

$$s_nk_n\alpha_n'(2\beta-s_n)\|Bz_n-Bp\|^2\le k_n\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n''\|e_n-p\|^2.$$

With the aid of the restrictions (b) and (d), we find that

$$\lim_{n\to\infty}\|Bz_n-Bp\|=0. \tag{3.3}$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}\|y_n-p\|^2 &= \big\|P_C(I-s_nB)z_n-P_C(I-s_nB)p\big\|^2\\ &\le \big\langle (I-s_nB)z_n-(I-s_nB)p, y_n-p\big\rangle\\ &= \frac{1}{2}\Big\{\big\|(I-s_nB)z_n-(I-s_nB)p\big\|^2+\|y_n-p\|^2\\ &\quad -\big\|(I-s_nB)z_n-(I-s_nB)p-(y_n-p)\big\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|z_n-p\|^2+\|y_n-p\|^2-\big\|z_n-y_n-s_n(Bz_n-Bp)\big\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|z_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n\|^2\\ &\quad +2s_n\langle z_n-y_n, Bz_n-Bp\rangle-s_n^2\|Bz_n-Bp\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|x_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n\|^2\\ &\quad +2s_n\langle z_n-y_n, Bz_n-Bp\rangle-s_n^2\|Bz_n-Bp\|^2\Big\},\end{aligned}$$

which implies that

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|z_n-y_n\|^2+2s_n\|z_n-y_n\|\|Bz_n-Bp\|.$$

Hence, we find from (3.2) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\|y_n-p\|^2+\alpha_n''\|e_n-p\|^2\\ &\le k_n\|x_n-p\|^2-\alpha_n'k_n\|z_n-y_n\|^2+2\alpha_n's_nk_n\|z_n-y_n\|\|Bz_n-Bp\|\\ &\quad+\alpha_n''\|e_n-p\|^2.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\alpha_n'k_n\|z_n-y_n\|^2 &\le k_n\|x_n-p\|^2-\|x_{n+1}-p\|^2+2s_nk_n\|z_n-y_n\|\|Bz_n-Bp\|\\ &\quad+\alpha_n''\|e_n-p\|^2.\end{aligned}$$

From the restrictions (b) and (d), we find from (3.3) that

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{3.4}$$

It follows from (3.1) that

$$\begin{aligned}\|z_n-p\|^2 &= \big\|T_{r_n}(I-r_nA)x_n-p\big\|^2\\ &\le \|x_n-p\|^2-r_n(2\alpha-r_n)\|Ax_n-Ap\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\|y_n-p\|^2+\alpha_n''\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\alpha_n'k_n\|z_n-p\|^2+\alpha_n''\|e_n-p\|^2\\ &\le k_n\|x_n-p\|^2-\alpha_n'r_n(2\alpha-r_n)k_n\|Ax_n-Ap\|^2+\alpha_n''\|e_n-p\|^2.\end{aligned}$$

This implies that

$$\alpha_n'r_n(2\alpha-r_n)k_n\|Ax_n-Ap\|^2\le k_n\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n''\|e_n-p\|^2.$$

Using the restrictions (b) and (d), we obtain that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.5}$$

Since T_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|z_n-p\|^2 &= \big\|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)p\big\|^2\\ &\le \big\langle (I-r_nA)x_n-(I-r_nA)p, z_n-p\big\rangle\\ &= \frac{1}{2}\Big(\big\|(I-r_nA)x_n-(I-r_nA)p\big\|^2+\|z_n-p\|^2\\ &\quad-\big\|(I-r_nA)x_n-(I-r_nA)p-(z_n-p)\big\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x_n-p\|^2+\|z_n-p\|^2-\big\|x_n-z_n-r_n(Ax_n-Ap)\big\|^2\Big)\end{aligned}$$

$$= \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \big(\|x_n - z_n\|^2 \\ - 2r_n\langle x_n - z_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2\big)\big),$$

which implies that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2r_n\|x_n - z_n\|\|Ax_n - Ap\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + \alpha_n' k_n\|y_n - p\|^2 + \alpha_n''\|e_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + \alpha_n' k_n\|z_n - p\|^2 + \alpha_n''\|e_n - p\|^2 \\ &\leq k_n\|x_n - p\|^2 - \alpha_n' k_n\|x_n - z_n\|^2 + 2r_n\alpha_n' k_n\|x_n - z_n\|\|Ax_n - Ap\| \\ &\quad + \alpha_n''\|e_n - p\|^2,\end{aligned}$$

which yields that

$$\alpha_n' k_n\|x_n - z_n\|^2 \leq k_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n\alpha_n'\|x_n - z_n\|\|Ax_n - Ap\| \\ + \alpha_n''\|e_n - p\|^2.$$

Using the restrictions (b) and (d), we find from (3.5) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to ξ. Let T be a maximal monotone mapping defined by

$$Tx = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, y) \in \operatorname{Graph}(T)$, we have $y - Bx \in N_C x$. Since $y_n \in C$, by the definition of N_C, we have $\langle x - y_n, y - Bx\rangle \geq 0$. Since $y_n = P_C(I - s_n B)z_n$, we see that $\langle x - y_n, y_n - (I - s_n B)z_n\rangle \geq 0$ and hence

$$\left\langle x - y_n, \frac{y_n - z_n}{s_n} + Bz_n\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle x - y_{n_i}, y\rangle &\geq \langle x - y_{n_i}, Bx\rangle \\ &\geq \langle x - y_{n_i}, Bx\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{s_{n_i}} + Bz_{n_i}\right\rangle\end{aligned}$$

$$= \langle x - y_{n_i}, Bx - By_{n_i} \rangle + \langle x - y_{n_i}, By_{n_i} - Bz_{n_i} \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{s_{n_i}} \right\rangle$$
$$\geq \langle x - y_{n_i}, By_{n_i} - Bz_{n_i} \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{s_{n_i}} \right\rangle.$$

Since y_{n_i} converges weakly to ξ and B is $\frac{1}{\beta}$-Lipschitz continuous, we see that $\langle x - \xi, y \rangle \geq 0$. Notice that T is maximal monotone and hence $0 \in T\xi$. This shows that $\xi \in \mathrm{VI}(C, B)$. From (3.6), we see that z_{n_i} converges weakly to ξ. It follows that

$$F(z_n, z) + \langle Ax_n, z - z_n \rangle + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C.$$

From condition (A2), we see that

$$\langle Ax_n, z - z_n \rangle + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq F(z, z_n), \quad \forall z \in C.$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, z - z_{n_i} \rangle + \left\langle z - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(z, z_{n_i}), \quad \forall z \in C. \tag{3.8}$$

For t with $0 < t \leq 1$ and $z \in C$, let $u_t = tz + (1 - t)\xi$. Since $z \in C$ and $\xi \in C$, we have $u_t \in C$. In view of (3.8), we find that

$$\langle u_t - z_{n_i}, Au_t \rangle \geq \langle u_t - z_{n_i}, Au_t \rangle - \langle Ax_{n_i}, u_t - z_{n_i} \rangle - \left\langle u_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle$$
$$+ F(u_t, z_{n_i})$$
$$= \langle u_t - z_{n_i}, Au_t - Az_{n_i} \rangle + \langle u_t - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle$$
$$- \left\langle u_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(u_t, z_{n_i}).$$

Using (3.6), we have $\lim_{i\to\infty} Az_{n_i} - Ax_{n_i} = 0$. Since A is monotone, we see that $\langle u_t - z_{n_i}, Au_t - Az_{n_i} \rangle \geq 0$. It follows from condition (A4) that

$$\langle u_t - \xi, Au_t \rangle \geq F(u_t, \xi). \tag{3.9}$$

Using conditions (A1) and (A4), we see from (3.9) that

$$0 = F(u_t, u_t) \leq tF(u_t, z) + (1 - t)F(u_t, \xi)$$
$$\leq tF(u_t, z) + (1 - t)\langle u_t - \xi, Au_t \rangle$$
$$= tF(u_t, u) + (1 - t)t\langle z - \xi, Au_t \rangle,$$

which yields that

$$F(u_t, z) + (1 - t)\langle z - \xi, Au_t \rangle \geq 0.$$

Letting $t \to 0$, we find

$$F(\xi, z) + \langle z - \xi, A\xi \rangle \geq 0,$$

which implies that $\xi \in \mathrm{EP}(F, A)$.

Now, we are in a position to show $\xi \in F(S)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we may assume that $\lim_{n\to\infty} \|x_n - p\| = d > 0$. Put $\lambda_n = \beta_n y_n + (1 - \beta_n)S^n y_n$. It follows from (3.2) that $\limsup_{n\to\infty} \|x_n - p + \alpha_n''(e_n - \lambda_n)\| \leq d$ and $\limsup_{n\to\infty} \|\lambda_n - p + \alpha_n''(e_n - \lambda_n)\| \leq d$. On the other hand, we have

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \big\|\alpha_n\big((x_n - p) + \alpha_n''(e_n - \lambda_n)\big) + (1 - \alpha_n)\big(\lambda_n - p + \alpha_n''(e_n - \lambda_n)\big)\big\| = d.$$

Using Lemma 2.3, we obtain that $\lim_{n\to\infty} \|\lambda_n - x_n\| = 0$. Note that

$$S^n y_n - x_n = \frac{\lambda_n - x_n}{1 - \beta_n} + \frac{\beta_n(x_n - y_n)}{1 - \beta_n}.$$

Hence, we have $\lim_{n\to\infty} \|S^n y_n - x_n\| = 0$. Note that $\|S^n x_n - x_n\| \leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\|$. Since S is Lipschitz continuous, we have $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$. Further, we find that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Using Lemma 2.2, we see that $\xi \in F(S)$. This proves that $\eta \in \Omega$.

Finally, we show that the sequence $\{x_n\}$ converges weakly to ξ. Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to η. In the same way, we find $\eta \in \Omega$. If $\eta \neq \xi$, we see from the Opial condition [35] that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \xi\| &= \liminf_{i\to\infty} \|x_{n_i} - \xi\| < \liminf_{i\to\infty} \|x_{n_i} - \eta\| \\
&= \liminf_{n\to\infty} \|x_n - \eta\| = \liminf_{j\to\infty} \|x_{n_j} - \eta\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \xi\| = \lim_{n\to\infty} \|x_n - \xi\|.
\end{aligned}$$

This derives a contradiction. Hence, we have $\eta = \xi$. This implies that $x_n \rightharpoonup \xi \in \Omega$. This completes the proof. □

Remark 3.2 The key of the weak convergence of the algorithm is due to the fact that A is inverse-strongly monotone, which yields that $I - r_n A$ is nonexpansive. The nonexpansivity of the mapping $I - r_n A$ plays an important role in this theorem. Therefore, it is of interest to relax the monotonicity of A such that the algorithm is still weakly convergent.

Next, we give some subresults of Theorem 3.1. If S is asymptotically nonexpansive, we find the following result.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone mapping, and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $S : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $\Omega = F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F, A)$ is not empty. Let $\{\alpha_n\}$, $\{\alpha_n'\}$,* *and* $\{\alpha''_n\}$ *be real number sequences in* $(0,1)$. *Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real number sequences. Let* $\{x_n\}$ *be a sequence generated in the following process*:

$$\begin{cases} x_1 \in C, \\ F(z_n,z) + \langle Ax_n, z - z_n\rangle + \frac{1}{r_n}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = P_C(z_n - s_n B z_n), \\ x_{n+1} = \alpha_n x_n + \alpha'_n S^n y_n + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* C. *Assume that the control sequences satisfy the following restrictions*:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;
(b) $0 < p \leq \alpha_n \leq p < 1$ *and* $\sum_{n=1}^{\infty} \alpha''_n < \infty$;
(c) $0 < s \leq s_n \leq s' < 2\beta$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where p, q, s, s', r, r' *are real constants. Then* $\{x_n\}$ *converges weakly to some point in* Ω.

Further, if S is an identity mapping, we have the following result.

Corollary 3.4 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping, and let* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Assume that* $\Omega = \mathrm{VI}(C,B) \cap \mathrm{EP}(F,A)$ *is not empty. Let* $\{\alpha_n\}$, $\{\alpha'_n\}$, *and* $\{\alpha''_n\}$ *be real number sequences in* $(0,1)$. *Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real number sequences. Let* $\{x_n\}$ *be a sequence generated in the following process*:

$$\begin{cases} x_1 \in C, \\ F(y_n,z) + \langle Ax_n, z - y_n\rangle + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0, \\ x_{n+1} = \alpha_n x_n + \alpha'_n P_C(y_n - s_n B y_n) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* C. *Assume that the control sequences satisfy the following restrictions*:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;
(b) $0 < p \leq \alpha_n \leq q < 1$ *and* $\sum_{n=1}^{\infty} \alpha''_n < \infty$;
(c) $0 < s \leq s_n \leq s' < 2\beta$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where p, q, s, s', r, r' *are real constants. Then* $\{x_n\}$ *converges weakly to some point in* Ω.

Next, we give a result on variational inequality (2.1).

Corollary 3.5 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping, and let* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Assume that* $\Omega = \mathrm{VI}(C,B) \cap \mathrm{VI}(C,A)$ *is not empty. Let* $\{\alpha_n\}$, $\{\alpha'_n\}$, *and* $\{\alpha''_n\}$ *be real number sequences in* $(0,1)$. *Let* $\{r_n\}$ *and* $\{s_n\}$ *be two positive real number sequences. Let* $\{x_n\}$ *be a sequence generated in the following process*:

$$\begin{cases} x_1 \in C, \\ z_n = P_C(x_n - s_n A x_n), \\ x_{n+1} = \alpha_n x_n + \alpha'_n P_C(z_n - s_n B z_n) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;
(b) $0 < p \le \alpha_n \le q < 1$ *and* $\sum_{n=1}^{\infty} \alpha''_n < \infty$;
(c) $0 < s \le s_n \le s' < 2\beta$ *and* $0 < r \le r_n \le r' < 2\alpha$,

where p, q, s, s', r, r' are real constants. Then $\{x_n\}$ converges weakly to some point in Ω.

Proof Putting $F \equiv 0$, we see that

$$\langle Ax_n, z - z_n\rangle + \frac{1}{r_n}\langle z - z_n, z_n - x_n\rangle \ge 0, \quad \forall z \in C$$

is equivalent to

$$\langle x_n - r_n Ax_n - z_n, z_n - z\rangle \ge 0, \quad \forall z \in C.$$

This implies that $z_n = P_C(x_n - r_n Ax_n)$. Let $\beta_n = 0$ and S be the identity. Then we can obtain from Theorem 3.1 the desired results immediately. □

Finally, we consider solving common fixed points of a pair of strict pseudocontractions.

Corollary 3.6 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1 : C \to C$ be an α-strict pseudocontraction, and let $T_2 : C \to C$ be a β-strict pseudocontraction. Assume that $\Omega = F(T_1) \cap F(T_2)$ is not empty. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, and $\{\alpha''_n\}$ be real number sequences in $(0,1)$. Let $\{r_n\}$ and $\{s_n\}$ be two positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following process*:

$$\begin{cases} x_1 \in C, \\ z_n = (1 - r_n)x_n + r_n T_2 x_n, \\ y_n = (1 - s_n)x_n + s_n T_1 x_n, \\ x_{n+1} = \alpha_n x_n + \alpha'_n y_n + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;
(b) $0 < p \le \alpha_n \le q < 1$ *and* $\sum_{n=1}^{\infty} \alpha''_n < \infty$;
(c) $0 < s \le s_n \le s' < 1 - \alpha$ *and* $0 < r \le r_n \le r' < 1 - \beta$,

where p, q, s, s', r, r' are real constants. Then $\{x_n\}$ converges weakly to some point in Ω.

Proof Put $F \equiv 0$, $A = I - T_2$ and $B = I - T_1$. It follows that A is $\frac{1-\alpha}{2}$-inverse-strongly monotone and B is $\frac{1-\beta}{2}$-inverse-strongly monotone. We also have $F(T_1) = \mathrm{VI}(C, B)$ and $F(T_2) = \mathrm{VI}(C, A)$. In view of Theorem 3.1, we find the desired result immediately. □

Competing interests

The author declares that they have no competing interests.

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