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On fractional differential equations and inclusions with nonlocal and average-valued (integral) boundary conditions

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Abstract

This paper is concerned with the existence of solutions for boundary value problems of fractional differential equations and inclusions supplemented with nonlocal and average-valued (integral) boundary conditions. The existence results for the single-valued case (equations) are obtained by means of fixed point theorems due to O'Regan and Sadovski, whereas the existence of solutions for the multivalued case (inclusions) is established via nonlinear alternative for contractive maps. The obtained results are well illustrated by examples.

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1 Introduction

The study of fractional differential equations has recently attracted the attention of many researchers and modelers. The interest in the subject owes to its widespread applications in a variety of applied sciences and engineering disciplines such as biological sciences, ecology, aerodynamics, control theory, viscoelasticity, electro-dynamics of complex medium, electron-analytical chemistry, environmental issues, *et cetera*. The recent trend in the mathematical modeling of several phenomena indicates the popularity of fractional calculus modeling tools due to the nonlocal characteristic of fractional-order differential and integral operators, which are capable of tracing the past history of many materials and processes; see, for instance, [1–7] and the references therein.

Differential inclusions, regarded as a generalization of differential equations and inequalities, have very important and interesting applications in optimal control theory and stochastic processes [8]. In fact, the tools of differential inclusions facilitate the investigation of dynamical systems having velocities not uniquely determined by the state of the system.

Boundary value problems of fractional-order differential equations and inclusions supplemented with several kinds of conditions such as classical, nonlocal, multipoint, periodic/antiperiodic, fractional-order, and integral boundary conditions have extensively been investigated by many researchers. In particular, the study of nonlocal boundary value problems finds interesting applications in physical and chemical processes, where the

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classical initial/boundary conditions fail to describe some peculiar phenomena occurring inside the domain. On the other hand, integral boundary conditions help to formulate computational fluid dynamics (blood flow) problems in a better way as such conditions allow one to describe the cross-section of vessels in a more realistic arbitrary manner instead of always assuming circular type cross-section [9]. Also, ill-posed parabolic backward problems in time partial differential equations can be regularized with the aid of integral boundary conditions; see, for example, mathematical models for bacterial self-regularization [10]. For details and examples, we refer the reader to a variety of results [11–26]. In a recent article [27], the authors studied a boundary value problem of fractional differential equations with nonlocal and average-type integral boundary conditions given by

$$
\begin{cases}
{}^{c}D^{\alpha}x(t) = f(t, x(t), {}^{c}D^{\beta}x(t)), & 0 < t < 1, \\
x(0) + x'(0) = h(x), \quad \int_{0}^{\eta} x(t)\,dt = \xi, & 0 < \eta < 1, \xi \in \mathbb{R}, \\
x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0,
\end{cases}
\tag{1.1}
$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ denote the Caputo fractional derivatives of orders α and β with $n - 1 < \alpha < n$ ($n \geq 2$) and $0 < \beta < 1$, and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions. Applying the Leray-Schauder nonlinear alternative, Krasnoselskii's fixed point theorem and Banach's fixed point theorem together with Hölder inequality, some existence results for problem (1.1) were obtained.

The objective of the present paper is to continue the study initiated in [27] and provide a variety in the existence criteria for solutions of the problem at hand. Precisely, we establish two more existence results for problem (1.1), which are based on fixed point theorems due to O'Regan and Sadovski (Section 3). Then we switch onto investigating the multivalued analogue of (1.1)

$$
\begin{cases}
{}^{c}D^{\alpha}x(t) \in F(t, x(t), {}^{c}D^{\beta}x(t)), & 0 < t < 1, \\
x(0) + x'(0) = h(x), \quad \int_{0}^{\eta} x(t)\,dt = \xi, & 0 < \eta < 1, \\
x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0,
\end{cases}
\tag{1.2}
$$

where $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. In Section 4, we discuss the existence of solutions for problem (1.2) by means of the nonlinear alternative for contractive maps.

2 Preliminaries

In this section, we present some basic definitions on fractional calculus and an auxiliary lemma [1, 3].

Definition 2.1 The fractional integral of order r with the lower limit zero for a function f is defined as

$$
I^{r}f(t) = \frac{1}{\Gamma(r)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-r}}\,ds, \quad t > 0, r > 0,
$$

provided that the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function defined by $\Gamma(r) = \int_{0}^{\infty} t^{r-1} e^{-t}\,dt$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $r > 0$, $n - 1 < r < n$, $n \in N$, of a function f is defined as

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-r-1} f(s) \, ds,$$

provided that the function f has absolutely continuous derivatives up to order $(n - 1)$.

Definition 2.3 The Caputo derivative of order r of a function $f : [0, \infty) \to R$ is defined as

$$^cD^r f(t) = D^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n - 1 < r < n.$$

Remark 2.1 If $f(t) \in C^n[0, \infty)$, then

$$^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} \, ds = I^{n-r} f^{(n)}(t), \quad t > 0, n - 1 < q < n.$$

We need the following known lemma [27].

Lemma 2.1 *Let $y \in AC[0, 1]$ and $x \in AC^n[0, 1]$. Then the linear problem*

$$\begin{cases} ^cD^\alpha x(t) = y(t), & 0 < t < 1, \\ x(0) + x'(0) = h(x), & \int_0^\eta x(t) \, dt = \xi, \quad 0 < \eta < 1, \\ x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0, \end{cases} \tag{2.1}$$

is equivalent to the fractional integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + \frac{2(1-t)}{\eta(2-\eta)} \xi + \frac{2t-\eta}{2-\eta} h(x)$$

$$+ \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} y(s) \, ds. \tag{2.2}$$

3 Existence results for a single-valued problem

Define the space $X = \{x | x \in C([0, 1], \mathbb{R}) \text{ and } ^cD^\beta x \in C([0, 1], \mathbb{R})\}$ equipped with the norm $\|x\|_X = \|x\| + \|^cD^\beta x\| = \sup_{t \in [0,1]} |x(t)| + \sup_{t \in [0,1]} |^cD^\beta x(t)|$. Clearly, $(X, \| \cdot \|_X)$ is a Banach space [28].

In view of Lemma 2.1, we introduce the operator $F : X \to X$ by

$$F(x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds + \frac{2(1-t)}{\eta(2-\eta)} \xi + \frac{2t-\eta}{2-\eta} h(x)$$

$$+ \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds,$$

which can be expressed as

$$(Fx)(t) = (F_1 x)(t) + (F_2 x)(t), \quad t \in [0, 1], \tag{3.1}$$

where $F_{1,2} : X \to X$ are given by

$$(F_1 x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x(s), {}^c D^\beta x(s)\right) ds$$
$$+ \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f\left(s, x(s), {}^c D^\beta x(s)\right) ds, \tag{3.2}$$

$$(F_2 x)(t) = \frac{2(1-t)}{\eta(2-\eta)} \xi + \frac{2t-\eta}{2-\eta} h(x). \tag{3.3}$$

For computational convenience, we set the notations

$$\Lambda_1 := \frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{2}{\eta\Gamma(2-\beta)} \frac{1}{\Gamma(\alpha+2)} \tag{3.4}$$

and

$$\Lambda_2 := \frac{2|\xi|}{\eta}\left(1 + \frac{1}{\Gamma(2-\beta)}\right), \qquad \Lambda_3 = 2 + \eta + \frac{2}{\Gamma(2-\beta)}. \tag{3.5}$$

Our first existence result relies on a fixed point theorem of O'Regan [29].

Lemma 3.1 *Denote by U an open set in a closed convex set C of a Banach space E. Assume that $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, where $F_1 : \bar{U} \to E$ is continuous and completely continuous, and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(z) < z$ for $z > 0$ and $\|F_2(x) - F_2(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in \bar{U}$). Then, either*
  (C1) *F has a fixed point $u \in \bar{U}$; or*
  (C2) *there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.*

In the proof of the next result, we use the notations

$$\Omega_r = \left\{ x \in X : \|x\|_X < r \right\}$$

and

$$M_r = \max\left\{ \left| f(t,x,y) \right| : (t,x) \in [0,1] \times [-r,r] \times [-r,r] \right\}.$$

Theorem 3.1 *Assume that*:

(A_1) *$h : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous function satisfying the condition*

$$\left| h(u) - h(v) \right| \leq \ell \|u - v\|, \quad \forall u, v \in C([0,1], \mathbb{R}),$$

with $\ell > 0$ such that $\ell\Lambda_3 < 1$, where Λ_3 is given by (3.5);
(A_2) *$h(0) = 0$;*
(A_3) *there exist a nonnegative function $p \in C([0,1], \mathbb{R})$ and nondecreasing functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, such that*

$$\left| f(t,x,y) \right| \leq p(t)\left(\psi_1\left(|x|\right) + \psi_2\left(|y|\right)\right)$$

for all $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$;

(A_4) $\sup_{r\in(0,\infty)} \frac{r}{\Lambda_1\|p\|[\psi_1(r)+\psi_2(r)]+\Lambda_2} > \frac{1}{1-\ell\Lambda_3}$, *where* Λ_1 *is given by* (3.4) *and* Λ_i, *i* = 2, 3, *are defined by* (3.5) (*note that* $\ell\Lambda_3 < 1$ *by* (A_1)).

Then the boundary value problem (1.1) *has at least one solution on* [0, 1].

Proof By assumption (A_4), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{\Lambda_1\|p\|[\psi_1(r_0)+\psi_2(r_0)]+\Lambda_2} > \frac{1}{1-\ell\Lambda_3}. \tag{3.6}$$

We shall show that the operators F_1 and F_2 defined by (3.2) and (3.3), respectively, satisfy all the conditions of Lemma 3.1.

Step 1. *The operator* F_1 *is completely continuous.* We first show that $F_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$
\begin{aligned}
\|F_1x\| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,x(s),{}^cD^\beta x(s)\right)\right|ds \\
&\quad + \frac{2(t-1)}{\eta(2-\eta)}\int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)}\left|f\left(s,x(s),{}^cD^\beta x(s)\right)\right|ds \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,x(s),{}^cD^\beta x(s)\right)\right|ds \\
&\quad + \frac{2(t-1)}{\eta(2-\eta)}\int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)}\left|f\left(s,x(s),{}^cD^\beta x(s)\right)\right|ds \\
&\le M_r\left[\frac{1}{\Gamma(\alpha+1)}+\frac{2}{\eta\Gamma(\alpha+2)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|{}^cD^\beta(F_1x)\right\| &\le M_r\int_0^t \frac{(t-s)^{\alpha-1-\beta}}{\Gamma(\alpha-\beta)}ds + M_r\frac{2t^{1-\beta}}{\eta\Gamma(2-\beta)}\int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)}ds \\
&\le M_r\int_0^t \frac{(t-s)^{\alpha-1-\beta}}{\Gamma(\alpha-\beta)}ds + M_r\frac{2}{\eta\Gamma(2-\beta)}\int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)}ds \\
&\le M_r\left\{\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{2}{\eta\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha+2)}\right\}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\|F_1x\|_X &= \|F_1x\| + \left\|{}^cD^\beta(F_1x)\right\| \\
&\le M_r\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{2}{\eta\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{2}{\eta\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha+2)}\right\},
\end{aligned}
$$

which shows that the operator $F_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

Further, for $0 \le t_1 < t_2 \le 1$, we have

$$
\begin{aligned}
\left|(F_1x)(t_2)-(F_1x)(t_1)\right| &\le \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]f\left(s,x(s),{}^cD^\beta x(s)\right)ds\right. \\
&\quad \left.+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f\left(s,x(s),{}^cD^\beta x(t)\right)ds\right|
\end{aligned}
$$

$$+ \frac{2(t_2 - t_1)}{\eta(2 - \eta)} \int_0^{\eta} \frac{(\eta - s)^{\alpha}}{\Gamma(\alpha + 1)} \left| f\left(s, x(s), {}^cD^{\beta}x(s)\right) \right| ds$$

$$\leq \frac{M_r}{\Gamma(\alpha + 1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) + \frac{2M_r(t_2 - t_1)}{\eta\Gamma(\alpha + 2)}$$

and

$$\left| {}^cD^{\beta}(F_1x)(t_2) - {}^cD^{\beta}(F_1x)(t_1) \right| \leq \frac{M_r}{\Gamma(\alpha - \beta + 1)} \left(t_2^{\alpha - \beta - 1} - t_1^{\alpha - \beta - 1}\right)$$

$$+ \frac{2M_r}{\eta\Gamma(\alpha + 2)\Gamma(2 - \beta)} \left(t_2^{1-\beta} - t_1^{1-\beta}\right).$$

Obviously, $\|(F_1x)(t_2) - (F_1x)(t_1)\|_X \to 0$ as $t_2 \to t_1$. Hence, the operator F_1 is equicontinuous. Thus, it follows by the Arzelá-Ascoli theorem that $F_1(\bar{\Omega}_{r_0})$ is relatively compact.

Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\|_X \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ uniformly on $[0, 1]$. From the uniform continuity of $f(t, x, y)$ on the compact set $[0, 1] \times [-r_0, r_0] \times [-r_0, r_0]$ it follows that $\|f(t, x_n(t), {}^cD^{\beta}x_n(t)) - f(t, x(t), {}^cD^{\beta}x(t))\| \to 0$ uniformly on $[0, 1]$. Hence, $\|F_1x_n - F_1x\|_X \to 0$ as $n \to \infty$, which proves the continuity of F_1. This completes the proof of Step 1.

Step 2. The operator $F_2 : \bar{\Omega}_{r_0} \to X$ *is contractive.* This is a consequence of (A_2). Indeed, for $x, y \in X$, we have

$$\left| (F_2x)(t) - (F_2y)(t) \right| = \left| \frac{2t - \eta}{2 - \eta} \right| \left| h(x) - h(y) \right|$$

$$\leq (2 + \eta) \left| h(x) - h(y) \right|$$

$$\leq (2 + \eta)\ell \|x - y\|,$$

which, by taking the supremum for $t \in [0, 1]$, yields

$$\left\| (F_2x) - (F_2y) \right\| \leq (2 + \eta)\ell \|x - y\|.$$

Also, we have

$$\left\| {}^cD^{\beta}F_2x - {}^cD^{\beta}F_2y \right\| \leq \frac{2}{\Gamma(2 - \beta)} \ell \|x - y\|.$$

Consequently, we get

$$\|F_2x - F_2y\|_X \leq \left(2 + \eta + \frac{2}{\Gamma(2 - \beta)}\right) \ell \|x - y\|_X,$$

which, in view of (A_1), implies that F_2 is a contraction.

Step 3. The set $F(\bar{\Omega}_{r_0})$ *is bounded.* Assumptions (A_1) and (A_2) imply that

$$\left| (F_2x)(t) \right| \leq \frac{2(1 - t)}{\eta(2 - \eta)} |\xi| + \frac{|2t - \eta|}{2 - \eta} \left| h(x) \right|$$

$$\leq \frac{2}{\eta} |\xi| + (2 + \eta)\left(\left| h(x) - h(0) \right| + \left| h(0) \right|\right)$$

$$\leq \frac{2}{\eta}|\xi| + (2 + \eta)\ell\|x\|$$

$$\leq \frac{2}{\eta}|\xi| + (2 + \eta)\ell r_0$$

and

$$\left|{}^cD^\beta(F_2x)(t)\right| \leq \frac{1}{\Gamma(2 - \beta)}\left(\frac{2|\xi|}{\eta} + 2\ell r_0\right).$$

Hence,

$$\|F_2x\|_X \leq \frac{2|\xi|}{\eta}\left(1 + \frac{1}{\Gamma(2 - \beta)}\right) + \ell r_0\left(2 + \eta + \frac{2}{\Gamma(2 - \beta)}\right)$$

for any $x \in \bar\Omega_{r_0}$. This, together with the boundedness of the set $F_1(\bar\Omega_{r_0})$, implies that the set $F(\bar\Omega_{r_0})$ is bounded.

Step 4. *Finally, we will show that case* (C2) *in Lemma* 3.1 *does not hold.* On the contrary, suppose that (C2) holds. Then, we have that there exist $\lambda \in (0, 1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda Fx$. So, we have $\|x\|_X = r_0$ and

$$x(t) = \lambda \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}f\left(s, x(s), {}^cD^\beta x(s)\right)ds + \lambda\frac{2(1 - t)}{\eta(2 - \eta)}\xi + \lambda\frac{2t - \eta}{2 - \eta}h(x)$$

$$+ \lambda\frac{2(t - 1)}{\eta(2 - \eta)}\int_0^\eta \frac{(\eta - s)^\alpha}{\Gamma(\alpha + 1)}f\left(s, x(s), {}^cD^\beta x(s)\right)ds.$$

Further, we have that

$$|x(t)| \leq \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}p(s)\left(\psi_1\left(|x(s)|\right) + \psi_2\left(|{}^cD^\beta x(s)|\right)\right)ds + \frac{2}{\eta}|\xi| + (2 + \eta)\ell\|x\|$$

$$+ \frac{2}{\eta}\int_0^\eta \frac{(\eta - s)^\alpha}{\Gamma(\alpha + 1)}p(s)\left(\psi_1\left(|x(s)|\right) + \psi_2\left(|{}^cD^\beta x(s)|\right)\right)ds$$

$$\leq \int_0^1 \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)}p(s)\left(\psi_1\left(\|x\|_X\right) + \psi_2\left(\|x\|_X\right)\right)ds + \frac{2}{\eta}|\xi| + (2 + \eta)\ell\|x\|_X$$

$$+ \frac{2}{\eta}\int_0^1 \frac{(1 - s)^\alpha}{\Gamma(\alpha + 1)}p(s)\left(\psi_1\left(\|x\|_X\right) + \psi_2\left(\|x\|_X\right)\right)ds$$

$$\leq \|p\|\left[\psi_1(r_0) + \psi_2(r_0)\right]\left\{\frac{1}{\Gamma(\alpha + 1)} + \frac{2}{\eta\Gamma(\alpha + 2)}\right\} + \frac{2}{\eta}|\xi| + (2 + \eta)\ell r_0,$$

$$\left|{}^cD^\beta x(t)\right| \leq \int_0^t \frac{(t - s)^{\alpha - 1 - \beta}}{\Gamma(\alpha - \beta)}p(s)\left(\psi_1\left(|x|\right) + \psi_2\left(|{}^cD^\beta x(s)|\right)\right)ds$$

$$+ \frac{t^{1 - \beta}}{\Gamma(2 - \beta)}\frac{2}{2 - \eta}\left|h(x) - \frac{\xi}{\eta}\right|$$

$$+ \frac{2}{\eta(2 - \eta)}\frac{t^{1 - \beta}}{\Gamma(2 - \beta)}\int_0^\eta \frac{(\eta - s)^\alpha}{\Gamma(\alpha + 1)}p(s)\left(\psi_1\left(|x|\right) + \psi_2\left(|{}^cD^\beta x(s)|\right)\right)ds$$

$$\leq \int_0^t \frac{(t - s)^{\alpha - 1 - \beta}}{\Gamma(\alpha - \beta)}p(s)\left(\psi_1\left(\|x\|_X\right) + \psi_2\left(\|x\|_X\right)\right)ds + \frac{2}{\Gamma(2 - \beta)}\left(\ell\|x\|_X + \frac{|\xi|}{\eta}\right)$$

$$+ \frac{2}{\eta\Gamma(2 - \beta)}\int_0^1 \frac{(1 - s)^\alpha}{\Gamma(\alpha + 1)}p(s)\left(\psi_1\left(\|x\|_X\right) + \psi_2\left(\|x\|_X\right)\right)ds$$

$$\leq \|p\| \big[\psi_1(r_0) + \psi_2(r_0)\big] \left\{ \frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{2}{\eta \Gamma(2 - \beta)} \frac{1}{\Gamma(\alpha + 2)} \right\}$$
$$+ \frac{2}{\Gamma(2 - \beta)} \left(\ell r_0 + \frac{|\xi|}{\eta} \right).$$

Thus, we have

$$r_0 \leq \|p\| \big[\psi_1(r_0) + \psi_2(r_0)\big] \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{2}{\eta \Gamma(\alpha + 2)} + \frac{1}{\Gamma(\alpha - \beta + 1)} \right.$$
$$\left. + \frac{2}{\eta \Gamma(2 - \beta)} \frac{1}{\Gamma(\alpha + 2)} \right\} + \frac{2|\xi|}{\eta} \left(1 + \frac{1}{\Gamma(2 - \beta)} \right) + \ell r_0 \left(2 + \eta + \frac{2}{\Gamma(2 - \beta)} \right)$$
$$= \Lambda_1 \|p\| \big[\psi_1(r_0) + \psi_2(r_0)\big] + \Lambda_2 + \ell \Lambda_3 r_0,$$

which yields a contradiction,

$$\frac{r_0}{\Lambda_1 \|p\| [\psi_1(r_0) + \psi_2(r_0)] + \Lambda_2} \leq \frac{1}{1 - \ell \Lambda_3}.$$

Thus, the operators F_1 and F_2 satisfy all the conditions of Lemma 3.1. Hence, the operator F has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of problem (1.1). This completes the proof. □

Our second existence result is based on Sadovskii's fixed point theorem. Let us first recall some auxiliary material before proceeding further.

Definition 3.1 Let M be a bounded set in metric space (X, d). The Kuratowskii measure of noncompactness $\alpha(M)$ is defined as $\inf\{\epsilon : M$ covered by a finitely many sets such that the diameter of each set $\leq \epsilon\}$.

Definition 3.2 [30] Let $\Phi : \mathcal{D}(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset \mathcal{D}(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 3.2 ([31], Example 11.7) *The map $K + C$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing, if*

(i) *$K, C : \mathcal{D} \subseteq X \to X$ are operators on the Banach space X;*
(ii) *K is k-contractive, that is,*

$$\|Kx - Ky\| \leq k \|x - y\|$$

for all $x, y \in \mathcal{D}$ and fixed $k \in [0, 1)$;
(iii) *C is compact.*

Lemma 3.3 [32] *Let B be a convex, bounded, and closed subset of a Banach space X, and $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

Theorem 3.2 *Assume that (A_1), (A_2), and (A_3) hold. Then problem (1.1) has at least one solution on $[0, 1]$.*

Proof Let $B_r = \{x \in X : \|x\| \leq r\}$ be a closed bounded and convex subset of X, where r will be fixed later. We define the map $F : B_r \to X$ by

$$(Fx)(t) = (F_1x)(t) + (F_2x)(t), \quad t \in [0,1],$$

where F_1 and F_2 are defined by (3.2) and (3.3), respectively. Notice that problem (1.1) is equivalent to the fixed point problem $F(x) = x$.

Step 1. $(Fx)B_r \subset B_r$.

For that, select $r \geq \frac{\Lambda_1 \|p\| [\psi_1(r) + \psi_2(r)] + \Lambda_2}{1 - \ell\Lambda_3}$, where Λ_i, $i = 1, 2, 3$, are defined by (3.4) and (3.5). As in Theorem 3.1, Step 1, we can prove that

$$\|F_1x\|_X \leq \Lambda_1 \|p\| [\psi_1(r) + \psi_2(r)],$$

and, as in Step 3, we can get

$$\|F_2x\| \leq \Lambda_2 + \ell r\Lambda_3.$$

Consequently,

$$\|Fx\|_X = \|Fx\| + \left\|{}^cD^\beta(Fx)\right\| \leq \Lambda_1 \|p\| [\psi_1(r) + \psi_2(r)] + \Lambda_2 + \ell r\Lambda_3 < r,$$

which implies that $(Fx)(B_r) \subset B_r$.

Step 2. F_1 is compact.

This was proved in Theorem 3.1, Step 1.

Step 3. F_2 is continuous and γ-contractive.

To show the continuity of F_2 for $t \in [0,1]$, let us consider a sequence x_n converging to x. Then, as in Step 2 of the proof of Theorem 3.1, we can show that

$$\|F_2x_n - F_2x\|_X \leq \left(2 + \eta + \frac{2}{\Gamma(2 - \beta)}\right)\ell\|x_n - x\|,$$

which implies that F_2 is continuous. Also, F_2 is γ-contractive by Theorem 3.1 (Step 2) with $\gamma = (2 + \eta + \frac{2}{\Gamma(2-\beta)})\ell < 1$.

Step 4. F is condensing.

Since T_2 is a continuous γ-contraction and T_1 is compact, by Lemma 3.2, $F : B_r \to B_r$ with $F = F_1 + F_2$ is a condensing map on B_r.

From the previous four steps we conclude by Lemma 3.3 that the map F has a fixed point, which, in turn, implies that problem (1.1) has a solution. $\qquad\square$

Example 3.1 Consider the following boundary value problem:

$$^cD^{5/2}x(t) = \frac{e^{-2t}}{\sqrt{9 + t}}\left(a\sin x(t) + b\frac{{}^cD^{3/4}x(t)}{1 + {}^cD^{3/4}x(t)} + \frac{1}{3}\right), \quad 0 < t < 1, \tag{3.7}$$

$$x(0) + x'(0) = \frac{1}{10}x(4/5), \qquad x''(0) = 0, \qquad \int_0^{2/3} x(t)\,dt = \frac{1}{2}, \tag{3.8}$$

where $h(x) = \frac{1}{10}x(4/5)$, $\eta = 2/3$, $\xi = 1/2$, and a and b are suitably chosen real numbers.

Observe that

$$
\begin{aligned}
\left| f\left(t, x(t), {}^{c}D^{3/4}x(t)\right) \right| &= \left| \frac{e^{-2t}}{\sqrt{9+t}} \left(a\sin x(t) + b\frac{{}^{c}D^{3/4}x(t)}{1 + {}^{c}D^{3/4}x(t)} + \frac{1}{3} \right) \right| \\
&\le p(t)\left(\psi_1\left(|x|\right) + \psi_2\left(|y|\right) \right)
\end{aligned}
$$

with $p(t) = e^{-2t}$, $\psi_1(|x|) = a|x|$, $\psi_2(|{}^{c}D^{3/4}x(t)|) = b + 1/3$, and $\ell = 1/10$ since

$$
\left| h(u) - h(v) \right| \le \frac{1}{10}\|u - v\|.
$$

With the given values, we find that $\Lambda_1 \simeq 1.199538$, $\Lambda_2 \simeq 3.154894$, $\Lambda_3 \simeq 3.769929$. Further, $\ell < (2 + \eta + 2/\Gamma(2 - \beta))^{-1}$ holds since $(2 + \eta + 2/\Gamma(2 - \beta))^{-1} \simeq 0.205204 > \frac{1}{10} = \ell$, and (A_4) is satisfied for $0 < a < 1/2$ and for any finite real value of b since $\frac{1}{1 - \ell\Lambda_3} \simeq 1.605118$, and $\sup_{r \in (0,\infty)} \frac{r}{\Lambda_1\|p\|[\psi_1(r) + \psi_2(r)] + \Lambda_2} \simeq \frac{1}{1.199538a}$. Thus, all the conditions of Theorem 3.1 are satisfied, and consequently there exists one solution for problem (3.7)-(3.8) on $[0,1]$.

Remark 3.1 Notice that Example 3.1 also illustrates Theorem 3.2 since assumptions (A_1), (A_2), and (A_3) clearly hold.

4 Existence results for multivalued problem (1.2)

Let us recall some basic definitions on multivalued maps [33, 34].

For a normed space $(X, \|\cdot\|)$, let $P_{\mathrm{cl}}(X) = \{Y \in F(X) : Y \text{ is closed}\}$, $P_{\mathrm{b}}(X) = \{Y \in F(X) : Y \text{ is bounded}\}$, $P_{\mathrm{cp}}(X) = \{Y \in F(X) : Y \text{ is compact}\}$, and $P_{\mathrm{cp,c}}(X) = \{Y \in F(X) : Y \text{ is compact and convex}\}$. A multivalued map $G : X \to F(X)$ is convex (closed)-valued if $G(x)$ is convex (closed) for all $x \in X$. A map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_{\mathrm{b}}(X)$ (*i.e.*, $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). A map G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$; G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_{\mathrm{b}}(X)$. If a multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is, $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$. A map G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of a multivalued operator G will be denoted by Fix G. A multivalued map $G : [0;1] \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

Definition 4.1 A function $x \in AC^{n}([0,1], \mathbb{R})$ satisfying the conditions $x(0) + x'(0) = h(x)$, $\int_0^{\eta} x(t)\,dt = \xi$, $x''(0) = x'''(0) = \cdots = x^{(n-1)}(0) = 0$ is said to be a solution of problem (1.2) if there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, x(t), {}^{c}D^{\beta}x(t))$ a.e. on $[0,1]$ and

$$
\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \frac{2(1-t)}{\eta(2-\eta)}\xi + \frac{2t-\eta}{2-\eta}h(x) \\
&\quad + \frac{2(t-1)}{\eta(2-\eta)} \int_0^{\eta} \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} f(s)\,ds.
\end{aligned} \tag{4.1}
$$

Definition 4.2 A multivalued map $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be a Carathéodory function if

(i) $t \mapsto F(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$;

(ii) $(x, y) \mapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in [0, 1]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $a > 0$, there exists $\varphi_a \in L^1([0, 1], \mathbb{R}^+)$ such that

$$\|F(t, x, y)\| = \sup\{|v| : v \in F(t, x, y)\} \leq \varphi_a(t)$$

for all $\|x\|, \|y\| \leq a$ and for a.e. $t \in [0, 1]$.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t), {}^c D^\beta y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

The following lemma will be used in the sequel.

Lemma 4.1 [35] *Let X be a Banach space. Let $F : [0, 1] \times X \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map, and let Θ be a linear continuous mapping from $L^1([0, 1], X)$ to $C([0, 1], X)$. Then the operator*

$$\Theta \circ S_F : C([0, 1], X) \to \mathcal{P}_{cp,c}(C([0, 1], X)), \qquad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}),$$

is a closed graph operator in $C([0, 1], X) \times C([0, 1], X)$.

To prove our main result in this section, we use the following form of the nonlinear alternative for contractive maps ([36], Corollary 3.8).

Theorem 4.1 *Let X be a Banach space, and $\mathcal{D}$ a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{\mathcal{D}} \to \mathcal{P}_{cp,c}(X)$ be two multivalued operators satisfying*

(a) *Z_1 is contraction, and*

(b) *Z_2 is u.s.c. and compact.*

Then, if $G = Z_1 + Z_2$, then either

(i) *G has a fixed point in $\bar{\mathcal{D}}$, or*

(ii) *there are a point $u \in \partial \mathcal{D}$ and $\lambda \in (0, 1)$ with $u \in \lambda G(u)$.*

Theorem 4.2 *Assume that (A_1) and (A_2) hold. In addition, we suppose that*:

(H_1) *$F : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is an L^1-Carathéodory multivalued map*;

(H_2) *there exist continuous nondecreasing functions $\psi_i : [0, \infty) \to (0, \infty)$, $i = 1, 2$, and a function $p \in C([0, 1], \mathbb{R}^+)$ such that*

$$\|F(t, x, y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t, x, y)\} \leq p(t)[\psi_1(|x|) + \psi_2(|y|)]$$

for each $(t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$;

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1 - \ell \Lambda_3)M}{\Lambda_1 \|p\|[\psi_1(M) + \psi_2(M)] + \Lambda_2} > 1, \quad \ell \Lambda_3 < 1, \tag{4.2}$$

where Λ_i, $i = 1, 2, 3$, are defined in (3.4) and (3.5).

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof To transform problem (1.1) into a fixed point problem, we introduce the operator $\mathcal{N} : X \to \mathcal{P}(X)$ as follows:

$$
\mathcal{N}(x) = \left\{
\begin{array}{l}
h \in X : \\[2mm]
h(t) = \left\{
\begin{array}{l}
\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \frac{2(1-t)}{\eta(2-\eta)}\xi + \frac{2t-\eta}{2-\eta}h(x) \\[4mm]
+ \dfrac{2(t-1)}{\eta(2-\eta)} \displaystyle\int_0^\eta \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} f(s)\,ds, \quad f \in S_{F,x}
\end{array}
\right\}
\end{array}
\right\}.
$$

Now, we define the operator $\mathcal{A}_1 : X \to X$ by

$$
\mathcal{A}_1 x(t) = \frac{2(1-t)}{\eta(2-\eta)}\xi + \frac{2t-\eta}{2-\eta}h(x) \tag{4.3}
$$

and the multivalued operator $\mathcal{A}_2 : X \to \mathcal{P}(X)$ by

$$
\mathcal{A}_2(x) = \left\{
\begin{array}{l}
h \in C\big([0,1],\mathbb{R}\big) : \\[2mm]
h(t) = \left\{ \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \displaystyle\int_0^\eta \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} f(s)\,ds \right.
\end{array}
\right\}. \tag{4.4}
$$

Observe that $\mathcal{N} = \mathcal{A}_1 + \mathcal{A}_2$. We shall show that the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions of Theorem 4.1 on $[0,1]$. Also, we establish that the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are such that $\mathcal{A}_1, \mathcal{A}_2 : B_r \to \mathcal{P}_{\mathrm{cp,c}}(X)$, where $B_r = \{x \in X : \|x\|_X \le r\}$ is a bounded set in $C([0,1],\mathbb{R})$. First, we prove that $\mathcal{A}_2$ is compact-valued on B_r. Note that the operator $\mathcal{A}_2$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator from $L^1([0,1],\mathbb{R})$ into X defined by

$$
\mathcal{L}(v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} v(s)\,ds.
$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by the definition of $S_{F,x}$, we have $v_n(t) \in F(t,x(t),{}^cD^\beta x(t))$ for almost all $t \in [0,1]$. Since $F(t,x(t),{}^cD^\beta x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on $[0,1]$.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equicontinuous sequence. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
\big|\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)\big| &\le \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] v_n(s)\,ds \right. \\
&\quad + \left. \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} v_n(s)\,ds \right| \\
&\quad + \frac{2(t_2-t_1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^{\alpha}}{\Gamma(\alpha+1)} \big|v_n(s)\big|\,ds \\
&\le \big[\psi_1(r) + \psi_2(r)\big] \left\{ \frac{1}{\Gamma(\alpha+1)}\big(t_2^\alpha - t_1^\alpha\big) + \frac{2(t_2-t_1)}{\eta\Gamma(\alpha+2)} \right\}.
\end{aligned}
$$

Continuing this process, we have

$$\left| \left(\mathcal{L}'(v_n)(t_2)\right) - \left(\mathcal{L}'(v_n)(t_1)\right) \right| \leq \|p\| \left[\psi_1(r) + \psi_2(r) \right] \left\{ \frac{1}{\Gamma(\alpha - \beta + 1)} \left(t_2^{\alpha - \beta - 1} - t_1^{\alpha - \beta - 1} \right) \right.$$

$$\left. + \frac{2}{\eta \Gamma(\alpha + 2)\Gamma(2 - \beta)} \left(t_2^{1-\beta} - t_1^{1-\beta} \right) \right\}.$$

We observe that the right-hand sides of the above inequalities tend to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equicontinuous, and by using the Arzelá-Ascoli theorem we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{A}_2(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{A}_2(x)$ is compact.

Now, we show that $\mathcal{A}_2(x)$ is convex for all $x \in X$. Let $z_1, z_2 \in \mathcal{A}_2(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$z_i(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s)\, ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f_i(s)\, ds, \quad i = 1, 2$$

for almost all $t \in [0,1]$. Let $0 \leq \lambda \leq 1$. Then, we have

$$\left[\lambda z_1 + (1 - \lambda) z_2 \right](t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\lambda f_1(s) + (1-\lambda) f_2(s) \right] ds$$

$$+ \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} \left[\lambda f_1(s) + (1-\lambda) f_2(s) \right] ds.$$

Since F has convex values, $S_{F,u}$ is convex, and $\lambda f_1(s) + (1-\lambda) f_2(s) \in S_{F,x}$. Thus,

$$\lambda z_1 + (1 - \lambda) z_2 \in \mathcal{A}_2(x).$$

Consequently, $\mathcal{A}_2$ is convex-valued. Obviously, $\mathcal{A}_1$ is compact and convex-valued.

The rest of the proof consists of several steps and claims.

Step 1. *We show that $\mathcal{A}_1$ is a contraction on $C([0,1], \mathbb{R})$.* The proof is similar to that for the operator F_2 in Step 2 of Theorem 3.1.

Step 2. *$\mathcal{A}_2$ is upper semicontinuous and compact.* This will be established in several claims.

CLAIM I: *$\mathcal{A}_2$ maps bounded sets into bounded sets in X.* Let $B_r = \{x \in X : \|x\|_X \leq r\}$ be a bounded set in X. Then, for each $h \in \mathcal{A}_2(x)$, $x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(q)} f(s)\, ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f(s)\, ds.$$

Then, for $t \in [0,1]$, we have

$$\|h\| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s) \right| ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} \left| f(s) \right| ds$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f(s) \right| ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} \left| f(s) \right| ds$$

$$\leq \|p\| \left[\psi_1(\|x\|) + \psi_2(\|{}^c D^\beta x\|) \right] \left[\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta \Gamma(\alpha+2)} \right]$$

$$\leq \|p\| \big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big] \bigg[\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)}\bigg]$$

$$\leq \|p\| \big[\psi_1(r) + \psi_2(r)\big] \bigg[\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)}\bigg].$$

Considering

$$h'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)\,ds + \frac{2}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f(s),$$

we obtain

$$^cD^\beta h(t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(\tau)\,d\tau + \frac{2}{\eta} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds\bigg)$$

$$= \int_0^t \frac{(t-s)^{\alpha-1-\beta}}{\Gamma(\alpha-\beta)} f(s)\,ds + \frac{2t^{1-\beta}}{\eta\Gamma(2-\beta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds,$$

and therefore

$$\big\|^cD^\beta h\big\| \leq \|p\| \big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big] \int_0^t \frac{(t-s)^{\alpha-1-\beta}}{\Gamma(\alpha-\beta)}\,ds$$

$$+ \frac{2}{\eta\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)}\,ds$$

$$\leq \|p\| \big[\psi_1(r) + \psi_2(r)\big] \bigg\{\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{2}{\eta\Gamma(2-\beta)} \frac{1}{\Gamma(\alpha+2)}\bigg\}.$$

Consequently,

$$\|\mathcal{A}_2 x\|_X = \|\mathcal{A}_2 x\| + \big\|^cD^\beta(\mathcal{A}_2 x)\big\|$$

$$\leq \|p\| \big[\psi_1(r) + \psi_2(r)\big] \bigg\{\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-\beta+1)}$$

$$+ \frac{2}{\eta\Gamma(2-\beta)} \frac{1}{\Gamma(\alpha+2)}\bigg\},$$

and thus the operator $\mathcal{A}_2(B_r)$ is uniformly bounded.

Claim II: $\mathcal{A}_2$ *maps bounded sets into equicontinuous sets.* Now let $0 \leq t_1 < t_2 \leq 1$. Then we have the following facts:

$$\big|(\mathcal{A}_2 x)(t_2) - (\mathcal{A}_2 x)(t_1)\big| \leq \bigg|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] f(s)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)\,ds\bigg|$$

$$+ \frac{2}{\eta}(t_2-t_1) \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} \big|f(s)\big|\,ds$$

$$\leq \|p\| \big[\psi_1(r) + \psi_2(r)\big] \bigg\{\frac{1}{\Gamma(\alpha+1)}\big(t_2^\alpha - t_1^\alpha\big) + \frac{2(t_2-t_1)}{\eta\Gamma(\alpha+2)}\bigg\}$$

and

$$
\left| {}^{c}D^{\beta}(\mathcal{A}_2 x)(t_2) - {}^{c}D^{\beta}(\mathcal{A}_2)(t_1) \right| \leq \|p\| \left[\psi_1(r) + \psi_2(r) \right] \left\{ \frac{1}{\Gamma(\alpha - \beta + 1)} \left(t_2^{\alpha - \beta - 1} - t_1^{\alpha - \beta - 1} \right) \right.
$$
$$
\left. + \frac{2}{\eta \Gamma(\alpha + 2)\Gamma(2 - \beta)} \left(t_2^{1-\beta} - t_1^{1-\beta} \right) \right\}.
$$

Hence, we have

$$
\left\| (\mathcal{A}_2 x)(t_2) - (\mathcal{A}_2 x)(t_1) \right\|_X \to 0 \quad \text{as } t_2 \to t_1.
$$

Thus, $\mathcal{A}_2$ is equicontinuous. Therefore, by the Ascoli-Arzelá theorem it follows that $\mathcal{A}_2 : X \to \mathcal{P}(X)$ is completely continuous.

CLAIM III: $\mathcal{A}_2$ *has a closed graph.* Let $x_n \to x_*$, $h_n \in \mathcal{A}_2(x_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{A}_2(x_*)$. Associated with $h_n \in \mathcal{A}_2(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$
h_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(q)} f_n(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f_n(s)\,ds.
$$

Thus, it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$
h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(q)} f_*(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f_*(s)\,ds.
$$

Let us consider the linear operator $\Theta : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ given by

$$
f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(q)} f(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f(s)\,ds.
$$

Observe that

$$
\left\| h_n(t) - h_*(t) \right\|
$$
$$
= \left\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left(f_n(s) - f_*(s) \right) ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} \left(f_n(s) - f_*(s) \right) ds \right\| \to 0
$$

as $n \to \infty$.

Thus, it follows by Lemma 4.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$
h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(q)} f_*(s)\,ds + \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)} f_*(s)\,ds
$$

for some $f_* \in S_{F,x_*}$. Hence, $\mathcal{A}_2$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{A}_2$ is upper semicontinuous.

Thus, the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions of Theorem 4.1, and hence its conclusion implies that either condition (i) or condition (ii) holds. We show that conclusion (ii) is not possible. If $x \in \lambda \mathcal{A}_1(x) + \lambda \mathcal{A}_2(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such

that

$$x(t) = \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds + \lambda \frac{2(1-t)}{\eta(2-\eta)}\xi + \lambda \frac{2t-\eta}{2-\eta}h(x)$$
$$+ \lambda \frac{2(t-1)}{\eta(2-\eta)} \int_0^\eta \frac{(\eta-s)^\alpha}{\Gamma(\alpha+1)}f(s)\,ds, \quad t \in [0,1].$$

Following the method of the proof of Claim I, we can obtain

$$\|x\| \le \|p\|\big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big]\left[\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)}\right]$$
$$+ \frac{2}{\eta}|\xi| + (2+\eta)\ell\|x\|_X$$

and

$$\big\|{}^cD^\beta x\big\| \le \big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big]\left\{\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{2}{\eta\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha+2)}\right\}$$
$$+ \frac{2}{\Gamma(2-\beta)}\left(\frac{|\xi|}{\eta} + \ell\|x\|_X\right).$$

Consequently,

$$\|x\|_X = \|x\| + \big\|{}^cD^\beta x\big\|$$
$$\le \|p\|\big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big]\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{2}{\eta\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha-\beta+1)}\right.$$
$$+ \left.\frac{2}{\eta\Gamma(2-\beta)}\frac{1}{\Gamma(\alpha+2)}\right\} + \frac{2}{\eta}|\xi|\left(1 + \frac{1}{\Gamma(2-\beta)}\right)$$
$$+ \left(2 + \eta + \frac{2}{\Gamma(2-\beta)}\right)\ell\|x\|_X.$$

Thus,

$$\|x\|_X \le \Lambda_1\|p\|\big[\psi_1\big(\|x\|_X\big) + \psi_2\big(\|x\|_X\big)\big] + \Lambda_2 + \ell\Lambda_3\|x\|_X. \tag{4.5}$$

If condition (ii) of Theorem 4.1 holds, then there exist $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda\mathcal{N}(x)$. Then, x is a solution of (1.1) with $\|x\|_X = M$. Now, by inequality (4.5) we get

$$\frac{(1-\ell\Lambda_3)M}{\Lambda_1\|p\|[\psi_1(M) + \psi_2(M)] + \Lambda_2} \le 1,$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in $[0,1]$ by Theorem 4.1, and consequently problem (1.1) has a solution. This completes the proof. □

Example 4.1 Consider the fractional differential inclusion

$${}^cD^{5/2}x(t) \in F\big(t, x(t), {}^cD^{3/4}x(t)\big) \tag{4.6}$$

supplemented with the boundary conditions (3.8), where

$$
F\big(t, x(t), {}^{c}D^{3/4}x(t)\big) = \left[\frac{1}{\sqrt{225+t}} \left(x(t) + \tan^{-1}\big({}^{c}D^{3/4}x(t)\big) + \frac{\pi}{2} \right), \right.
$$
$$
\left. \frac{1}{4}\sin x(t) + \frac{{}^{c}D^{3/4}x(t)}{9(1 + {}^{c}D^{3/4}x(t))} + \frac{1}{10} \right].
$$

Clearly,

$$
\big| F\big(t, x, {}^{c}D^{3/4}x(t)\big) \big| \leq p(t)\big(\psi_1(|x|) + \psi_2\big(\big|{}^{c}D^{3/4}x(t)\big|\big)\big),
$$

where $p(t) = 1/\sqrt{225+t}$, $\psi_1(|x|) = |x|$, $\psi_2(|x|) = \pi$. Using the data of Example 3.1 and condition (H$_3$), we find that $M > M_1 \simeq 6.272351$. Hence, the hypothesis of Theorem 4.2 is satisfied, which implies that the fractional differential inclusion (4.6) together with (3.8) has a solution on $[0,1]$.

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