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On joint distributions of order statistics for a discrete case

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Abstract

In this study, the joint distributions of order statistics of *innid* discrete random variables are expressed. Also, the joint distributions are obtained in the form of an integral. Then, the results related to *pf* and *df* are given.

MSC: 62G30; 62E15

Keywords: order statistics; discrete random variable; probability function; distribution function

1 Introduction

The joint probability density function (*pdf*) and marginal *pdf* of order statistics of independent but not necessarily identically distributed (*innid*) random variables was derived by Vaughan and Venables [1] by means of permanents. In addition, Balakrishnan [2] and Bapat and Beg [3] obtained the joint *pdf* and distribution function (*df*) of order statistics of *innid* random variables by means of permanents. Balasubramanian *et al.* [4] obtained the distribution of single order statistics in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, \ldots, X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian *et al.* [5] generalized their previous results [4] to the case of the joint distribution function of several order statistics. Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West [6]. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \leq r \leq n+1)$ was obtained by Childs and Balakrishnan [7] by adding another independent random variable to the original n variables $X_1, X_2, \ldots, X_n$. Also, Balasubramanian *et al.* [8] established the identities satisfied by the distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In 1991, Beg [9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer *et al.* [10] derived the expressions for the distribution and density functions by Ryser's method and the distributions of maxima and minima based on permanents.

The notion of distribution theory has been applied in various branches of science for investigations. Recently, the notion of a uniform distribution has been applied in sequence spaces and the notion of statistical convergent sequences has been studied and investigated from different aspects by Rath and Tripathy [11], Tripathy [12, 13], Tripathy and Baruah [14], Tripathy and Dutta [15], Tripathy and Sarma [16], Tripathy and Sen [17], and others.

A multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution was defined by Corley [18]. Guilbaud [19] expressed the probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of independent and identically distributed (*iid*) random vectors and thus also for order statistics of random variables. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller [20].

Several identities and recurrence relations for *pdf* and *df* of order statistics of *iid* random variables were established by numerous authors including Arnold *et al.* [21], Balasubramanian and Beg [22], David [23], and Reiss [24]. Furthermore, Arnold *et al.* [21], David [23], Gan and Bain [25], and Khatri [26] obtained the probability function (*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan [27] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. In 1986, Nagaraja [28] explored the behavior of higher order conditional probabilities of order statistics in an attempt to understand the structure of discrete order statistics. Later, Nagaraja [29] considered some results on order statistics of a random sample taken from a discrete population.

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, the joint distributions of order statistics of *innid* discrete random variables are expressed in the form of an integral. As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables.

From now on, the subscripts and superscripts are defined at first usage, and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1, \mathrm{a}_2, \ldots$ are column vectors, then $[\underset{m_1}{\mathrm{a}_1}\ \underset{m_2}{\mathrm{a}_2}\ \cdots]$ will denote the matrix obtained by taking m_1 copies of a_1, m_2 copies of a_2 and so on. $\operatorname{per} \mathrm{A}$ denotes the permanent of a square matrix A; the permanent is defined just like the determinant, except that all signs in the expansion are positive.

Let $X_1, X_2, \ldots, X_n$ be *innid* discrete random variables and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics obtained by arranging the n X_i's in the increasing order of magnitude. Let F_i and f_i be *df* and *pf* of X_i $(i = 1, 2, \ldots, n)$, respectively.

The *df* and *pf* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_p:n}$, $1 \leq r_1 < r_2 < \cdots < r_n \leq n$ $(p = 1, 2, \ldots, n)$ will be given. For notational convenience, we write $\sum_{z_1, z_2, \ldots, z_p}$, $\sum_{m_p, k_p, \ldots, m_1, k_1}$, $\int$ and $\int_{\mathrm{V}}$ instead of $\sum_{z_1=0}^{x_1} \sum_{z_2=z_1}^{x_2} \sum_{z_3=z_2}^{x_3} \cdots \sum_{z_p=z_{p-1}}^{x_p}$, $\sum_{m_p=0}^{n-r_p} \sum_{k_p=0}^{r_p-1-r_{p-1}} \cdots \sum_{m_2=0}^{r_3-1-r_2} \sum_{k_2=0}^{r_2-1-r_1} \sum_{m_1=0}^{r_2-1-r_1} \sum_{k_1=0}^{r_1-1}$, $\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} \int_{F_{\varsigma_4^{(1)}}(x_2-)}^{F_{\varsigma_4^{(1)}}(x_2)} \cdots \int_{F_{\varsigma_{2p}^{(1)}}(x_p-)}^{F_{\varsigma_{2p}^{(1)}}(x_p)}$ and $\int_0^{F_{\varsigma_2^{(1)}}(x_1)} \int_{v_{\varsigma_2^{(1)}}}^{F_{\varsigma_4^{(1)}}(x_2)} \cdots \int_{v_{\varsigma_{2(p-1)}^{(1)}}}^{F_{\varsigma_{2p}^{(1)}}(x_p)}$ in the expressions below, respectively $(x_i = 0, 1, 2, \ldots)$ $(z_0 = 0)$.

2 Theorems for distribution and probability functions

In this section, the theorems related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_p:n}$ will be given. We will now express the following theorem for the joint *pf* of order statistics of *innid* discrete random variables.

Theorem 1

$$\begin{aligned} &f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) \\ &\quad= \sum_{m_p,k_p,\ldots,m_1,k_1} \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{2p}}} \left(\prod_{w=1}^{p+1} \prod_{i_{2w-1}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}} \left[F_{s_{2w-1}^{(i_{2w-1})}}(x_w-) - F_{s_{2w-1}^{(i_{2w-1})}}(x_{w-1}) \right] \right) \\ &\qquad \times \prod_{w=1}^{p} \prod_{i_{2w}=1}^{k_w+1+m_w} f_{s_{2w}^{(i_{2w})}}(x_w), \end{aligned} \tag{1}$$

where $x_1 < x_2 < \cdots < x_p$, $\sum_{n_{s_1},n_{s_2},\ldots,n_{s_{2p}}}$ *denotes the sum over* $\bigcup_{\ell=1}^{2p} s_\ell$ *for which* $s_\upsilon \cap s_\vartheta = \phi$ *for* $\upsilon \neq \vartheta$, $S = \bigcup_{\ell=1}^{2p+1} s_\ell$, $S = \{1,2,\ldots,n\}$, $r_0 = 0$, $r_{p+1} = n+1$, $m_0 = 0$, $k_{p+1} = 0$, $m_{w-1} + k_w \leq r_w - r_{w-1} - 1$, $F_{s_1^{(i_1)}}(x_0) = 0$, $F_{s_{2p+1}^{(i_{2p+1})}}(x_{p+1}-) = 1$, $F_i(x_w-) = P(X_i < x_w)$ $(w = 1,2,\ldots,p+1)$, n_{s_ℓ} *is the cardinality of* s_ℓ *and*

$$s_\ell = \begin{cases} \{s_\ell^{(1)}, s_\ell^{(2)}, \ldots, s_\ell^{(k_{\frac{\ell}{2}}+1+m_{\frac{\ell}{2}})}\}, & \text{if } \ell \text{ even}, \\ \{s_\ell^{(1)}, s_\ell^{(2)}, \ldots, s_\ell^{(r_{\frac{\ell+1}{2}}-1-k_{\frac{\ell+1}{2}}-m_{\frac{\ell-1}{2}}-r_{\frac{\ell-1}{2}})}\}, & \text{if } \ell \text{ odd}. \end{cases}$$

Proof Consider the event $\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, \ldots, X_{r_p:n} = x_p\}$.

The above event can be realized in mutually exclusive ways as follows: $r_1 - 1 - k_1$ observations are less than x_1, $k_w + 1 + m_w$ $(w = 1,2,\ldots,p)$ observations are equal to x_w, $r_\xi - 1 - k_\xi - m_{\xi-1} - r_{\xi-1}$ $(\xi = 2,3,\ldots,p)$ observations are in interval $(x_{\xi-1}, x_\xi)$ and $n - m_p - r_p$ observations exceed x_p. The probability function of the above event can be written as

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = P\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, \ldots, X_{r_p:n} = x_p\}. \tag{2}$$

Identity (2) can be expressed as

$$\begin{aligned} &f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) \\ &\quad= \sum_{m_p,k_p,\ldots,m_1,k_1} \mathrm{C}\,\mathrm{per}\Big[\underbrace{\mathrm{F}(x_1-)}_{r_1-1-k_1}\ \underbrace{\mathrm{f}(x_1)}_{k_1+1+m_1}\ \underbrace{\mathrm{F}(x_2-)-\mathrm{F}(x_1)}_{r_2-1-k_2-m_1-r_1}\ \underbrace{\mathrm{f}(x_2)}_{k_2+1+m_2}\ \cdots\ \underbrace{\mathrm{f}(x_p)}_{k_p+1+m_p}\ \underbrace{1-\mathrm{F}(x_p)}_{n-m_p-r_p}\Big] \\ &\quad= \sum_{m_p,k_p,\ldots,m_1,k_1} \mathrm{C} \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{2p}}} \mathrm{per}\Big[\underbrace{\mathrm{F}(x_1-)}_{r_1-1-k_1}\Big][s_1/\cdot) \\ &\qquad \times \mathrm{per}\Big[\underbrace{\mathrm{f}(x_1)}_{k_1+1+m_1}\Big][s_2/\cdot)\,\mathrm{per}\Big[\underbrace{\mathrm{F}(x_2-)-\mathrm{F}(x_1)}_{r_2-1-k_2-m_1-r_1}\Big][s_3/\cdot) \\ &\qquad \times \mathrm{per}\Big[\underbrace{\mathrm{f}(x_2)}_{k_2+1+m_2}\Big][s_4/\cdot)\cdots\mathrm{per}\Big[\underbrace{\mathrm{f}(x_p)}_{k_p+1+m_p}\Big][s_{2p}/\cdot)\,\mathrm{per}\Big[\underbrace{1-\mathrm{F}(x_p)}_{n-m_p-r_p}\Big][s_{2p+1}/\cdot), \end{aligned}$$

where $\mathrm{C} = (\prod_{w=1}^{p+1}[(r_w - 1 - k_w - m_{w-1} - r_{w-1})!]^{-1}) \prod_{w=1}^{p}[(k_w + 1 + m_w)!]^{-1}$, $\mathrm{F}(x_w) = (F_1(x_w), F_2(x_w), \ldots, F_n(x_w))'$ and $\mathrm{f}(x_w) = (f_1(x_w), f_2(x_w), \ldots, f_n(x_w))'$ are column vectors. $\mathrm{A}[s_\ell/\cdot)$ is the matrix obtained from A by taking rows whose indices are in s_ℓ. Using the expansion of the permanent in the above identity, we get (1). □

Identity (1) can also be written in the form of an integral as follows.

Theorem 2

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int\left(\prod_{w=1}^{p+1}\prod_{i_w=1}^{r_w-r_{w-1}-1}\left[\nu^{(w)}_{\varsigma^{(i_w)}_{2w-1}}-\nu^{(w-1)}_{\varsigma^{(i_w)}_{2w-1}}\right]\right)\prod_{w=1}^{p}d\nu^{(w)}_{\varsigma^{(1)}_{2\mathrm{w}}},\tag{3}$$

where $x_1<x_2<\cdots<x_p$, $\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}$ *denotes the sum over* $\bigcup_{\ell=1}^{2p}\varsigma_\ell$ *for which* $\varsigma_\upsilon\cap\varsigma_\vartheta=\phi$ *for* $\upsilon\neq\vartheta$, $S=\bigcup_{\ell=1}^{2p+1}\varsigma_\ell$, $\nu^{(t)}_{\varsigma^{(i_w)}_{2w-1}}=[\nu^{(t)}_{\varsigma^{(1)}_{2t}}-F_{\varsigma^{(1)}_{2t}}(x_t-)]\frac{f_{\varsigma^{(i_w)}_{2w-1}}(x_t)}{f_{\varsigma^{(1)}_{2t}}(x_t)}+F_{\varsigma^{(i_w)}_{2w-1}}(x_t-)$, $\nu^{(0)}_{\varsigma^{(i_1)}_{1}}=0$, $\nu^{(p+1)}_{\varsigma^{(i_{p+1})}_{2p+1}}=1$ *and*

$$\varsigma_\ell=\begin{cases}\{\varsigma^{(1)}_\ell\}, & \textit{if } \ell \textit{ even},\\ \{\varsigma^{(1)}_\ell,\varsigma^{(2)}_\ell,\ldots,\varsigma^{(r_{\frac{\ell+1}{2}}-r_{\frac{\ell-1}{2}}-1)}_\ell\}, & \textit{if } \ell \textit{ odd}.\end{cases}$$

Proof Consider the identity

$$\begin{aligned}&\sum_{n_{s_{2w-1}}=\vartheta-k_w-1-m_w}\left(\prod_{j=1}^{\vartheta-k_w-1-m_w}G_{s^{(j)}_{2w-1}}\right)\prod_{i=1}^{k_w+1+m_w}f_{s^{(i)}_{2w}}\\&=\frac{k_w!m_w!}{(k_w+1+m_w)!}\sum_{\substack{n_{\tau_{4w-3}}=\vartheta-k_w-1-m_w\\ n_{\tau_{4w-2}}=k_w\\ n_{\tau_{4w-1}}=1}}\left(\prod_{i_1=1}^{\vartheta-k_w-1-m_w}G_{\tau^{(i_1)}_{4w-3}}\right)\left(\prod_{i_2=1}^{k_w}f_{\tau^{(i_2)}_{4w-2}}\right)f_{\tau^{(1)}_{4w-1}}\prod_{i_3=1}^{m_w}f_{\tau^{(i_3)}_{4w}}\end{aligned}\tag{4}$$

and using (4) in (1), it can be written as

$$\begin{aligned}&f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)\\&=\sum_{m_p,k_p,\ldots,m_1,k_1}\left(\prod_{w=1}^{p}\frac{k_w!m_w!}{(k_w+1+m_w)!}\right)\sum_{n_{\tau_1},n_{\tau_2},\ldots,n_{\tau_{4p}}}\left[\prod_{w=1}^{p+1}\left(\prod_{i_{3(w-1)}=1}^{m_{w-1}}f_{\tau^{(i_{3(w-1)})}_{4(w-1)}}(x_{w-1})\right)\right.\\&\quad\times\left(\prod_{i_{3w-2}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}}\left[F_{\tau^{(i_{3w-2})}_{4w-3}}(x_w-)-F_{\tau^{(i_{3w-2})}_{4w-3}}(x_{w-1})\right]\right)\\&\quad\times\left.\prod_{i_{3w-1}=1}^{k_w}f_{\tau^{(i_{3w-1})}_{4w-2}}(x_w)\right]\prod_{w=1}^{p}f_{\tau^{(1)}_{4w-1}}(x_w),\end{aligned}$$

where $\sum_{n_{\tau_1},n_{\tau_2},\ldots,n_{\tau_{4p}}}$ denotes the sum over $\bigcup_{l=1}^{4p}\tau_l$ for which $\tau_\upsilon\cap\tau_\vartheta=\phi$ for $\upsilon\neq\vartheta$, $S=\bigcup_{l=1}^{4p+1}\tau_l$,

$$s_{2w}=\tau_{4w-2}\cup\tau_{4w-1}\cup\tau_{4w},\qquad s_{2w-1}=\tau_{4w-3}\quad\text{and}$$

$$\tau_l=\begin{cases}\{\tau^{(1)}_l,\tau^{(2)}_l,\ldots,\tau^{(m_{\frac{l}{4}})}_l\}, & \text{if } l\equiv 0 \pmod 4,\\ \{\tau^{(1)}_l,\tau^{(2)}_l,\ldots,\tau^{(r_{\frac{l+3}{4}}-1-m_{\frac{l-1}{4}}-k_{\frac{l+3}{4}}-r_{\frac{l-1}{4}})}_l\}, & \text{if } l\equiv 1 \pmod 4,\\ \{\tau^{(1)}_l,\tau^{(2)}_l,\ldots,\tau^{(k_{\frac{l+2}{4}})}_l\}, & \text{if } l\equiv 2 \pmod 4,\\ \{\tau^{(1)}_l\}, & \text{if } l\equiv 3 \pmod 4.\end{cases}$$

The above identity can be written as

$$\begin{aligned}
&f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)\\
&\quad=\sum_{m_p,k_p,\ldots,m_1,k_1}\left(\prod_{w=1}^{p}\frac{k_w!m_w!}{(k_w+1+m_w)!}\right)\\
&\qquad\times\sum_{n_{\tau_1},n_{\tau_2},\ldots,n_{\tau_{4p}}}\left(\prod_{w=1}^{p}\frac{(k_w+1+m_w)!}{k_w!m_w!}\right)\\
&\qquad\times\left[\int_0^1\int_0^1\cdots\int_0^1 y_1^{k_1}(1-y_1)^{m_1}y_2^{k_2}(1-y_2)^{m_2}\cdots y_p^{k_p}(1-y_p)^{m_p}\,dy_1\,dy_2\cdots dy_p\right]\\
&\qquad\times\left[\prod_{w=1}^{p+1}\left(\prod_{i_{3(w-1)}=1}^{m_{w-1}}f_{\tau^{(i_{3(w-1)})}_{4(w-1)}}(x_{w-1})\right)\right.\\
&\qquad\times\left(\prod_{i_{3w-2}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}}\left[F_{\tau^{(i_{3w-2})}_{4w-3}}(x_w-)-F_{\tau^{(i_{3w-2})}_{4w-3}}(x_{w-1})\right]\right)\\
&\qquad\times\left.\prod_{i_{3w-1}=1}^{k_w}f_{\tau^{(i_{3w-1})}_{4w-2}}(x_w)\right]\prod_{w=1}^{p}f_{\tau^{(1)}_{4w-1}}(x_w).
\end{aligned}$$

The following expression can be written from the above identity

$$\begin{aligned}
&f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)\\
&\quad=\sum_{m_p,k_p,\ldots,m_1,k_1}\sum_{n_{\tau_1},n_{\tau_2},\ldots,n_{\tau_{4p}}}\int_0^1\int_0^1\cdots\int_0^1\left[\prod_{w=1}^{p+1}\left(\prod_{i_{3(w-1)}=1}^{m_{w-1}}(1-y_{w-1})f_{\tau^{(i_{3(w-1)})}_{4(w-1)}}(x_{w-1})\right)\right.\\
&\qquad\times\left(\prod_{i_{3w-2}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}}\left[F_{\tau^{(i_{3w-2})}_{4w-3}}(x_w-)-F_{\tau^{(i_{3w-2})}_{4w-3}}(x_{w-1})\right]\right)\\
&\qquad\times\left.\prod_{i_{3w-1}=1}^{k_w}y_wf_{\tau^{(i_{3w-1})}_{4w-2}}(x_w)\right]\prod_{w=1}^{p}f_{\tau^{(1)}_{4w-1}}(x_w)\,dy_w
\end{aligned}$$

and here, if $v^{(w)}_{\tau^{(i_j)}_l}=y_wf_{\tau^{(i_j)}_l}(x_w)+F_{\tau^{(i_j)}_l}(x_w-)$, the following identity is obtained

$$\begin{aligned}
&f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)\\
&\quad=\sum_{m_p,k_p,\ldots,m_1,k_1}\sum_{n_{\tau_1},n_{\tau_2},\ldots,n_{\tau_{4p}}}\int_{F_{\tau^{(1)}_3}(x_1-)}^{F_{\tau^{(1)}_3}(x_1)}\int_{F_{\tau^{(1)}_7}(x_2-)}^{F_{\tau^{(1)}_7}(x_2)}\cdots\int_{F_{\tau^{(1)}_{4p-1}}(x_p-)}^{F_{\tau^{(1)}_{4p-1}}(x_p)}\left[\prod_{w=1}^{p+1}\left(\prod_{i_{3(w-1)}=1}^{m_{w-1}}\left[F_{\tau^{(i_{3(w-1)})}_{4(w-1)}}(x_{w-1})\right.\right.\right.\\
&\qquad\left.\left.-v^{(w-1)}_{\tau^{(i_{3(w-1)})}_{4(w-1)}}\right]\right)\left(\prod_{i_{3w-2}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}}\left[F_{\tau^{(i_{3w-2})}_{4w-3}}(x_w-)-F_{\tau^{(i_{3w-2})}_{4w-3}}(x_{w-1})\right]\right)\\
&\qquad\times\left.\prod_{i_{3w-1}=1}^{k_w}\left[v^{(w)}_{\tau^{(i_{3w-1})}_{4w-2}}-F_{\tau^{(i_{3w-1})}_{4w-2}}(x_w-)\right]\right]\prod_{w=1}^{p}dv^{(w)}_{\tau^{(1)}_{4w-1}}.
\end{aligned}\tag{5}$$

By considering

$$\sum_{\xi=0}^{l}\sum_{\zeta=0}^{l}\sum_{\substack{n_{\tau_{4(w-1)}}=\zeta\\ n_{\tau_{4w-3}}=l-\xi-\zeta}}\left(\prod_{i_{3(w-1)}=1}^{\zeta} g_{\tau_{4(w-1)}^{(i_{3(w-1)})}}\right)\left(\prod_{i_{3w-2}=1}^{l-\xi-\zeta} f_{\tau_{4w-3}^{(i_{3w-2})}}\right)\prod_{i_{3w-1}=1}^{\xi} h_{\tau_{4w-2}^{(i_{3w-1})}}$$

$$=\prod_{i_w=1}^{l}\left[g_{\varsigma_{2w-1}^{(i_w)}}+f_{\varsigma_{2w-1}^{(i_w)}}+h_{\varsigma_{2w-1}^{(i_w)}}\right], \tag{6}$$

where $\xi+\zeta\leq l$, and using (6) for each m_{w-1} and k_w in (5), we get

$$\begin{aligned}
&f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)\\
&\quad=\sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}}\int\left(\prod_{w=1}^{p+1}\prod_{i_w=1}^{r_w-r_{w-1}-1}\left[F_{\varsigma_{2w-1}^{(i_w)}}(x_{w-1})-v_{\varsigma_{2w-1}^{(i_w)}}^{(w-1)}+F_{\varsigma_{2w-1}^{(i_w)}}(x_w-)\right.\right.\\
&\qquad\left.\left.-F_{\varsigma_{2w-1}^{(i_w)}}(x_{w-1})+v_{\varsigma_{2w-1}^{(i_w)}}^{(w)}-F_{\varsigma_{2w-1}^{(i_w)}}(x_w-)\right]\right)\prod_{w=1}^{p} dv_{\varsigma_{2w}^{(1)}}^{(w)},
\end{aligned}$$

where $\varsigma_{2w-1}=\tau_{4(w-1)}\cup\tau_{4w-3}\cup\tau_{4w-2}$ and $\varsigma_{2w}=\tau_{4w-1}$. This completes the proof of the theorem. □

We have the following special cases obtained from (3). Consider by taking $p=2$, $n=3$, $r_1=1$, $r_2=2$ and $v_{\varsigma_5^{(1)}}^{(2)}=[v_{\varsigma_4^{(1)}}^{(2)}-F_{\varsigma_4^{(1)}}(x_2-)]\frac{f_{\varsigma_5^{(1)}}(x_2)}{f_{\varsigma_4^{(1)}}(x_2)}+F_{\varsigma_5^{(1)}}(x_2-)$, the following identity is obtained

$$\begin{aligned}
f_{1,2:3}(x_1,x_2)&=\sum_{n_{\varsigma_2},n_{\varsigma_4}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)}\left(\int_{F_{\varsigma_4^{(1)}}(x_2-)}^{F_{\varsigma_4^{(1)}}(x_2)}\left(1-v_{\varsigma_5^{(1)}}^{(2)}\right)dv_{\varsigma_4^{(1)}}^{(2)}\right)dv_{\varsigma_2^{(1)}}^{(1)}\\
&=\sum_{n_{\varsigma_2},n_{\varsigma_4}} f_{\varsigma_2^{(1)}}(x_1)\left\{f_{\varsigma_4^{(1)}}(x_2)+\frac{1}{2}f_{\varsigma_5^{(1)}}(x_2)F_{\varsigma_4^{(1)}}(x_2-)\right.\\
&\quad\left.-\frac{1}{2}f_{\varsigma_5^{(1)}}(x_2)F_{\varsigma_4^{(1)}}(x_2)-f_{\varsigma_4^{(1)}}(x_2)F_{\varsigma_5^{(1)}}(x_2-)\right\}\\
&=f_1(x_1)\left\{f_2(x_2)+\frac{1}{2}f_3(x_2)F_2(x_2-)-\frac{1}{2}f_3(x_2)F_2(x_2)-f_2(x_2)F_3(x_2-)\right\}\\
&\quad+f_1(x_1)\left\{f_3(x_2)+\frac{1}{2}f_2(x_2)F_3(x_2-)-\frac{1}{2}f_2(x_2)F_3(x_2)-f_3(x_2)F_2(x_2-)\right\}\\
&\quad+f_2(x_1)\left\{f_3(x_2)+\frac{1}{2}f_1(x_2)F_3(x_2-)-\frac{1}{2}f_1(x_2)F_3(x_2)-f_3(x_2)F_1(x_2-)\right\}\\
&\quad+f_2(x_1)\left\{f_1(x_2)+\frac{1}{2}f_3(x_2)F_1(x_2-)-\frac{1}{2}f_3(x_2)F_1(x_2)-f_1(x_2)F_3(x_2-)\right\}\\
&\quad+f_3(x_1)\left\{f_1(x_2)+\frac{1}{2}f_2(x_2)F_1(x_2-)-\frac{1}{2}f_2(x_2)F_1(x_2)-f_1(x_2)F_2(x_2-)\right\}\\
&\quad+f_3(x_1)\left\{f_2(x_2)+\frac{1}{2}f_1(x_2)F_2(x_2-)-\frac{1}{2}f_1(x_2)F_2(x_2)-f_2(x_2)F_1(x_2-)\right\}.
\end{aligned}$$

Moreover, the above identity in the *iid* case can be expressed by

$$f_{1,2:3}(x_1,x_2) = 6f(x_1)f(x_2) - 6f(x_1)f(x_2)F(x_2) + 3f(x_1)f^2(x_2).$$

This result is obtained if $i = 1$, $j = 2$, and $n = 3$ in equation (6) in Khatri [26].

If $x_1 = x_2 = \cdots = x_p = x$, it should be written as $\int\int\cdots\int$ instead of $\int$ in (3), where $\int\int\cdots\int$ is to be carried out over the region: $F_{\varsigma_2^{(1)}}(x_1-) \le \nu^{(1)}_{\varsigma_2^{(1)}} \le \nu^{(2)}_{\varsigma_4^{(1)}} \le \cdots \le \nu^{(p)}_{\varsigma_{2p}^{(1)}} \le F_{\varsigma_{2p}^{(1)}}(x_p-), F_{\varsigma_2^{(1)}}(x_1-) \le \nu^{(1)}_{\varsigma_2^{(1)}} \le F_{\varsigma_2^{(1)}}(x_1), F_{\varsigma_4^{(1)}}(x_2-) \le \nu^{(2)}_{\varsigma_4^{(1)}} \le F_{\varsigma_4^{(1)}}(x_2), \ldots, F_{\varsigma_{2p}^{(1)}}(x_p-) \le \nu^{(p)}_{\varsigma_{2p}^{(1)}} \le F_{\varsigma_{2p}^{(1)}}(x_p)$.

Further on considering $p = 2$, $n = 3$, $r_1 = 1$, $r_2 = 2$, and $\nu^{(2)}_{\varsigma_5^{(1)}} = [\nu^{(2)}_{\varsigma_4^{(1)}} - F_{\varsigma_4^{(1)}}(x_2-)]\frac{f_{\varsigma_5^{(1)}}(x_2)}{f_{\varsigma_4^{(1)}}(x_2)} + F_{\varsigma_5^{(1)}}(x_2-)$ in (3), if X_1, X_2, X_3 are *innid* discrete random variables and for $x_1 = x_2 = x$, then

$$\begin{aligned}
&f_{1,2:3}(x,x)\\
&\quad= \sum_{n_{\varsigma_2},n_{\varsigma_4}} \int_{F_{\varsigma_2^{(1)}}(x-)}^{F_{\varsigma_2^{(1)}}(x)} \left(\int_{\nu_{\varsigma_2^{(1)}}}^{F_{\varsigma_4^{(1)}}(x)} \left(1-\nu^{(2)}_{\varsigma_5^{(1)}}\right) d\nu^{(2)}_{\varsigma_4^{(1)}} \right) d\nu^{(1)}_{\varsigma_2^{(1)}}\\
&\quad= \sum_{n_{\varsigma_2},n_{\varsigma_4}} \Biggl\{ F_{\varsigma_4^{(1)}}(x)f_{\varsigma_2^{(1)}}(x) - \frac{1}{2}\bigl[F_{\varsigma_2^{(1)}}(x) + F_{\varsigma_2^{(1)}}(x-)\bigr]f_{\varsigma_2^{(1)}}(x) - \frac{1}{2}F^2_{\varsigma_4^{(1)}}(x)f_{\varsigma_2^{(1)}}(x)\frac{f_{\varsigma_5^{(1)}}(x)}{f_{\varsigma_4^{(1)}}(x)}\\
&\qquad + \frac{1}{6}\bigl[F^3_{\varsigma_2^{(1)}}(x) - F^3_{\varsigma_2^{(1)}}(x-)\bigr]\frac{f_{\varsigma_5^{(1)}}(x)}{f_{\varsigma_4^{(1)}}(x)} + F_{\varsigma_4^{(1)}}(x)F_{\varsigma_4^{(1)}}(x-)\frac{f_{\varsigma_2^{(1)}}(x)f_{\varsigma_5^{(1)}}(x)}{f_{\varsigma_4^{(1)}}(x)}\\
&\qquad - \frac{1}{2}\bigl[F_{\varsigma_2^{(1)}}(x) + F_{\varsigma_2^{(1)}}(x-)\bigr]f_{\varsigma_2^{(1)}}(x)F_{\varsigma_4^{(1)}}(x-)\frac{f_{\varsigma_5^{(1)}}(x)}{f_{\varsigma_4^{(1)}}(x)} - F_{\varsigma_4^{(1)}}(x)F_{\varsigma_5^{(1)}}(x-)f_{\varsigma_2^{(1)}}(x)\\
&\qquad + \frac{1}{2}\bigl[F_{\varsigma_2^{(1)}}(x) + F_{\varsigma_2^{(1)}}(x-)\bigr]f_{\varsigma_2^{(1)}}(x)F_{\varsigma_5^{(1)}}(x-)\Biggr\}\\
&\quad= \Biggl\{F_2(x)f_1(x) - \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x) - \frac{1}{2}F_2^2(x)f_1(x)\frac{f_3(x)}{f_2(x)} + \frac{1}{6}\bigl[F_1^3(x) - F_1^3(x-)\bigr]\frac{f_3(x)}{f_2(x)}\\
&\qquad + F_2(x)F_2(x-)\frac{f_1(x)f_3(x)}{f_2(x)} - \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x)F_2(x-)\frac{f_3(x)}{f_2(x)} - F_2(x)F_3(x-)f_1(x)\\
&\qquad + \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x)F_3(x-)\Biggr\}\\
&\qquad + \Biggl\{F_3(x)f_1(x) - \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x) - \frac{1}{2}F_3^2(x)f_1(x)\frac{f_2(x)}{f_3(x)}\\
&\qquad + \frac{1}{6}\bigl[F_1^3(x) - F_1^3(x-)\bigr]\frac{f_2(x)}{f_3(x)} + F_3(x)F_3(x-)\frac{f_1(x)f_2(x)}{f_3(x)}\\
&\qquad - \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x)F_3(x-)\frac{f_2(x)}{f_3(x)} - F_3(x)F_2(x-)f_1(x)\\
&\qquad + \frac{1}{2}\bigl[F_1(x)+F_1(x-)\bigr]f_1(x)F_2(x-)\Biggr\}\\
&\qquad + \Biggl\{F_1(x)f_2(x) - \frac{1}{2}\bigl[F_2(x)+F_2(x-)\bigr]f_2(x) - \frac{1}{2}F_1^2(x)f_2(x)\frac{f_3(x)}{f_1(x)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}\big[F_2^3(x)-F_2^3(x-)\big]\frac{f_3(x)}{f_1(x)}+F_1(x)F_1(x-)\frac{f_2(x)f_3(x)}{f_1(x)} \\
&-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_1(x-)\frac{f_3(x)}{f_1(x)}-F_1(x)F_3(x-)f_2(x) \\
&+\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_3(x-)\bigg\} \\
&+\bigg\{F_3(x)f_2(x)-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)-\frac{1}{2}F_3^2(x)f_2(x)\frac{f_1(x)}{f_3(x)} \\
&+\frac{1}{6}\big[F_2^3(x)-F_2^3(x-)\big]\frac{f_1(x)}{f_3(x)}+F_3(x)F_3(x-)\frac{f_2(x)f_1(x)}{f_3(x)} \\
&-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_3(x-)\frac{f_1(x)}{f_3(x)}-F_3(x)F_1(x-)f_2(x) \\
&+\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_1(x-)\bigg\} \\
&+\bigg\{F_2(x)f_3(x)-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)-\frac{1}{2}F_2^2(x)f_3(x)\frac{f_1(x)}{f_2(x)} \\
&+\frac{1}{6}\big[F_3^3(x)-F_3^3(x-)\big]\frac{f_1(x)}{f_2(x)}+F_2(x)F_2(x-)\frac{f_3(x)f_1(x)}{f_2(x)} \\
&-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_2(x-)\frac{f_1(x)}{f_2(x)}-F_2(x)F_1(x-)f_3(x) \\
&+\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_1(x-)\bigg\} \\
&+\bigg\{F_1(x)f_3(x)-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)-\frac{1}{2}F_1^2(x)f_3(x)\frac{f_2(x)}{f_1(x)} \\
&+\frac{1}{6}\big[F_3^3(x)-F_3^3(x-)\big]\frac{f_2(x)}{f_1(x)}+F_1(x)F_1(x-)\frac{f_3(x)f_2(x)}{f_1(x)} \\
&-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_1(x-)\frac{f_2(x)}{f_1(x)}-F_1(x)F_2(x-)f_3(x) \\
&+\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_2(x-)\bigg\}.
\end{aligned}
$$

Moreover, the above identity in the *iid* case can be expressed by

$$
\begin{aligned}
&= 6F(x)f(x)-3\big[F(x)+F(x-)\big]f(x)-3F^2(x)f(x)+\big[F^3(x)-F^3(x-)\big]+6F(x)F(x-)f(x) \\
&\quad -3\big[F(x)+F(x-)\big]F(x-)f(x)-6F(x)F(x-)f(x)+3\big[F(x)+F(x-)\big]f(x)F(x-) \\
&= 6F(x)f(x)-3F(x)f(x)-3F(x-)f(x)-3F^2(x)f(x)+F^3(x)-F^3(x-) \\
&= 3f^2(x)-3F^2(x)f(x)+f(x)\big[3F^2(x)-3F(x)f(x)+f^2(x)\big] \\
&= f^3(x)+3f^2(x)\big[1-F(x)\big].
\end{aligned}
$$

This result is obtained if $r = 1$, $s = 2$, and $n = 3$ in equation (2.4.3) in David [23].

Furthermore, if $x_1 \le x_2 \le \cdots \le x_p$, it should be written $\int\int\cdots\int$ instead of $\int$ in (3), where $\int\int\cdots\int$ is to be carried out over the region: $\nu^{(1)}_{\varsigma^{(1)}_2} \le \nu^{(2)}_{\varsigma^{(1)}_4} \le \cdots \le \nu^{(p)}_{\varsigma^{(1)}_{2p}}, F_{\varsigma^{(1)}_2}(x_1-) \le \nu^{(1)}_{\varsigma^{(1)}_2} \le F_{\varsigma^{(1)}_2}(x_1), F_{\varsigma^{(1)}_4}(x_2-) \le \nu^{(2)}_{\varsigma^{(1)}_4} \le F_{\varsigma^{(1)}_4}(x_2), \ldots, F_{\varsigma^{(1)}_{2p}}(x_p-) \le \nu^{(p)}_{\varsigma^{(1)}_{2p}} \le F_{\varsigma^{(1)}_{2p}}(x_p)$.

We now express the following theorem to obtain the joint *df* of order statistics of *innid* discrete random variables.

Theorem 3

$$\begin{aligned} &F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) \\ &\quad = \sum_{z_1,z_2,\ldots,z_p} \sum_{m_p,k_p,\ldots,m_1,k_1} \sum_{n_{s_1},n_{s_2},\ldots,n_{s_{2p}}} \left(\prod_{w=1}^{p+1} \prod_{i_{2w-1}=1}^{r_w-1-k_w-m_{w-1}-r_{w-1}} \left[F_{s_{2w-1}^{(i_{2w-1})}}(z_w-) - F_{s_{2w-1}^{(i_{2w-1})}}(z_{w-1}) \right] \right) \\ &\qquad \times \prod_{w=1}^{p} \prod_{i_{2w}=1}^{k_w+1+m_w} f_{s_{2w}^{(i_{2w})}}(z_w). \end{aligned} \tag{7}$$

Proof We have

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{z_1,z_2,\ldots,z_p} f_{r_1,r_2,\ldots,r_p:n}(z_1,z_2,\ldots,z_p) \tag{8}$$

and using (1) in (8), (7) is obtained. □

The identity (7) can also be written in the form of an integral as follows.

Theorem 4

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{n_{\varsigma_1},n_{\varsigma_2},\ldots,n_{\varsigma_{2p}}} \left(\int_{\mathrm{V}} \prod_{w=1}^{p+1} \prod_{i_w=1}^{r_w-r_{w-1}-1} \left[v^{(w)}_{\varsigma_{2w-1}^{(i_w)}} - v^{(w-1)}_{\varsigma_{2w-1}^{(i_w)}} \right] \right) \prod_{w=1}^{p} dv^{(w)}_{\varsigma_{2w}^{(1)}}. \tag{9}$$

Proof Using (3) in (8), (9) is obtained. □

3 Results for distribution and probability functions

In this section, the results related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_p:n}$ will be determined. We express the following result for *pf* of the rth order statistic of *innid* discrete random variables.

Result 1

$$\begin{aligned} f_{r_1:n}(x_1) &= \sum_{m_1=0}^{n-r_1} \sum_{k_1=0}^{r_1-1} \sum_{n_{s_1},n_{s_2}} \left(\prod_{i_1=1}^{r_1-1-k_1} F_{s_1^{(i_1)}}(x_1-) \right) \left(\prod_{i_2=1}^{k_1+1+m_1} f_{s_2^{(i_2)}}(x_1) \right) \prod_{i_3=1}^{n-m_1-r_1} \left[1 - F_{s_3^{(i_3)}}(x_1) \right] \\ &= \sum_{n_{\varsigma_1},n_{\varsigma_2}} \int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)} \left(\prod_{i_1=1}^{r_1-1} v^{(1)}_{\varsigma_1^{(i_1)}} \right) \left(\prod_{i_2=1}^{n-r_1} \left[1 - v^{(1)}_{\varsigma_3^{(i_2)}} \right] \right) dv^{(1)}_{\varsigma_2^{(1)}}. \end{aligned} \tag{10}$$

Proof In (1) and (3), if $p = 1$, (10) is obtained. □

In Result 2 and Result 3, the *pf*'s of minimum and maximum order statistics of *innid* discrete random variables are given respectively.

Result 2

$$\begin{aligned} f_{1:n}(x_1) &= \sum_{m_1=0}^{n-1}\sum_{n_{s_2}}\left(\prod_{i_2=1}^{1+m_1} f_{s_2^{(i_2)}}(x_1)\right)\prod_{i_3=1}^{n-m_1-1}\left[1-F_{s_3^{(i_3)}}(x_1)\right] \\ &= \sum_{n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)}\left(\prod_{i_2=1}^{n-1}\left[1-v_{\varsigma_3^{(i_2)}}^{(1)}\right]\right)dv_{\varsigma_2^{(1)}}^{(1)}. \end{aligned} \tag{11}$$

Proof In (10), if $r_1 = 1$, (11) is obtained. □

Result 3

$$\begin{aligned} f_{n:n}(x_1) &= \sum_{k_1=0}^{n-1}\sum_{n_{s_1},n_{s_2}}\left(\prod_{i_1=1}^{n-1-k_1} F_{s_1^{(i_1)}}(x_1-)\right)\prod_{i_2=1}^{k_1+1} f_{s_2^{(i_2)}}(x_1) \\ &= \sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_{F_{\varsigma_2^{(1)}}(x_1-)}^{F_{\varsigma_2^{(1)}}(x_1)}\left(\prod_{i_1=1}^{n-1} v_{\varsigma_1^{(i_1)}}^{(1)}\right)dv_{\varsigma_2^{(1)}}^{(1)}. \end{aligned} \tag{12}$$

Proof In (10), if $r_1 = n$, (12) is obtained. □

In the following result, we determine the joint *pf* of $X_{1:n}, X_{2:n}, \ldots, X_{p:n}$.

Result 4 *If* $x_1 \le x_2 \le \cdots \le x_p$,

$$\begin{aligned} f_{1,2,\ldots,p:n}(x_1,x_2,\ldots,x_p) &= \sum_{m_p=0}^{n-p}\sum_{n_{s_2},n_{s_4},\ldots,n_{s_{2p}}}\left(\prod_{i_{2p+1}=1}^{n-m_p-p}\left[1-F_{s_{2p+1}^{(i_{2p+1})}}(x_p)\right]\right)\prod_{w=1}^{p}\prod_{i_{2w}=1}^{1+m_w} f_{s_{2w}^{(i_{2w})}}(x_w) \\ &= \sum_{n_{\varsigma_2},n_{\varsigma_4},\ldots,n_{\varsigma_{2p}}}\int\int\cdots\int\left(\prod_{i_{p+1}=1}^{n-p}\left[1-v_{\varsigma_{2p+1}^{(i_{p+1})}}^{(p)}\right]\right)\prod_{w=1}^{p} dv_{\varsigma_{2w}^{(1)}}^{(w)}, \end{aligned} \tag{13}$$

where $\int\int\cdots\int$ *is to be carried out over the region*: $v_{\varsigma_2^{(1)}}^{(1)} \le v_{\varsigma_4^{(1)}}^{(2)} \le \cdots \le v_{\varsigma_{2p}^{(1)}}^{(p)}, F_{\varsigma_2^{(1)}}(x_1-) \le v_{\varsigma_2^{(1)}}^{(1)} \le F_{\varsigma_2^{(1)}}(x_1), F_{\varsigma_4^{(1)}}(x_2-) \le v_{\varsigma_4^{(1)}}^{(2)} \le F_{\varsigma_4^{(1)}}(x_2), \ldots, F_{\varsigma_{2p}^{(1)}}(x_p-) \le v_{\varsigma_{2p}^{(1)}}^{(p)} \le F_{\varsigma_{2p}^{(1)}}(x_p)$.

Proof In (1) and (3), if $r_1 = 1$, $r_2 = 2, \ldots, r_p = p$ and $\int\int\cdots\int$ instead of $\int$, (13) is obtained. □

Specially, in (13), by taking $p = 2$, $n = 3$ and $v_{\varsigma_5^{(1)}}^{(2)} = [v_{\varsigma_4^{(1)}}^{(2)} - F_{\varsigma_4^{(1)}}(x_2-)]\frac{f_{\varsigma_5^{(1)}}(x_2)}{f_{\varsigma_4^{(1)}}(x_2)} + F_{\varsigma_5^{(1)}}(x_2-)$, the following identity is obtained

$$\begin{aligned} f_{1,2:3}(x_1,x_2) &= \sum_{n_{s_2},n_{s_4}}\big(1-F_{s_5^{(1)}}(x_2)\big)f_{s_2^{(1)}}(x_1)f_{s_4^{(1)}}(x_2) + \sum_{n_{s_2}} f_{s_2^{(1)}}(x_1)f_{s_4^{(1)}}(x_2)f_{s_4^{(2)}}(x_2) \\ &= \big(1-F_1(x_2)\big)f_2(x_1)f_3(x_2) + \big(1-F_1(x_2)\big)f_3(x_1)f_2(x_2) + \big(1-F_2(x_2)\big)f_3(x_1)f_1(x_2) \\ &\quad + \big(1-F_2(x_2)\big)f_1(x_1)f_3(x_2) + \big(1-F_3(x_2)\big)f_1(x_1)f_2(x_2) \\ &\quad + \big(1-F_3(x_2)\big)f_2(x_1)f_1(x_2) + f_1(x_2)f_2(x_2)f_3(x_1) \\ &\quad + f_2(x_2)f_1(x_2)f_3(x_1) + f_3(x_2)f_1(x_2)f_2(x_1). \end{aligned}$$

Moreover, the above identity in the *iid* case can be expressed as

$$f_{1,2:3}(x_1,x_2) = 6f(x_1)f(x_2) - 6f(x_1)f(x_2)F(x_2) + 3f(x_1)f^2(x_2).$$

We now establish three results for the *df* of single order statistic of *innid* discrete random variables.

Result 5

$$\begin{aligned} F_{r_1:n}(x_1) &= \sum_{z_1=0}^{x_1}\sum_{m_1=0}^{n-r_1}\sum_{k_1=0}^{r_1-1}\sum_{n_{s_1},n_{s_2}}\left(\prod_{i_1=1}^{r_1-1-k_1} F_{s_1^{(i_1)}}(z_1-)\right)\left(\prod_{i_2=1}^{k_1+1+m_1} f_{s_2^{(i_2)}}(z_1)\right)\prod_{i_3=1}^{n-m_1-r_1}\left[1-F_{s_3^{(i_3)}}(z_1)\right] \\ &= \sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_0^{F_{\varsigma_2^{(1)}}(x_1)}\left(\prod_{i_1=1}^{r_1-1} v_{\varsigma_1^{(i_1)}}^{(1)}\right)\left(\prod_{i_2=1}^{n-r_1}\left[1-v_{\varsigma_3^{(i_2)}}^{(1)}\right]\right)dv_{\varsigma_2^{(1)}}^{(1)}. \end{aligned} \tag{14}$$

Proof In (7) and (9), if $p = 1$, (14) is obtained. □

Result 6

$$\begin{aligned} F_{1:n}(x_1) &= \sum_{z_1=0}^{x_1}\sum_{m_1=0}^{n-1}\sum_{n_{s_2}}\left(\prod_{i_2=1}^{1+m_1} f_{s_2^{(i_2)}}(z_1)\right)\prod_{i_3=1}^{n-m_1-1}\left[1-F_{s_3^{(i_3)}}(z_1)\right] \\ &= \sum_{n_{\varsigma_2}}\int_0^{F_{\varsigma_2^{(1)}}(x_1)}\left(\prod_{i_2=1}^{n-1}\left[1-v_{\varsigma_3^{(i_2)}}^{(1)}\right]\right)dv_{\varsigma_2^{(1)}}^{(1)}. \end{aligned} \tag{15}$$

Proof In (14), if $r_1 = 1$, (15) is obtained. □

Result 7

$$\begin{aligned} F_{n:n}(x_1) &= \sum_{z_1=0}^{x_1}\sum_{k_1=0}^{n-1}\sum_{n_{s_1},n_{s_2}}\left(\prod_{i_1=1}^{n-1-k_1} F_{s_1^{(i_1)}}(z_1-)\right)\prod_{i_2=1}^{k_1+1} f_{s_2^{(i_2)}}(z_1) \\ &= \sum_{n_{\varsigma_1},n_{\varsigma_2}}\int_0^{F_{\varsigma_2^{(1)}}(x_1)}\left(\prod_{i_1=1}^{n-1} v_{\varsigma_1^{(i_1)}}^{(1)}\right)dv_{\varsigma_2^{(1)}}^{(1)}. \end{aligned} \tag{16}$$

Proof In (14), if $r_1 = n$, (16) is obtained. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MG, YB and BY have contributed to all parts of the article. All authors read and approved the final manuscript.

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