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L^p and BMO bounds for weighted Hardy operators on the Heisenberg group

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Abstract

In the setting of the Heisenberg group $\mathbb{H}^n$, we characterize those nonnegative functions w defined on [0, 1] for which the weighted Hardy operator H_w is bounded on $L^p(\mathbb{H}^n)$, $1 \le p \le \infty$, and on $\mathrm{BMO}(\mathbb{H}^n)$. Meanwhile, the corresponding operator norm in each case is derived. Furthermore, we introduce a type of weighted multilinear Hardy operators and obtain the characterizations of their weights for which the weighted multilinear Hardy operators are bounded on the product of Lebesgue spaces in terms of Heisenberg group. In addition, the corresponding norms are worked out.

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Keywords: Heisenberg group; weighted Hardy operator; BMO; weighted multilinear Hardy operator

1 Introduction

The history of weighted Hardy operators can be traced back to the end of the 19th century when Hadamard [1] used the idea of fractional differentiation of an analytic function via differentiation of its Taylor series. Corresponding to fractional differentiation, we note that Hadamard dealt with fractional integration in the form of

$$J^\alpha f(x) = \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} f(x\xi)\, d\xi,$$

which led him further to consider generalized fractional integrals of the form

$$\int_0^1 g(x\xi)\nu(\xi)\, d\xi. \tag{1.1}$$

Notice that, if $g(x\xi) = \frac{x^\alpha}{\Gamma(\alpha)} f(x\xi)$, $\nu(\xi) = (1-\xi)^{\alpha-1}$, then (1.1) reduces to $J^\alpha f(x)$. However, Hadamard considered the case $\nu(\xi) = \frac{1}{\Gamma(\alpha)}(-\ln\xi)^{\alpha-1}$, he did not develop this idea. Many years later a substantial theory of generalized integration (1.1) was created by Dzherbashyan in [2] and [3]. It is clear that in $\mathbb{R}^1$ if $\nu(\xi) \equiv 1$, then (1.1) is precisely reduced to the classical Hardy operator H defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\, dt, \quad x \neq 0,$$

which is one of the fundamental integral averaging operator in real analysis. In 1984, Carton-Lebrun and Fosset [4] defined the weighted Hardy operators H_ψ as follows. Let $\psi : [0,1] \to [0,\infty)$ be a function. If f is a measurable complex-valued function on $\mathbb{R}^n$, then

$$H_\psi f(x) := \int_0^1 f(tx)\psi(t)\,dt, \quad x \in \mathbb{R}^n.$$

Sometimes H_ψ is called the generalized Hardy operator [5]. Xiao [6] gave the characterization of ψ for which H_ψ is bounded on either $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, or $\mathrm{BMO}(\mathbb{R}^n)$. Meanwhile, the corresponding operator norms were worked out. Rim and Lee [7] obtained the similar results on a p-adic field. For other results of the weighted Hardy operators on the Euclidean space one can refer to [8] and references therein. As we know, the weighted Hardy operators are closely related to Hausdorff operators; see [9]. In this paper, we will consider the weighted Hardy operators on the Heisenberg group.

The Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$, whose group law is given by

$$\begin{aligned}&(x_1, x_2, \dots, x_{2n}, x_{2n+1})(x_1', x_2', \dots, x_{2n}', x_{2n+1}')\\&\quad = \left(x_1 + x_1', x_2 + x_2', \dots, x_{2n} + x_{2n}', x_{2n+1} + x_{2n+1}' + 2\sum_{j=1}^{n}(x_j' x_{n+j} - x_j x_{n+j}')\right).\end{aligned}$$

This multiplication is non-commutative. By the definition, we can see that the identity element on $\mathbb{H}^n$ is $0 \in \mathbb{R}^{2n+1}$, while the reverse element of x is $-x$. The vector fields

$$\begin{aligned}X_j &= \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,\\X_{n+j} &= \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,\\X_{2n+1} &= \frac{\partial}{\partial x_{2n+1}},\end{aligned}$$

form a natural basis for the Lie algebra of left-invariant vector fields. The only non-trivial commutator relations between those fields are

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(x_1, x_2, \dots, x_{2n}, x_{2n+1}) := (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

The homogeneous norm is defined by

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{\frac{1}{4}},$$

where $x = (x_1, x_2, \dots, x_{2n}, x_{2n+1})$. From this one also can derive the distance function

$$d(p,q) := d\bigl(q^{-1}p, 0\bigr) = \bigl|q^{-1}p\bigr|_h.$$

This distance d is left-invariant in the sense that $d(p,q)$ remains unchanged when p and q are both left-translated by some fixed vector on $\mathbb{H}^n$. Furthermore, d satisfies the triangular inequality (p.320 in [10])

$$d(p,q) \leq d(p,x) + d(x,q), \quad p,x,q \in \mathbb{H}^n.$$

For $r > 0$ and $x \in \mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by

$$B(x,r) = \bigl\{y \in \mathbb{H}^n : d(x,y) < r\bigr\}$$

and

$$S(x,r) = \bigl\{y \in \mathbb{H}^n : d(x,y) = r\bigr\},$$

respectively.

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. We denote by $|E|$ the measure of any measurable set $E \subset \mathbb{H}^n$. Then

$$\bigl|\delta_r(E)\bigr| = r^Q|E|, \qquad d(\delta_r x) = r^Q\,dx,$$

where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$. We have

$$\bigl|B(x,r)\bigr| = \bigl|B(0,r)\bigr| = \Omega_Q r^Q,$$

where

$$\Omega_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})},$$

is the volume of the unit ball $B(0,1)$ on $\mathbb{H}^n$. The area of $S(0,1)$ on $\mathbb{H}^n$ is $\omega_Q = Q\Omega_Q$; see [11]. For more details as regards the Heisenberg group one can refer to [12].

Definition 1.1 Let $w : [0,1] \to [0,\infty)$ be a function, for a measurable function f on $\mathbb{H}^n$. We define the weighted Hardy operators H_w on $\mathbb{H}^n$ as

$$\mathsf{H}_w f(x) := \int_0^1 f(\delta_t x) w(t)\,dt.$$

Recall that the space $\mathrm{BMO}(\mathbb{H}^n)$ is defined to be the space of all locally integrable functions f on $\mathbb{H}^n$ such that

$$\|f\|_{\mathrm{BMO}(\mathbb{H}^n)} := \sup_{B\subset\mathbb{H}^n} \frac{1}{|B|} \int_B \bigl|f(x) - f_B\bigr|\,dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{H}^n$ and $f_B = \frac{1}{|B|}\int_B f(x)\,dx$.

In Section 2, we will characterize the nonnegative functions ω defined on $[0,1]$ for which the weighted Hardy operator H_w is bounded on $L^p(\mathbb{H}^n)$, $1 \leq p \leq \infty$, and on $\mathrm{BMO}(\mathbb{H}^n)$. Meanwhile, the corresponding operator norm in each case will be obtained. In Section 3, we will introduce a type of weighted multilinear Hardy operators and investigate the characterizations of their weights for which the weighted multilinear Hardy operators are bounded on the product of Lebesgue spaces in terms of Heisenberg group. In addition, the corresponding norms will be worked out. We will give an extension of [13] and [6] to the setting of the Heisenberg group $\mathbb{H}^n$ since it is a non-commutative nilpotent Lie group with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$, in which geometric motions are different from the Euclidean space $\mathbb{R}^n$ due to the loss of interchangeability. A new special function for the sufficient part of BMO bounds will be constructed.

2 Bounds for weighted Hardy operators on $\mathbb{H}^n$

Theorem 2.1 *Let $w : [0,1] \to (0,\infty)$ be a function and let $1 \leq p \leq \infty$. Then H_w is bounded on $L^p(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-\frac{Q}{p}} w(t)\,dt < \infty. \tag{2.1}$$

Moreover, if (2.1) *holds, then*

$$\|\mathsf{H}_w\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} = \int_0^1 t^{-\frac{Q}{p}} w(t)\,dt.$$

Proof Since the case $p = \infty$ is trivial, it suffices to consider $1 \leq p < \infty$. Suppose (2.1) holds. By Minkowski's inequality, we have

$$\begin{aligned}
\|\mathsf{H}_w f\|_{L^p(\mathbb{H}^n)} &= \left(\int_{\mathbb{H}^n} \left|\int_0^1 f(\delta_t x) w(t)\,dt\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \int_0^1 \left(\int_{\mathbb{H}^n} \left|f(\delta_t x)\right|^p dx\right)^{\frac{1}{p}} w(t)\,dt \\
&= \int_0^1 \left(\int_{\mathbb{H}^n} \left|f(y)\right|^p dy\right)^{\frac{1}{p}} t^{-\frac{Q}{p}} w(t)\,dt \\
&= \|f\|_{L^p(\mathbb{H}^n)} \int_0^1 t^{-\frac{Q}{p}} w(t)\,dt.
\end{aligned} \tag{2.2}$$

Therefore, H_w is bounded from $L^p(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$.

Conversely, suppose $1 \leq p < \infty$ and H_w is bounded on $L^p(\mathbb{H}^n)$. Then

$$C := \|\mathsf{H}_w\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} < \infty,$$

and for $f \in L^p(\mathbb{H}^n)$,

$$\|\mathsf{H}_w f\|_{L^p(\mathbb{H}^n)} \leq C\|f\|_{L^p(\mathbb{H}^n)}. \tag{2.3}$$

Now, for any $\varepsilon > 0$, take

$$f_\varepsilon = \begin{cases} 0, & |x|_h \leq 1, \\ |x|_h^{-\frac{Q}{p}-\varepsilon}, & |x|_h > 1. \end{cases}$$

Then $\|f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} = \frac{\omega_Q}{\varepsilon p}$, and

$$\mathsf{H}_w f_\varepsilon(x) = \begin{cases} 0, & |x|_h \le 1, \\ |x|_h^{-\frac{Q}{p}-\varepsilon} \int_{|x|_h^{-1}}^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt, & |x|_h > 1. \end{cases}$$

Putting $0 < \varepsilon < 1$, then by (2.3), we can see that

$$\begin{aligned} C^p\|f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} \ge \|\mathsf{H}_w f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} &= \int_{|x|_h>1} \left(|x|_h^{-\frac{Q}{p}-\varepsilon} \int_{|x|_h^{-1}}^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \right)^p dx \\ &\ge \int_{|x|_h>\frac{1}{\varepsilon}} \left(|x|_h^{-\frac{Q}{p}-\varepsilon} \int_\varepsilon^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \right)^p dx \\ &= \left(\int_{|x|_h>\frac{1}{\varepsilon}} |x|_h^{-Q-\varepsilon p}\,dx \right) \left(\int_\varepsilon^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \right)^p. \end{aligned}$$

By the change of variable $x = \delta_{\frac{1}{\varepsilon}} y$, we get

$$\begin{aligned} C^p\|f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} &\ge \left(\int_{|y|_h>1} |y|_h^{-Q-\varepsilon p} \varepsilon^{\varepsilon p}\,dy \right) \left(\int_\varepsilon^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \right)^p \\ &= \|f_\varepsilon\|^p_{L^p(\mathbb{H}^n)} \left(\varepsilon^\varepsilon \int_\varepsilon^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \right)^p. \end{aligned}$$

This implies that

$$\varepsilon^\varepsilon \int_\varepsilon^1 t^{-\frac{Q}{p}-\varepsilon} w(t)\,dt \le C.$$

Letting ε approach to 0, we have

$$\int_0^1 t^{-\frac{Q}{p}} w(t)\,dt \le C. \tag{2.4}$$

Moreover, when (2.3) is true, *i.e.* H_w is bounded on $L^p(\mathbb{H}^n)$, then by (2.2) and (2.4), we have

$$\|\mathsf{H}_w\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} = \int_0^1 t^{-\frac{Q}{p}} w(t)\,dt.$$

This completes the proof. □

On the Heisenberg group, the weighted Hardy operator can also turn into the n-dimensional Hardy operator, see [14, 15].

Proposition 2.1 *If f is a radial function and $w(t) = Qt^{Q-1}$ then $\mathsf{H}_w f(x) = \mathsf{H}f(x)$, where*

$$\mathsf{H}f(x) := \frac{1}{|B(0,|x|_h)|} \int_{B(0,|x|_h)} f(y)\,dy, \quad x \in \mathbb{H}^n \setminus \{0\}, \tag{2.5}$$

is the Hardy operator on the Heisenberg group.

Proof In fact, if f is a radial function, then

$$\begin{aligned}\mathsf{H}f(x) &= \frac{1}{|B(0,|x|_h)|}\int_{B(0,|x|_h)} f(y)\,dy \\ &= \frac{1}{|B(0,|x|_h)|}\int_0^1\int_{S(0,1)} f\left(\delta_{t|x|_h}y'\right)t^{Q-1}|x|_h^Q\,dt\,dy' \\ &= \frac{1}{\Omega_Q}\int_0^1\int_{S(0,1)} f(\delta_t x)t^{Q-1}\,dt\,dy' \\ &= \int_0^1 f(\delta_t x)Qt^{Q-1}\,dt = \mathsf{H}_w f(x). \qquad \square\end{aligned}$$

Denote $\mathcal{L}^p(\mathbb{H}^n) = \{f : f \text{ is radial and } f \in L^p(\mathbb{H}^n)\}$. By Theorem 2.1, we can get the following result.

Corollary 2.1 *Let* $1 < p \leq \infty$. *Then* H *is bounded on* $\mathcal{L}^p(\mathbb{H}^n)$. *Moreover,*

$$\|\mathsf{H}\|_{\mathcal{L}^p(\mathbb{H}^n)\to\mathcal{L}^p(\mathbb{H}^n)} = \frac{p}{p-1}, \quad 1 < p < \infty,$$

$$\|\mathsf{H}\|_{\mathcal{L}^\infty(\mathbb{H}^n)\to\mathcal{L}^\infty(\mathbb{H}^n)} = 1.$$

Theorem 2.2 *Let* $w : [0,1] \to (0,\infty)$ *be a function. Then* H_w *is bounded on* $\mathrm{BMO}(\mathbb{H}^n)$ *if and only if*

$$\int_0^1 w(t)\,dt < \infty. \tag{2.6}$$

Moreover, if (2.6) *holds, then*

$$\|\mathsf{H}_w\|_{\mathrm{BMO}(\mathbb{H}^n)\to\mathrm{BMO}(\mathbb{H}^n)} = \int_0^1 w(t)\,dt.$$

Proof For each $t > 0$ and ball $B(x_0,r) \subset \mathbb{H}^n$, let $tB(x_0,r)$ be the ball $B(\delta_t x_0, tr)$, then $|tB(x_0,r)| = t^Q|B(x_0,r)|$.

Suppose (2.6) holds. Let $f \in \mathrm{BMO}(\mathbb{H}^n)$ and let B be a ball. Then by Fubini's theorem, we have

$$\begin{aligned}(\mathsf{H}_w f)_B &= \frac{1}{|B|}\int_B \mathsf{H}_w f(x)\,dx \\ &= \frac{1}{|B|}\int_B\int_0^1 f(\delta_t x)w(t)\,dt\,dx \\ &= \int_0^1\left(\frac{1}{|B|}\int_B f(\delta_t x)\,dx\right)w(t)\,dt \\ &= \int_0^1\left(\frac{1}{|B|}\int_{tB} f(y)t^{-Q}\,dy\right)w(t)\,dt \\ &= \int_0^1 f_{tB}w(t)\,dt.\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left|\mathsf{H}_w f(x) - (\mathsf{H}_w f)_B\right| dx \\
&\quad = \frac{1}{|B|}\int_B \left|\int_0^1 f(\delta_t x) w(t)\,dt - \int_0^1 f_{tB} w(t)\,dt\right| dx \\
&\quad \le \frac{1}{|B|}\int_B \int_0^1 \left|f(\delta_t x) - f_{tB}\right| w(t)\,dt\,dx \\
&\quad = \int_0^1 \left(\frac{1}{|B|}\int_B \left|f(\delta_t x) - f_{tB}\right| dx\right) w(t)\,dt \\
&\quad = \int_0^1 \left(\frac{1}{|tB|}\int_{tB} \left|f(y) - f_{tB}\right| dy\right) w(t)\,dt \\
&\quad \le \|f\|_{\mathrm{BMO}(\mathbb{H}^n)} \int_0^1 w(t)\,dt,
\end{aligned} \tag{2.7}$$

which implies that H_w is bounded on $\mathrm{BMO}(\mathbb{H}^n)$.

Conversely, if H_w is bounded on $\mathrm{BMO}(\mathbb{H}^n)$. Choose

$$f_0(x) = \begin{cases} 1, & x_{2n+1} > 0, \\ 0, & x_{2n+1} = 0, \\ -1, & x_{2n+1} < 0. \end{cases}$$

Then $f_0 \in \mathrm{BMO}(\mathbb{H}^n)$ with $\|f_0\|_{\mathrm{BMO}(\mathbb{H}^n)} \neq 0$. Let

$$\mathsf{H}_w f_0(x) = \begin{cases} \int_0^1 w(t)\,dt, & x_{2n+1} > 0, \\ 0, & x_{2n+1} = 0, \\ -\int_0^1 w(t)\,dt, & x_{2n+1} < 0. \end{cases}$$

Then

$$\mathsf{H}_w f_0(x) = f_0(x) \int_0^1 w(t)\,dt.$$

Consequently,

$$\int_0^1 w(t)\,dt \le \|\mathsf{H}_w\|_{\mathrm{BMO}(\mathbb{H}^n)\to\mathrm{BMO}(\mathbb{H}^n)}. \tag{2.8}$$

Moreover, when (2.6) holds, then (2.7) and (2.8) imply that

$$\|\mathsf{H}_w\|_{\mathrm{BMO}(\mathbb{H}^n)\to\mathrm{BMO}(\mathbb{H}^n)} = \int_0^1 w(t)\,dt.$$

This completes the proof. □

Corollary 2.2 *Denote*

$$\mathrm{BMO}(\mathbb{H}^n) = \{f : f \textit{ is radial and } f \in \mathrm{BMO}(\mathbb{H}^n)\}.$$

Then H *is bounded on* $\mathrm{BMO}(\mathbb{H}^n)$ *and*

$$\|\mathsf{H}\|_{\mathrm{BMO}(\mathbb{H}^n)\to\mathrm{BMO}(\mathbb{H}^n)} = 1.$$

3 Bounds for weighted multilinear Hardy operators on $\mathbb{H}^n$

The study of multilinear averaging operators is traced back to the multilinear singular integral operator theory [16], and motivated not only the generalization of the theory of linear ones but also their natural appearance in analysis. For a more complete account on multilinear operators, we refer to [13, 17] and [18]. Very recently, Fu *et al.* [13] defined a kind of multilinear Hardy operators, we will investigate their estimates on the Heisenberg group.

Definition 3.1 Let $m \in \mathbb{N}$ and

$$\Phi : \overbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}^{m} \to [0,\infty)$$

be an integrable function. The weighted multilinear Hardy operator H_Φ^m on $\mathbb{H}^n$ is defined as

$$\mathsf{H}_\Phi^m(\vec{f})(x) := \int_{0<t_1,t_2,\ldots,t_m<1} \left(\prod_{i=1}^m f_i(\delta_{t_i}x)\right) \Phi(\vec{t})\, d\vec{t}, \quad x \in \mathbb{H}^n,$$

where $\vec{f} := (f_1,f_2,\ldots,f_m)$, $\Phi(\vec{t}) := \Phi(t_1,t_2,\ldots,t_m)$, $d\vec{t} := dt_1\, dt_2 \cdots dt_m$, and f_i, $i = 1,\ldots,m$, are complex-valued measurable functions on $\mathbb{H}^n$. When $m = 2$, H_Φ^m is referred to as bilinear.

Remark 3.1 If f_i, $i = 1,2,\ldots,m$, are radial functions and $\Phi(t_1,\ldots,t_m) = Q^m \prod_{i=1}^m t_i^{Q-1}$, then $\mathsf{H}_\Phi^m f(x) = \prod_{i=1}^m \mathsf{H} f_i(x)$, where H is given by (2.5).

In fact, if f_i, $i = 1,2,\ldots,m$, are radial functions, then

$$\begin{aligned}\prod_{i=1}^m \mathsf{H} f_i(x) &= \frac{1}{\Omega_Q^m |x|_h^{mQ}} \prod_{i=1}^m \int_{|y_i|_h<|x|_h} f_i(y_i)\, dy_i \\ &= \frac{1}{\Omega_Q^m} \prod_{i=1}^m \int_0^1 \int_{S(0,1)} f_i\big(\delta_{t_i|x|_h} y_i'\big) t_i^{Q-1}\, dt_i\, dy_i' \\ &= \frac{1}{\Omega_Q^m} \prod_{i=1}^m \int_0^1 \int_{S(0,1)} f_i(\delta_{t_i}x) t_i^{Q-1}\, dt_i\, dy_i' \\ &= Q^m \prod_{i=1}^m \int_0^1 f_i(\delta_{t_i}x) t_i^{Q-1}\, dt \\ &= \int_{0<t_1,t_2,\ldots,t_m<1} \left(\prod_{i=1}^m f_i(\delta_{t_i}x)\right) Q^m \prod_{i=1}^m t_i^{Q-1}\, dt = \mathsf{H}_\Phi \vec{f}(x).\end{aligned}$$

Theorem 3.1 *Suppose* $\Phi : \overbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}^{m} \to [0,\infty)$ *is a function and* $m \geq 2$. *Let* $1 \leq p, p_i \leq \infty$, $i = 1,\ldots,m$ *and* $1/p = 1/p_1 + \cdots + 1/p_m$. *Then* H_Φ^m *is bounded from*

$L^{p_1}(\mathbb{H}^n) \times \cdots \times L^{p_m}(\mathbb{H}^n)$ *to* $L^p(\mathbb{H}^n)$ *if and only if*

$$A_m := \int_{0<t_1,t_2,\dots,t_m<1} \left(\prod_{i=1}^{m} t_i^{-\frac{Q}{p_i}}\right) \Phi(\vec{t})\, d\vec{t} < \infty. \tag{3.1}$$

Moreover, if (3.1) *holds, then*

$$\left\| \mathsf{H}_\Phi^m \right\|_{L^{p_1}(\mathbb{H}^n)\times\cdots\times L^{p_m}(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} = A_m. \tag{3.2}$$

Proof For simplicity, we only consider the case $m = 2$. A similar procedure works for the other $m \geq 3$.

Since the case $p = \infty$ and $p_i = \infty$, $i = 1,\dots,m$ is trivial, it suffices to consider $1 \leq p, p_i < \infty$, $i = 1,\dots,m$.

Suppose (3.1) holds. Using Minkowski's inequality and the change of variables $\delta_{t_1} x = y_1$, $\delta_{t_2} x = y_2$, we have

$$\begin{aligned} \left\| \mathsf{H}_\Phi^2(f_1,f_2) \right\|_{L^p(\mathbb{H}^n)} &= \left(\int_{\mathbb{H}^n} \left| \int_{0<t_1,t_2<1} f_1(\delta_{t_1}x) f_2(\delta_{t_2}x) \Phi(t_1,t_2)\, dt_1\, dt_2 \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \int_{0<t_1,t_2<1} \left(\int_{\mathbb{H}^n} \left| f_1(\delta_{t_1}x) f_2(\delta_{t_2}x) \right|^p dx \right)^{\frac{1}{p}} \Phi(t_1,t_2)\, dt_1\, dt_2. \end{aligned}$$

By Hölder's inequality with $1/p = 1/p_1 + 1/p_2$, we get

$$\begin{aligned} \left\| \mathsf{H}_\Phi^2(f_1,f_2) \right\|_{L^p(\mathbb{H}^n)} &\leq \int_{0<t_1,t_2<1} \prod_{i=1}^{2} \left(\int_{\mathbb{H}^n} \left| f_i(\delta_{t_i}x) \right|^{p_i} dx \right)^{\frac{1}{p_i}} \Phi(t_1,t_2)\, dt_1\, dt_2 \\ &= \|f_1\|_{L^{p_1}(\mathbb{H}^n)} \|f_2\|_{L^{p_2}(\mathbb{H}^n)} \int_{0<t_1,t_2<1} \prod_{i=1}^{2} t_i^{-\frac{Q}{p_i}} \Phi(t_1,t_2)\, dt_1\, dt_2. \end{aligned}$$

Thus H_Φ^2 maps $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$, and

$$\|\mathsf{H}_\Phi^2\|_{L^{p_1}(\mathbb{H}^n)\times L^{p_2}(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} \leq \int_{0<t_1,t_2<1} \prod_{i=1}^{2} t_i^{-\frac{Q}{p_i}} \Phi(t_1,t_2)\, dt_1\, dt_2 = A_2. \tag{3.3}$$

Conversely, suppose that H_Φ^2 is a bounded operator from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ to $L^p(\mathbb{H}^n)$. For sufficiently small $\varepsilon \in (0,1)$, we set

$$f_i^\varepsilon(x) = \begin{cases} 0, & |x|_h \leq 1, \\ |x|_h^{-\frac{Q}{p_i}-\frac{p}{p_i}\varepsilon}, & |x|_h > 1, \end{cases} \qquad i = 1,2.$$

A standard integral calculation gives

$$\left\| f_i^\varepsilon \right\|_{L^{p_i}(\mathbb{H}^n)}^{p_i} = \frac{w_Q}{\varepsilon p}, \quad i = 1,2.$$

And

$$\mathsf{H}_\Phi^2\left(f_1^\varepsilon, f_2^\varepsilon\right)(x) = \begin{cases} 0, & |x|_h \leq 1, \\ |x|_h^{-\frac{Q}{p}-\varepsilon} \int_{\frac{1}{|x|_h}}^{1} \int_{\frac{1}{|x|_h}}^{1} t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\, dt_1\, dt_2, & |x|_h > 1. \end{cases}$$

Consequently, we have

$$\begin{aligned}
&\big\| \mathsf{H}_\Phi^2(f_1^\varepsilon, f_2^\varepsilon)\big\|_{L^p(\mathbb{H}^n)} \\
&= \left\{ \int_{|x|_h>1} |x|_h^{-Q-\varepsilon p} \left(\int_{\frac{1}{|x|_h}}^1 \int_{\frac{1}{|x|_h}}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2 \right)^p dx \right\}^{\frac{1}{p}} \\
&\geq \left\{ \int_{|x|_h>\frac{1}{\varepsilon}} |x|_h^{-Q-\varepsilon p} \left(\int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2 \right)^p dx \right\}^{\frac{1}{p}} \\
&= \left(\int_{|y|_h>1} |y|_h^{-Q-\varepsilon p} \varepsilon^{Q+\varepsilon p} \varepsilon^{-Q}\,dy \right)^{\frac{1}{p}} \left(\int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2 \right) \\
&= \left(\frac{w_Q}{\varepsilon p} \right)^{\frac{1}{p}} \varepsilon^\varepsilon \int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2 \\
&= \left(\frac{w_Q}{\varepsilon p} \right)^{\frac{1}{p_1}} \left(\frac{w_Q}{\varepsilon p} \right)^{\frac{1}{p_2}} \varepsilon^\varepsilon \int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2 \\
&= \|f_1^\varepsilon\|_{L^{p_1}(\mathbb{H}^n)} \|f_2^\varepsilon\|_{L^{p_2}(\mathbb{H}^n)} \varepsilon^\varepsilon \int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2.
\end{aligned}$$

Therefore,

$$\big\| \mathsf{H}_\Phi^2 \big\|_{L^{p_1}(\mathbb{H}^n)\times L^{p_2}(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} \geq \varepsilon^\varepsilon \int_{\varepsilon}^1 \int_{\varepsilon}^1 t_1^{-\frac{Q}{p_1}-\frac{p}{p_1}\varepsilon} t_2^{-\frac{Q}{p_2}-\frac{p}{p_2}\varepsilon} \Phi(t_1,t_2)\,dt_1\,dt_2.$$

Since $\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$, we obtain

$$A_2 \leq \|\mathsf{H}_\Phi^2\|_{L^{p_1}(\mathbb{H}^n)\times L^{p_2}(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} < \infty.$$

This inequality and (3.3) yield (3.2). The proof is complete. □

4 Bounds for weighted Cesàro operators on $\mathbb{H}^n$

Given a nonnegative function $w : [0,1] \to (0,\infty)$. For a measurable complex-valued function f on $\mathbb{H}^n$, the adjoint operator of the weighted Hardy operator, the weighted Cesàro operator is defined as

$$\mathsf{C}_\omega f(x) := \int_0^1 f(\delta_{1/t}x) t^{-Q} \omega(t)\,dt, \quad x \in \mathbb{H}^n,$$

which satisfies

$$\int_{\mathbb{H}^n} f(x)(\mathsf{H}_\omega g)(x)\,dx = \int_{\mathbb{H}^n} g(x)(\mathsf{C}_\omega f)(x)\,dx.$$

Here $f \in L^p(\mathbb{H}^n)$, $g \in L^q(\mathbb{H}^n)$, $1 < p < \infty$, $q = p/(p-1)$, H_ω is bounded on $L^p(\mathbb{H}^n)$, and C_ω is bounded on $L^q(\mathbb{H}^n)$.

Theorem 4.1 *Let $w : [0,1] \to (0,\infty)$ be a function and let $1 \leq q \leq \infty$. Then C_w is bounded on $L^q(\mathbb{H}^n)$ if and only if*

$$\int_0^1 t^{-Q(1-1/q)} w(t)\,dt < \infty. \tag{4.1}$$

Moreover, if (4.1) *holds, then*

$$\|\mathsf{C}_w\|_{L^q(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} = \int_0^1 t^{-Q(1-1/q)} w(t)\,dt.$$

Theorem 4.2 *Let* $w : [0,1] \to (0,\infty)$ *be a function. Then* C_w *is bounded on* $\mathrm{BMO}(\mathbb{H}^n)$ *if and only if*

$$\int_0^1 t^{-Q} w(t)\,dt < \infty. \tag{4.2}$$

Moreover, if (4.2) *holds, then*

$$\|\mathsf{C}_w\|_{\mathrm{BMO}(\mathbb{H}^n)\to \mathrm{BMO}(\mathbb{H}^n)} = \int_0^1 t^{-Q} w(t)\,dt.$$

We also define the weighted multilinear Cesàro operator C_Φ^m on $\mathbb{H}^n$ as

$$\mathsf{C}_\Phi^m(\vec{f})(x) := \int_{0<t_1,t_2,\dots,t_m<1} \left(\prod_{i=1}^m f_i(\delta_{1/t_i}x)\right) \Phi(\vec{t})\,d\vec{t}, \quad x \in \mathbb{H}^n.$$

Theorem 4.3 *Suppose* $\Phi : \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{m} \to [0,\infty)$ *is a function and* $m \geq 2$. *Let* $1 \leq q, q_i \leq \infty$, $i = 1,\dots,m$, *and* $1/q = 1/q_1 + \cdots + 1/q_m$. *Then* C_Φ^m *is bounded from* $L^{q_1}(\mathbb{H}^n) \times \cdots \times L^{q_m}(\mathbb{H}^n)$ *to* $L^q(\mathbb{H}^n)$ *if and only if*

$$C_m := \int_{0<t_1,t_2\cdots,t_m<1} \left(\prod_{i=1}^m t_i^{-Q(1-1/q_i)}\right) \Phi(\vec{t})\,d\vec{t} < \infty. \tag{4.3}$$

Moreover, if (4.3) *holds, then*

$$\|\mathsf{C}_\Phi^m\|_{L^{q_1}(\mathbb{H}^n)\times\cdots\times L^{q_m}(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} = C_m. \tag{4.4}$$

The proof of the theorem in Section 4 is immediate from the proof of Section 2 and Section 3.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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