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# On equivalent conditions of two sequences to be R-dual

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**Abstract**

The concept of R-duals was introduced by Casazza, Kutyniok, and Lammers in 2004. In this paper, we give a condition when a Parseval frame can be dilated to an orthonormal basis of a given separable Hilbert space $H$. This is advantageous for deriving a condition for a sequence $\{\omega_j\}_{j\in J}$ to be an R-dual of a given frame $\{f_j\}_{j\in J}$.

## 1 Introduction

The concept of R-duals was first introduced by Casazza *et al.* in [1]. Although it is defined in general frame theory, there is a natural connection with Gabor frame theory. And it is still an open problem whether the duality principle in Gabor analysis actually can be derived from the theory of the R-dual. Lots of scholars have done much research in this area. Reference [2] introduces various alternative R-duals and shows their relations with Gabor frames. References [3] and [4] consider R-dual in Banach space. In [5], the authors give an equivalent condition for a sequence $\{\omega_j\}_{j\in J}$ to be an R-dual of a given frame $\{f_j\}_{j\in J}$. However, we think there is a mistake in their proof. The correction of it will be discussed in Section 3.

The dilation viewpoint on frames is introduced by Larson and Han in [6], which has a natural relation with the R-dual. They point out that any Parseval frame can be dilated to an orthonormal basis. But given a Hilbert space $H$ and a Parseval frame of a subspace of $H$, can the Parseval frame be dilated to an orthonormal basis for $H$? This will be discussed in Section 2.

In the entire paper, we let $H$ denote a separable Hilbert space, with the inner product $\langle\cdot,\cdot\rangle$, and $J$ be a countable index set.

**Definition 1** A sequence $\{f_j\}_{j\in J}$ of elements in $H$ is a frame for $H$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le B\|f\|^2, \quad f \in H.$$

The constants $A$, $B$ are called a lower and upper frame bounds for the frame. A frame is A-tight, if $A = B$. If $A = B = 1$, it is called a Parseval frame (a normalized tight frame in [6]).

**Definition 2** A sequence $\{\omega_j\}_{j\in J}$ in $H$ is a Riesz sequence if there exist constants $C, D > 0$ such that

$$C\sum_{j\in J}|c_j|^2 \le \left\|\sum_{j\in J} c_j\omega_j\right\|^2 \le D\sum_{j\in J}|c_j|^2$$

for all finite sequence $\{c_j\}_{j\in J}$. The numbers $C, D$ are called Riesz bounds. A Riesz sequence is a Riesz basis for $H$ if it is complete in $H$.

For more information as regards frames and Riesz bases we refer to the monograph [7]. We now state the definition of the R-dual sequence.

**Definition 3** [1] Let $\{e_i\}_{i\in J}$ and $\{h_i\}_{i\in J}$ denote two orthonormal bases for $H$, and let $\{f_i\}_{i\in J}$ be any sequence in $H$ for which

$$\sum_{i\in J}\left|\langle f_i, e_j\rangle\right|^2 < \infty, \quad \forall j \in J.$$

The R-dual of $\{f_i\}_{i\in J}$ with respect to the orthonormal bases $\{e_i\}_{i\in j}$ and $\{h_i\}_{i\in J}$ is the sequence $\{\omega_j\}_{j\in J}$ given by

$$\omega_j = \sum_{i\in J}\langle f_i, e_j\rangle h_i, \quad j \in J. \tag{1.1}$$

It is well known from [1] that $\{f_i\}_{i\in J}$ is a frame for $H$ with bounds $A$, $B$ if and only if $\{\omega_j\}_{j\in J}$ is a Riesz sequence in $H$ with bounds $A, B$. But given two sequence $\{f_i\}_{i\in J}$ and $\{\omega_j\}_{j\in J}$, under what conditions can we find orthonormal bases $\{e_i\}_{i\in J}$ and $\{h_i\}_{i\in J}$ for $H$ such that (1.1) holds? This is the main question we want to answer in this paper. It will be discussed in Section 3 explicitly. Assume that $\{f_i\}_{i\in J}$ is a frame for $H$. Define a sequence $\{n_i\}_{i\in J}$ by

$$n_i = \sum_{k\in J}\langle e_k, f_i\rangle\tilde{\omega}_k, \quad i \in J, \tag{1.2}$$

where $\{\tilde{\omega}_j\}_{j\in J}$ is the canonical dual Riesz sequence of $\{\omega_j\}_{j\in J}$. The construction of $\{n_i\}_{i\in J}$ comes from [5]. It plays an important role in this paper.

**Proposition 1** [5] *Let $\{\omega_j\}_{j\in J}$ be a Riesz basis for the subspace $W$ of $H$, with dual Riesz basis $\{\tilde{\omega}_k\}_{k\in J}$. Let $\{e_i\}_{i\in J}$ be an orthonormal basis for $H$. Given any sequence $\{f_i\}_{i\in J}$ in $H$, the following hold*:

(i) *There exists a sequence $\{h_i\}_{i\in J}$ in $H$ such that*

$$f_i = \sum_{j\in J}\langle \omega_j, h_i\rangle e_j, \quad \forall i \in J. \tag{1.3}$$

(ii) *The sequence $\{h_i\}_{i\in J}$ satisfying* (1.3) *is characterized as*

$$h_i = m_i + n_i, \tag{1.4}$$

*where $n_i$ is given by* (1.2) *and $m_i \in W^\perp$.*

(iii) *If* $\{\omega_j\}_{j\in J}$ *is a Riesz basis for* $H$, *then* (1.3) *has the unique solution*

$$h_i = n_i, \quad i \in J.$$

In [5], Christensen *et al.* give a solution to the main question.

**Theorem 1** [5] *Let* $\{\omega_j\}_{j\in J}$ *be a Riesz sequence spanning a proper subspace* $W$ *of* $H$ *and* $\{e_i\}_{i\in J}$ *an orthonormal basis for* $H$. *Given any frame* $\{f_i\}_{i\in J}$ *for* $H$, *the following are equivalent*:

(i) $\{\omega_j\}_{j\in J}$ *is an R-dual of* $\{f_i\}_{i\in J}$ *w.r.t.* $\{e_i\}_{i\in J}$ *and some orthonormal basis* $\{h_i\}_{i\in J}$.
(ii) *There exists an orthonormal basis* $\{h_i\}_{i\in J}$ *for* $H$ *satisfying* (1.3).
(iii) *The sequence* $\{n_i\}_{i\in J}$ *in* (1.2) *is a Parseval frame.*

We point out that, in fact, (iii) is not equivalent to the other items in Theorem 1. In order to clarify this, we need the following proposition from [6].

**Proposition 2** [6] *Let* $J$ *be a countable* (*or finite*) *index set. Suppose that* $\{x_n : n \in J\}$ *is a Parseval frame for* $W$. *Then there exist a Hilbert space* $K \supseteq W$ *and an orthonormal basis* $\{e_n : n \in J\}$ *for* $K$ *such that* $Pe_n = x_n$, *where* $P$ *is the orthogonal projection from* $K$ *onto* $W$.

## 2 A dilation theorem

In this section, a dilation theorem is given, which will be used in Section 3. Firstly, we give an example to show that Theorem 1 is not strictly right.

**Example 1** In this example, we choose the index set $J = \mathbb{N}$, the natural number set. Suppose $\{z_i\}_{i\in J}$ is an orthonormal basis for $H$. Define $f_i = 2z_i$ and $\omega_i = 2z_{2i}$ for all $i \in J$. Then the sequence $\{f_i\}_{i\in J}$ is a Parseval frame with frame bounds 2 and $\{\omega_j\}_{j\in J}$ is a Riesz sequence with bounds 2 as well. The canonical dual $\{\tilde{\omega}_j\}_{j\in J}$ of $\{\omega_j\}_{j\in J}$ equals $\{\frac{1}{2}z_{2j}\}_{j\in J}$. Let

$$n_i = \sum_{k\in J} \langle z_k, f_i\rangle \tilde{\omega}_k = \sum_{k\in J} \langle z_k, 2z_i\rangle \frac{1}{2} z_{2k} = z_{2i}.$$

Obviously, $\{n_i\}_{i\in J}$ is a Parseval frame, but $\{\omega_j\}_{j\in J}$ cannot be an R-dual of $\{f_i\}_{i\in J}$. If not, by (ii) of Proposition 1, an orthonormal basis $\{h_i\}_{i\in J}$ for $H$ can be characterized by

$$h_i = m_i + n_i,$$

where $m_i \in W^\perp$ for all $i \in J$. Since $n_i \in W$, we have

$$1 = \|h_i\|^2 = \|m_i + n_i\|^2 = \|m_i\|^2 + \|n_i\|^2 = \|m_i\|^2 + \|z_{2i}\|^2.$$

Since $\|z_{2i}\| = 1$, one has $m_i = 0$ for all $i \in J$. Therefore $h_i = n_i = z_{2i}$. This contradicts $\{h_i\}_{i\in J}$ being an orthonormal basis for $H$. Thus (iii) of Proposition 1 is not right.

In fact, given any orthonormal sequence (of course a Parseval frame), it cannot be dilated to any orthonormal basis but itself. Generally, we have the following theorem.

**Theorem 2** *Given two separable Hilbert spaces $H \supseteq M$, suppose that $\{x_n\}_{n\in J}$ is a Parseval frame for $W$. Then there exists an orthonormal basis $\{e_n\}_{n\in J}$ for $H$ s.t. $Pe_n = x_n$ if and only if*

$$\dim(\ker T) = \dim\bigl(W^\perp\bigr), \tag{2.1}$$

*where $P$ is an orthogonal projection from $H$ onto $W$, $T$ is the synthesis operator of $\{x_i\}_{i\in J}$.*

*Proof* First we treat sufficiency. Since

$$\sum_{i\in J} c_i x_i = \sum_{i\in J} c_i Pe_i = P\sum_{i\in J} c_i e_i,$$

for any $\{c_i\}_{i\in J} \in \ell^2(J)$, a sequence $\{c_i\}_{i\in J} \in \ker T$ if and only if $\sum_{i\in J} c_i e_i \in W^\perp$. So (2.1) holds.

Now we treat necessity. Suppose (2.1) holds, from the proof of the Proposition 2, there exist a Hilbert space $K = \ell^2(J)$, an orthogonal projection $P$, and an orthonormal basis $\{e_i\}_{i\in J}$ for $K$, such that

$$Pe_i = \theta(x_i), \tag{2.2}$$

where $\theta$ is the analysis operator of $\{x_i\}$. Since $\theta$ is injective, it has inverse restricted to $\theta(W)$. For simplicity, we just denote it by $\theta^{-1}$.

For any $\{c_i\}_{i\in J} \in \ell^2(J)$, since

$$\sum_{i\in J} c_i\langle x, x_i\rangle = \biggl\langle x, \sum_{i\in J} \overline{c}_i x_i\biggr\rangle,$$

we have

$$\dim\ker T = \dim\bigl(\theta(W)\bigr)^\perp. \tag{2.3}$$

Together with (2.1), we have

$$\dim W^\perp = \dim(\ker T) = \dim\bigl(\theta(W)\bigr)^\perp.$$

Therefore, there is an unitary operator $\eta$ from $W^\perp$ onto $(\theta(W))^\perp$. Combining with $\theta$, we can define a unitary operator $U$ from $H$ onto $K$:

$$Ut = U(t_1 + t_2) = \theta t_1 + \eta t_2, \quad t_1 \in W, t_2 \in W^\perp.$$

One can easily get

$$U^{-1}y = U^{-1}(y_1 + y_2) = \theta^{-1}y_1 + \eta^{-1}y_2, \quad y_1 \in \theta(W), y_2 \in \theta(W)^\perp.$$

Therefore, $U^* = U^{-1}$. In fact, for $t \in H$ and $y \in K$,

$$\begin{aligned}\langle Ut, y\rangle &= \bigl\langle U(t_1 + t_2), y_1 + y_2\bigr\rangle \\ &= \langle \theta t_1, y_1\rangle + \langle \eta t_2, y_2\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle t_1, \theta^{-1}y_1\rangle + \langle t_2, \eta^{-1}y_2\rangle \\
&= \langle t, \theta^{-1}y_1 + \eta^{-1}y_2\rangle \\
&= \langle t, U^{-1}y\rangle \\
&= \langle t, U^{*}y\rangle,
\end{aligned}
$$

where the third equation is due to the Parseval frame property of $\{x_n\}_{n\in J}$ and unitarity of $\eta$. Because of the unitarity of $U$, also $\epsilon_i = U^{-1}e_i$ is an orthonormal basis for $H$.

Now, taking $U^{-1}$ on the two sides of (2.2), we have

$$U^{-1}Pe_i = U^{-1}PUU^{-1}e_i = U^{-1}PU\epsilon_i = x_i.$$

We claim that $U^{-1}PU$ is also an orthogonal projection. In fact, by the properties of $U$ and $P$, we have

$$\left(U^{-1}PU\right)^2 = U^{-1}P^2U = U^{-1}PU$$

and

$$\left(U^{-1}PU\right)^* = \left(U^*PU\right)^* = U^*PU = U^{-1}PU.$$

Thus we get as desired the complete proof. □

## 3 Conditions of R-dual

In this section, we discuss under what conditions $\{\omega_i\}_{i\in J}$ can be an R-dual of $\{f_i\}_{i\in J}$. At first, we give two lemmata which will be used later.

**Lemma 1** *Let* $\{n_i\}_{i\in J}$ *be defined as* (1.2), $W$ *the close span of* $\{\omega_j\}_{j\in J}$, *then* $\overline{\mathrm{span}}\{n_i\}_{i\in J} = W$.

*Proof* Since $n_i = \sum_{k\in J}\langle e_k, f_i\rangle\tilde{\omega}_k$, we have

$$\overline{\mathrm{span}}\{n_i\}_{i\in J} \subseteq \overline{\mathrm{span}}\{\tilde{\omega}_i\}_{i\in J} = W.$$

In the opposite direction, since $\{f_i\}_{i\in J}$ is a frame for $H$, there exists a sequence $\{c_\ell\}_{\ell\in J} \in \ell^2(J)$ such that $e_m = \sum_{\ell\in J} c_\ell f_\ell$ for $m \in J$. Then one has

$$\sum_{\ell\in J}\bar{c}_\ell n_\ell = \sum_{\ell\in J}\bar{c}_\ell\sum_{k\in J}\langle e_k, f_\ell\rangle\tilde{\omega}_k = \sum_{k\in J}\left\langle e_k, \sum_{\ell\in J} c_\ell f_\ell\right\rangle\tilde{\omega}_k = \sum_{k\in J}\langle e_k, e_m\rangle\tilde{\omega}_k = \tilde{\omega}_m.$$

Thus $W \subseteq \overline{\mathrm{span}}\{n_i\}_{i\in J}$. We have the desired result. □

Define $S_\omega f = \sum_{k\in J}\langle f, \omega_k\rangle\omega_k$ and $S_{\tilde{\omega}} f = \sum_{k\in J}\langle f, \tilde{\omega}_k\rangle\tilde{\omega}_k$, for $f \in W$. Then $S_{\tilde{\omega}}^{-\frac{1}{2}}\tilde{\omega}_k$ is an orthonormal basis for $W$. Since $\langle\omega_k, S_\omega^{-1}\omega_\ell\rangle = \delta_{k,\ell}$ by [7], one has $\tilde{\omega}_k = S_\omega^{-1}\omega_k$. Furthermore, we have

$$S_{\tilde{\omega}} f = \sum_{k\in J}\langle f, S_\omega^{-1}\omega_k\rangle S_\omega^{-1}\omega_k = S_\omega^{-1}S_\omega S_\omega^{-1} f = S_\omega^{-1} f, \quad \forall f \in W.$$

This means the operator equation $S_{\tilde{\omega}} = S_\omega^{-1}$ holds.

Let $\epsilon_k = S_{\tilde{\omega}}^{-\frac{1}{2}}\tilde{\omega}$, then $\tilde{\omega}_k = S_{\tilde{\omega}}^{\frac{1}{2}}\epsilon_k$. Let $\{e_k\}_{k\in J}$ be an orthonormal basis for $H$, define an antiunitary operator $\Lambda : H \longrightarrow W$ by

$$\Lambda f = \Lambda\left(\sum_{k\in J} c_k e_k\right) = \sum_{k\in J}\bar{c}_k\epsilon_k, \quad \text{for } f = \sum_{k\in J} c_k e_k \in H.$$

Obviously, the inverse of $\Lambda$ is also an antiunitary operator and

$$\Lambda^{-1}g = \Lambda^{-1}\left(\sum_{k\in J} c_k\epsilon_k\right) = \sum_{k\in J}\bar{c}_k e_k, \quad \forall g \in W.$$

Furthermore, the antiunitary operator $\Lambda$ has the following property.

**Lemma 2** *Let $\Lambda$ be defined as above, then $\langle \Lambda f, g\rangle = \langle \Lambda^{-1}g, f\rangle$ for any $f \in H$ and $g \in W$.*

*Proof* By the definition of $\Lambda$, one has

$$\langle \Lambda f, g\rangle = \left\langle \sum_{k\in J}\langle e_k, f\rangle\epsilon_k, g\right\rangle = \sum_{k\in J}\langle e_k, f\rangle\langle \epsilon_k, g\rangle = \left\langle \sum_{k\in J}\langle \epsilon_k, g\rangle e_k, f\right\rangle = \left\langle \Lambda^{-1}g, f\right\rangle. \qquad \square$$

**Theorem 3** *There exists an orthonormal basis $\{e_i\}_{i\in J}$ such that $\{n_i\}_{i\in J}$ is a Parseval frame if and only if there exists an antiunitary operator $\Lambda$ such that $S_\omega = \Lambda S\Lambda^{-1}$, where $S$ is the frame operator of $\{f_i\}_{i\in J}$.*

*Proof* By the definition of $\{n_i\}_{i\in J}$ and Lemma 2, we have

$$\begin{aligned}
\sum_{i\in J}\left|\langle f, n_i\rangle\right|^2 &= \sum_{i\in J}\left|\left\langle f, \sum_{k\in J}\langle e_k, f_i\rangle\tilde{\omega}_k\right\rangle\right|^2 \\
&= \sum_{i\in J}\left|\sum_{k\in J}\langle f_i, e_k\rangle\langle f, \tilde{\omega}_k\rangle\right|^2 \\
&= \sum_{k\in J}\sum_{\ell\in J}\left(\sum_{i\in J}\langle f_i, e_k\rangle\langle e_\ell, f_i\rangle\right)\langle f, \tilde{\omega}_k\rangle\langle \tilde{\omega}_\ell, f\rangle \\
&= \sum_{k\in J}\sum_{\ell\in J}\langle e_\ell, Se_k\rangle\left\langle f, S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda e_k\right\rangle\left\langle S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda e_\ell, f\right\rangle \\
&= \sum_{k\in J}\sum_{\ell\in J}\langle e_\ell, Se_k\rangle\left\langle e_k, \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle\left\langle \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f, e_\ell\right\rangle \\
&= \sum_{k\in J}\left\langle e_k, \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle\left\langle \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f, Se_k\right\rangle \\
&= \left\langle \sum_{k\in J}\left\langle \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f, Se_k\right\rangle e_k, \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle \\
&= \left\langle S\Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f, \Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle \\
&= \left\langle f, S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda S\Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle.
\end{aligned} \tag{3.1}$$

Suppose $\{n_i\}_{i\in J}$ is a Parseval frame; then we have

$$\sum_{i\in J}\left|\langle f,n_i\rangle\right|^2=\|f\|^2,\quad \forall f\in W. \tag{3.2}$$

By (3.1), it becomes

$$\sum_{i\in J}\left|\langle f,n_i\rangle\right|^2=\left\langle f,S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda S\Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}f\right\rangle=\langle f,f\rangle. \tag{3.3}$$

For arbitrary complex numbers $a$ and $b$, we have

$$\Lambda S\Lambda^{-1}(af+bg)=\Lambda S\left(\overline{a}\Lambda^{-1}f+\overline{b}\Lambda^{-1}g\right)=a\Lambda S\Lambda^{-1}f+b\Lambda S\Lambda^{-1}g.$$

Thus $\Lambda S\Lambda^{-1}$ is a linear operator, so is the operator $S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda S\Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}$. This means $S_{\tilde{\omega}}^{\frac{1}{2}}\Lambda S\Lambda^{-1}S_{\tilde{\omega}}^{\frac{1}{2}}=I$ by (3.3), *i.e.*

$$S_\omega=\Lambda S\Lambda^{-1}.$$

On the other hand, assume there exists an antiunitary operator $\Lambda$ such that $S_\omega=\Lambda S\Lambda^{-1}$. Define $e_j=\Lambda^{-1}\epsilon_j=\Lambda^{-1}S_\omega^{-\frac{1}{2}}\tilde{\omega}_k$, then (3.1) means

$$n_i=\sum_{k\in J}\langle e_k,f_i\rangle\tilde{\omega_k}$$

is a Parseval frame. □

**Theorem 4** *Suppose $\{f_i\}_{i\in J}$ is a frame for a separable Hilbert space $H$ and $\{\omega_j\}_{j\in J}$ is a Riesz sequence in $H$. $\{f_i\}_{i\in J}$ is an R-dual of $\{\omega_j\}_{j\in J}$ if and only if the following two conditions hold*:

(i) *there exists an antiunitary operator $\Lambda$ s.t. $S_w=\Lambda S\Lambda^{-1}$*;

(ii) $\dim(\ker T)=\dim(W^\perp)$.

*Proof* By Proposition 1, $\{f_i\}_{i\in J}$ is an R-dual of $\{\omega_j\}_{j\in J}$ if and only if $\{n_i\}_{i\in J}$ can be dilated to an orthonormal basis for $H$. By Theorem 2, this is equivalent to $\{n_i\}_{i\in J}$ being a Parseval frame and (ii) holding. Using Theorem 3, we see that $\{f_i\}_{i\in J}$ is an R-dual of $\{\omega_j\}_{j\in J}$ if and only (i) and (ii) hold. □

We appreciate one reviewer having pointed out that Theorem 4 is of exactly the same type as the characterizations of type II/III in [2]. In the special case, if $\{f_i\}_{i\in\mathbb{N}}$ is an A-tight frame for a separable Hilbert space $H$ with infinite dimension and $\{\omega_j\}_{j\in\mathbb{N}}$ is an A-tight Riesz sequence where $\mathbb{N}$ denotes the natural number set, then there must be an antiunitary operator $\Lambda$ form $H$ onto $W$. So we have $S=AI_H$, $S_W=AI_W$, and

$$S_W=AI_W=\Lambda AI_H\Lambda^{-1}=\Lambda S\Lambda^{-1}.$$

Thus the condition (i) of Theorem 4 holds automatically. And we get the following corollary, first given in [2].

**Corollary 1** [2] *Let $\{f_i\}_{i\in J}$ be a tight frame for $H$ and let $\{\omega_j\}_{j\in J}$ be a tight Riesz sequence in $H$ with the same bound. Denote the synthesis operator for $\{f_i\}_{i\in J}$ by $T$. Then $\{\omega_j\}_{j\in J}$ is an R-dual of $\{f_i\}_{i\in J}$ if and only if* $\dim(\ker T) = \dim(W^{\perp})$ *holds.*

**Remark 1** Since $S_W f = \sum_{j\in J}\langle f, \omega_j\rangle\omega_j$ and

$$\Lambda S\Lambda^{-1}f = \sum_{j\in J}\langle f_j, \Lambda^{-1}f\rangle\Lambda f_j = \sum_{j\in J}\langle f, \Lambda f_j\rangle\Lambda f_j,$$

(i) of Theorem 4 is equivalent to there existing an antiunitary operator such that

$$\sum_{j\in J}\langle f, \omega_j\rangle\omega_j = \sum_{j\in J}\langle f, \Lambda f_j\rangle\Lambda f_j.$$

**Remark 2** For parameters $a, b \in \mathbb{R}$, define the operators $T_a$ and $E_b$ on $L_2(\mathbb{R})$by $T_a f(x) = f(x-a)$ and $E_b f(x) = e^{2\pi ibx}f(x)$, respectively. From [8], we know that if $ab < 1$ and $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ has an infinite excess. If $ab > 1$, then $\{E_{mb}T_{na}\}_{m,n\in\mathbb{Z}}$ has an infinite deficit. This demonstrates that, if we want to solve the open problem, we only need (i) of Theorem 4 to hold. By Remark 1, this is equivalent to finding an antiunitary operator $\Lambda$ such that

$$\sum_{m,n}\langle f, \Lambda E_{mb}T_{na}g\rangle\Lambda E_{mb}T_{na}g = \sum_{m,n}\left\langle f, \frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\right\rangle\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g.$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors? contributions**

All authors contributed to each part of this work equally and read and approved the final manuscript.

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