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Solutions of the third order Cauchy difference equation on groups

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Abstract

Let $f: G \to H$ be a function, where $(G, \cdot)$ is a group and $(H, +)$ is an abelian group. In this paper, the following third order Cauchy difference of f: $C^{(3)}f(x_1, x_2, x_3, x_4) = f(x_1x_2x_3x_4) - f(x_1x_2x_3) - f(x_1x_2x_4) - f(x_1x_3x_4) - f(x_2x_3x_4) + f(x_1x_2) + f(x_1x_3) + f(x_1x_4) + f(x_2x_3) + f(x_2x_4) + f(x_3x_4) - f(x_1) - f(x_2) - f(x_3) - f(x_4)$ $(\forall x_1, x_2, x_3, x_4 \in G)$, is studied. We first give some special solutions of $C^{(3)}f = 0$ on free groups. Then sufficient and necessary conditions on finite cyclic groups and symmetric groups are also obtained.

MSC: 39B52; 39A70

Keywords: Cauchy difference; free group; symmetric group; cyclic group

1 Introduction

It is well known from [1] that Jensen's functional equation

$$f(x+y) + f(x-y) = 2f(x), \tag{1.1}$$

with the additional condition $f(0) = 0$, is equivalent to Cauchy's equation

$$f(x+y) = f(x) + f(y)$$

on the real line. Let $(G, \cdot)$ be a group, $(H, +)$ be an abelian group. Let $e \in G$ and $0 \in H$ denote the identity elements. The study of (1.1) was extended to groups for f maps G into H in [2], where the general solution for a free group H with two generators and $G = GL_2(Z)$ was given, respectively. Later, the results were generalized to all free groups and $G = GL_n(Z)$, $n \geq 3$ (see [3]). Since functional equations involve Cauchy difference, which made it become much more interesting [4–7]. For a function $f: G \to H$, its Cauchy difference, $C^{(m)}f$, is defined by

$$C^{(0)}f = f, \tag{1.2}$$

$$C^{(1)}f(x_1, x_2) = f(x_1x_2) - f(x_1) - f(x_2), \tag{1.3}$$

$$\begin{aligned} &C^{(m+1)}f(x_1, x_2, \ldots, x_{m+2}) \\ &\quad = C^{(m)}f(x_1x_2, x_3, \ldots, x_{m+2}) - C^{(m)}f(x_1, x_3, \ldots, x_{m+2}) - C^{(m)}f(x_2, x_3, \ldots, x_{m+2}). \end{aligned} \tag{1.4}$$

The first order Cauchy difference $C^{(1)}f$ will be abbreviated as Cf. In [8], by using the reduction formulas and relations, as given in [2, 3], the general solution of the second order

Cauchy difference equation was provided on free groups. Particularly, the authors also gave the expression of general solutions on symmetric group and finite cyclic group.

In this paper, we consider the following functional equation:

$$\begin{aligned} & f(x_1x_2x_3x_4) - f(x_1x_2x_3) - f(x_1x_2x_4) - f(x_1x_3x_4) - f(x_2x_3x_4) \\ & \quad + f(x_1x_2) + f(x_1x_3) + f(x_1x_4) + f(x_2x_3) + f(x_2x_4) + f(x_3x_4) \\ & \quad - f(x_1) - f(x_2) - f(x_3) - f(x_4) = 0 \quad (\forall x_1, x_2, x_3, x_4 \in G). \end{aligned} \tag{1.5}$$

It follows from (1.4) that (1.5) is equivalent to the vanishing third order Cauchy difference equation

$$C^{(3)}f = 0.$$

The purpose of this paper is to determine the solutions of (1.5) on some given groups. Clearly, the general solution of (1.5) will be denoted by

$$\operatorname{Ker} C^{(3)}(G, H) = \{f : G \to H \mid f \text{ satisfies (1.5)}\}. \tag{1.6}$$

Remark 1 (1) $\operatorname{Ker} C^{(3)}(G, H)$ is an abelian group under the pointwise addition of functions; (2) $\operatorname{Hom}(G, H) \subseteq \operatorname{Ker} C^{(3)}(G, H)$.

2 Properties of solution

Lemma 1 *Suppose that* $f \in \operatorname{Ker} C^{(3)}(G, H)$. *Then*

$$f(e) = 0, \tag{2.1}$$

$$Cf(x, e) = 0, \qquad Cf(e, y) = 0, \tag{2.2}$$

$$C^{(2)}f(e, y, z) = 0, \qquad C^{(2)}f(x, e, z) = 0, \qquad C^{(2)}f(x, y, e) = 0, \tag{2.3}$$

$$C^{(2)}f \textit{ is a homomorphism with respect to each variable,} \tag{2.4}$$

$$f(x^n) = nf(x) + \frac{n(n-1)}{2} Cf(x, x) + \frac{n(n-1)(n-2)}{6} C^{(2)}f(x, x, x), \tag{2.5}$$

for all $x, y, z, \in G$ *and* $n \in \mathbb{Z}$.

Proof Putting $x_1 = e$ in (1.5) we get (2.1). Then from (2.1) we obtain (2.2)-(2.3). Furthermore, by the definition of $C^{(2)}f$, we have

$$C^{(2)}f(x, yw, z) = f(xywz) - f(xyw) - f(xz) - f(ywz) + f(x) + f(yw) + f(z),$$

and

$$\begin{aligned} C^{(2)}f(x, y, z) + C^{(2)}f(x, w, z) = {} & f(xyz) - f(xy) - f(xz) - f(yz) + f(x) + f(y) + f(z) \\ & + f(xwz) - f(xw) - f(xz) - f(wz) + f(x) + f(w) + f(z). \end{aligned}$$

One can easily check that

$$C^{(2)}f(x, yw, z) - \left(C^{(2)}f(x, y, z) + C^{(2)}f(x, w, z)\right) = C^{(3)}f(x, y, w, z) = 0.$$

Hence, the above relations imply that $Cf(x,\cdot,z)$ is a homomorphism. Similarly, the fact is also true for both $Cf(\cdot,y,z)$ and $Cf(x,y,\cdot)$. This proves (2.4).

We now consider (2.5). Actually, it is trivial for $n = 0,1,2$ by (2.1) and the definition of Cf. Suppose that (2.5) holds for all natural numbers smaller than $n \geq 4$, then

$$\begin{aligned}
f(x^n) &= f(x^{n-2}\cdot x\cdot x) = 2f(x^{n-1}) + f(x^2) - f(x^{n-2}) - 2f(x) + C^{(2)}f(x^{n-2},x,x)\\
&= 2\biggl[(n-1)f(x) + \frac{(n-1)(n-2)}{2}Cf(x,x)\\
&\quad + \frac{(n-1)(n-2)(n-3)}{6}C^{(2)}f(x,x,x)\biggr] + 2f(x) + Cf(x,x)\\
&\quad - \biggl[(n-2)f(x) + \frac{(n-2)(n-3)}{2}Cf(x,x)\\
&\quad + \frac{(n-2)(n-3)(n-4)}{6}C^{(2)}f(x,x,x)\biggr] - 2f(x) + (n-2)C^{(2)}f(x,x,x)\\
&= nf(x) + \frac{n(n-1)}{2}Cf(x,x) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(x,x,x),
\end{aligned}$$

where the definition of $C^{(2)}f$ and (2.4) are used in the second equation. This gives (2.5) for all $n \geq 0$. On the other hand, for any fixed integer $n > 0$, by (1.4) and (2.1), we have

$$\begin{aligned}
C^{(2)}f(x^n,x^{-n},x^n) &= f(x^n) - f(e) - f(x^{2n}) - f(e) + f(x^n) + f(x^{-n}) + f(x^n)\\
&= 3f(x^n) + f(x^{-n}) - f(x^{2n}),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
f(x^{-n}) &= f(x^{2n}) - 3f(x^n) + C^{(2)}f(x^n,x^{-n},x^n)\\
&= 2nf(x) + \frac{2n(2n-1)}{2}Cf(x,x) + \frac{2n(2n-1)(2n-2)}{6}C^{(2)}f(x,x,x)\\
&\quad - 3\biggl[nf(x) + \frac{n(n-1)}{2}Cf(x,x) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(x,x,x)\biggr]\\
&\quad - n^3C^{(2)}f(x,x,x)\\
&= -nf(x) + \frac{-n(-n-1)}{2}Cf(x,x) + \frac{-n(-n-1)(-n-2)}{6}C^{(2)}f(x,x,x),
\end{aligned}$$

from (2.4) and the above claim for $n > 0$. This confirms (2.5) for $n < 0$. □

Remark 2 For any function $f: G \to H$, the following statements are pairwise equivalent:

(i) The function $f \in \operatorname{Ker} C^{(3)}(G,H)$;
(ii) $C^{(2)}f(\cdot,y,z)$ is a homomorphism;
(iii) $C^{(2)}f(x,\cdot,z)$ is a homomorphism;
(iv) $C^{(2)}f(x,y,\cdot)$ is a homomorphism.

Before presenting Proposition 1, we first introduce the following useful lemma, which was given in [8].

Lemma 2 (Lemma 2.4 in [8]) *The following identity is valid for any function $f: G \to H$ and $l \in \mathbb{N}$:*

$$f(x_1 x_2 \cdots x_l) = \sum_{m \le l} \sum_{1 \le i_1 < i_2 < \cdots < i_m \le l} C^{(m-1)} f(x_{i_1}, x_{i_2}, \ldots, x_{i_m}). \tag{2.6}$$

Proposition 1 *Suppose that $f \in \operatorname{Ker} C^{(3)}(G, H)$. Then*

$$\begin{aligned} f\left(x_1^{n_1} x_2^{n_2} \cdots x_l^{n_l}\right) = {} & \sum_{1 \le i \le l} \left[n_i f(x_i) + \frac{n_i(n_i-1)}{2} Cf(x_i, x_i) \right. \\ & \left. + \frac{n_i(n_i-1)(n_i-2)}{6} C^{(2)} f(x_i, x_i, x_i) \right] + \sum_{1 \le i_1 < i_2 \le l} Cf\left(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}\right) \\ & + \sum_{1 \le i_1 < i_2 < i_3 \le l} n_{i_1} n_{i_2} n_{i_3} C^{(2)} f(x_{i_1}, x_{i_2}, x_{i_3}), \end{aligned} \tag{2.7}$$

for $n_i \in \mathbb{Z}$ and all $x_i \in G$, $i = 1, 2, \ldots, l$ such that $x_j \ne x_{j+1}$, $j = 1, 2, \ldots, l-1$.

Proof Replacing x_i in (2.6) by $x_i^{n_i}$ we have

$$f\left(x_1^{n_1} x_2^{n_2} \cdots x_l^{n_l}\right) = \sum_{m \le l} \sum_{1 \le i_1 < i_2 < \cdots < i_m \le l} C^{(m-1)} f\left(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, \ldots, x_{i_m}^{n_{i_m}}\right).$$

The vanishing of $C^{(m-1)} f$ for $m \ge 4$ yields

$$\begin{aligned} f\left(x_1^{n_1} x_2^{n_2} \cdots x_l^{n_l}\right) = {} & \sum_{1 \le i \le l} f\left(x_i^{n_i}\right) + \sum_{1 \le i_1 < i_2 \le l} Cf\left(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}\right) \\ & + \sum_{1 \le i_1 < i_2 < i_3 \le l} C^{(2)} f\left(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}\right). \end{aligned}$$

Therefore, by (2.5) and (2.4), we have

$$\begin{aligned} & f\left(x_i^{n_i}\right) = n_i f(x_i) + \frac{n_i(n_i-1)}{2} Cf(x_i, x_i) + \frac{n_i(n_i-1)(n_i-2)}{6} C^{(2)} f(x_i, x_i, x_i), \\ & C^{(2)} f\left(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}\right) = n_{i_1} n_{i_2} n_{i_3} C^{(2)} f(x_{i_1}, x_{i_2}, x_{i_3}), \end{aligned}$$

which is (2.7). This completes the proof. □

Remark 3 In particular, if $l = 1$, then Proposition 1 holds.

3 Solution on a free group

In this section, we study the solutions on a free group. We first solve (1.5) for the free group G on a single letter a.

Theorem 1 *Let G be the free group on one letter a. Then $f \in \operatorname{Ker} C^{(3)}(G, H)$ if and only if it is given by*

$$f\left(a^n\right) = nf(a) + \frac{n(n-1)}{2} Cf(a, a) + \frac{n(n-1)(n-2)}{6} C^{(2)} f(a, a, a), \quad \forall n \in \mathbb{Z}. \tag{3.1}$$

Proof Necessity. It can be obtained from (2.5) in Lemma 1.

Sufficiency. Taking (3.1) as the definition of f on $G = \langle a\rangle$. By Remark 2, we only need to verify that $C^{(2)}f$ is a homomorphism with respect to each variable and thus f belongs to $\operatorname{Ker} C^{(3)}(G,H)$. Let

$$x = a^m, \qquad y = a^n, \qquad z = a^p$$

be any three elements of G. Then it follows from (1.4) and (3.1) that

$$\begin{aligned}
C^{(2)}f(x,y,z) &= C^{(2)}f\left(a^m,a^n,a^p\right)\\
&= f\left(a^{m+n+p}\right) - f\left(a^{m+n}\right) - f\left(a^{m+p}\right) - f\left(a^{n+p}\right) + f\left(a^m\right) + f\left(a^n\right) + f\left(a^p\right)\\
&= \Bigg[(m+n+p)f(a) + \frac{(m+n+p)(m+n+p-1)}{2}Cf(a,a)\\
&\quad + \frac{(m+n+p)(m+n+p-1)(m+n+p-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad - \Bigg[(m+n)f(a) + \frac{(m+n)(m+n-1)}{2}Cf(a,a)\\
&\quad + \frac{(m+n)(m+n-1)(m+n-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad - \Bigg[(m+p)f(a) + \frac{(m+p)(m+p-1)}{2}Cf(a,a)\\
&\quad + \frac{(m+p)(m+p-1)(m+p-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad - \Bigg[(n+p)f(a) + \frac{(n+p)(n+p-1)}{2}Cf(a,a)\\
&\quad + \frac{(n+p)(n+p-1)(n+p-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad + \Bigg[mf(a) + \frac{m(m-1)}{2}Cf(a,a) + \frac{m(m-1)(m-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad + \Bigg[nf(a) + \frac{n(n-1)}{2}Cf(a,a) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(a,a,a)\Bigg]\\
&\quad + \Bigg[pf(a) + \frac{p(p-1)}{2}Cf(a,a) + \frac{p(p-1)(p-2)}{6}C^{(2)}f(a,a,a)\Bigg].
\end{aligned}$$

By a tedious calculation, we have

$$C^{(2)}f\left(a^m,a^n,a^p\right) = mnpC^{(2)}f(a,a,a),$$

which leads to the result that $C^{(2)}f$ is a homomorphism with respect to each variable. □

At the end of this section, for the free group on an alphabet $\langle\mathscr{A}\rangle$ with $|\mathscr{A}| \geq 2$, we discuss some special solutions of (1.5).

An element $x \in \mathscr{A}$ can be written in the form

$$x = a_1^{n_1}a_2^{n_2}\cdots a_l^{n_l}, \quad \text{where } a_i \in \mathscr{A}, n_i \in \mathbb{Z}. \tag{3.2}$$

For each fixed $a \in \mathscr{A}$ and fixed pair of distinct $a, b \in \mathscr{A}$, define the functions W, W_2, W_3:

$$W(x;a) = \sum_{a_i=a} n_i, \tag{3.3}$$

$$W_2(x;a,b) = \sum_{i<j, a_i=a, a_j=b} n_i n_j, \tag{3.4}$$

$$W_3(x;a,b) = \sum_{i>j, a_i=a, a_j=b} n_i n_j, \tag{3.5}$$

along with (3.2). Referring to [2, 3], the above functions are well defined. Furthermore, they satisfy the following relations:

$$W(xy;a) = W(x;a) + W(y;a), \tag{3.6}$$

$$W_2(x;a,b) = W_3(x;b,a). \tag{3.7}$$

Proposition 2 *For any fixed $a \in \mathscr{A}$ and fixed pair of distinct a, b in $\mathscr{A}$, the following assertions hold*:

(i) $W(\cdot;a)$ *belongs to* $\operatorname{Ker} C^{(3)}(\langle\mathscr{A}\rangle, \mathbb{Z})$;

(ii) $W_2(\cdot;a,b)$ *belongs to* $\operatorname{Ker} C^{(3)}(\langle\mathscr{A}\rangle, \mathbb{Z})$;

(iii) $W_3(\cdot;a,b)$ *belongs to* $\operatorname{Ker} C^{(3)}(\langle\mathscr{A}\rangle, \mathbb{Z})$.

Proof Claim (i) follows from the fact that $x \mapsto W(x;a)$ is a morphism from $\langle\mathscr{A}\rangle$ to $\mathbb{Z}$ by (3.6).

Now we consider assertion (ii). Let x, y, z, w in the free group be written as

$$x = a_1^{r_1} a_2^{r_2} \cdots a_l^{r_l}, \qquad y = b_1^{s_1} b_2^{s_2} \cdots b_p^{s_p},$$
$$z = c_1^{t_1} c_2^{t_2} \cdots c_q^{t_q}, \qquad w = d_1^{m_1} d_2^{m_2} \cdots d_v^{m_v}.$$

Then

$$\begin{aligned}
W_2(xyzw;a,b) = {} & \sum_{i<j, a_i=a, a_j=b} r_i r_j + \sum_{i<j, b_i=a, b_j=b} s_i s_j + \sum_{i<j, c_i=a, c_j=b} t_i t_j + \sum_{i<j, d_i=a, d_j=b} m_i m_j \\
& + \sum_{a_i=a, b_j=b} r_i s_j + \sum_{a_i=a, c_j=b} r_i t_j + \sum_{a_i=a, d_j=b} r_i m_j \\
& + \sum_{b_i=a, c_j=b} s_i t_j + \sum_{b_i=a, d_j=b} s_i m_j + \sum_{c_i=a, d_j=b} t_i m_j, \\
W_2(xyz;a,b) = {} & \sum_{i<j, a_i=a, a_j=b} r_i r_j + \sum_{i<j, b_i=a, b_j=b} s_i s_j + \sum_{i<j, c_i=a, c_j=b} t_i t_j \\
& + \sum_{a_i=a, b_j=b} r_i s_j + \sum_{a_i=a, c_j=b} r_i t_j + \sum_{b_i=a, c_j=b} s_i t_j, \\
W_2(xyw;a,b) = {} & \sum_{i<j, a_i=a, a_j=b} r_i r_j + \sum_{i<j, b_i=a, b_j=b} s_i s_j + \sum_{i<j, d_i=a, d_j=b} m_i m_j \\
& + \sum_{a_i=a, b_j=b} r_i s_j + \sum_{a_i=a, d_j=b} r_i m_j + \sum_{b_i=a, d_j=b} s_i m_j,
\end{aligned}$$

$$\begin{aligned}
W_2(xzw;a,b) &= \sum_{i<j,a_i=a,a_j=b} r_ir_j + \sum_{i<j,c_i=a,c_j=b} t_it_j + \sum_{i<j,d_i=a,d_j=b} m_im_j \\
&\quad + \sum_{a_i=a,c_j=b} r_it_j + \sum_{a_i=a,d_j=b} r_im_j + \sum_{c_i=a,d_j=b} t_im_j, \\
W_2(yzw;a,b) &= \sum_{i<j,b_i=a,b_j=b} s_is_j + \sum_{i<j,c_i=a,c_j=b} t_it_j + \sum_{i<j,d_i=a,d_j=b} m_im_j \\
&\quad + \sum_{b_i=a,c_j=b} s_it_j + \sum_{b_i=a,d_j=b} s_im_j + \sum_{c_i=a,d_j=b} t_im_j, \\
W_2(xy;a,b) &= \sum_{i<j,a_i=a,a_j=b} r_ir_j + \sum_{i<j,b_i=a,b_j=b} s_is_j + \sum_{a_i=a,b_j=b} r_is_j, \\
W_2(xz;a,b) &= \sum_{i<j,a_i=a,a_j=b} r_ir_j + \sum_{i<j,c_i=a,c_j=b} t_it_j + \sum_{a_i=a,c_j=b} r_it_j, \\
W_2(xw;a,b) &= \sum_{i<j,a_i=a,a_j=b} r_ir_j + \sum_{i<j,d_i=a,d_j=b} m_im_j + \sum_{a_i=a,d_j=b} r_im_j, \\
W_2(yz;a,b) &= \sum_{i<j,b_i=a,b_j=b} s_is_j + \sum_{i<j,c_i=a,c_j=b} t_it_j + \sum_{b_i=a,c_j=b} s_it_j, \\
W_2(yw;a,b) &= \sum_{i<j,b_i=a,b_j=b} s_is_j + \sum_{i<j,d_i=a,d_j=b} m_im_j + \sum_{b_i=a,d_j=b} s_im_j, \\
W_2(zw;a,b) &= \sum_{i<j,c_i=a,c_j=b} t_it_j + \sum_{i<j,d_i=a,d_j=b} m_im_j + \sum_{c_i=a,d_j=b} t_im_j, \\
W_2(x;a,b) &= \sum_{i<j,a_i=a,a_j=b} r_ir_j, \qquad W_2(y;a,b) = \sum_{i<j,b_i=a,b_j=b} s_is_j, \\
W_2(z;a,b) &= \sum_{i<j,c_i=a,c_j=b} t_it_j, \qquad W_2(w;a,b) = \sum_{i<j,d_i=a,d_j=b} m_im_j.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&W_2(xyzw;a,b) - W_2(xyz;a,b) - W_2(xyw;a,b) - W_2(xzw;a,b) - W_2(yzw;a,b) \\
&\quad + W_2(xy;a,b) + W_2(xz;a,b) + W_2(xw;a,b) + W_2(yz;a,b) + W_2(yw;a,b) \\
&\quad + W_2(zw;a,b) - W_2(x;a,b) - W_2(y;a,b) - W_2(z;a,b) - W_2(w;a,b) = 0.
\end{aligned}$$

This concludes assertion (ii). Claim (iii) follows from (3.7) directly. □

4 Solution on symmetric group S_n

The symmetric group on a finite set X is the group whose elements are all bijective functions from X to X and whose group operation is that of function composition. If $X = \{1, 2, \dots, n\}$, then it is called the symmetric group of degree n and denoted S_n.

In this section, we consider (1.5) for $G = S_n$.

Lemma 3 *If $f \in \operatorname{Ker} C^{(3)}(S_n, H)$, then*

$$C^{(2)}f(x_1x_2\cdots x_m, y, z) = C^{(2)}f(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(m)}, y, z), \tag{4.1}$$

$$C^{(2)}f(x, y_1y_2\cdots y_m, z) = C^{(2)}f(x, y_{\pi(1)}y_{\pi(2)}\cdots y_{\pi(m)}, z), \tag{4.2}$$

$$C^{(2)}f(x, y, z_1z_2\cdots z_m) = C^{(2)}f(x, y, z_{\pi(1)}z_{\pi(2)}\cdots z_{\pi(m)}), \tag{4.3}$$

for all $x, y, z, x_i, y_i, z_i \in S_n$, $i = 1, 2, \dots, m$, *and all rearrangements* π.

Proof Note that $C^{(2)}f(\cdot, y, z)$ is a homomorphism and H is an abelian group, which yields

$$\begin{aligned} &C^{(2)}f(x_1x_2\cdots x_n, y, z)\\ &\quad = C^{(2)}f(x_1, y, z) + C^{(2)}f(x_2, y, z) + \cdots + C^{(2)}f(x_n, y, z)\\ &\quad = C^{(2)}f(x_{\pi(1)}, y, z) + C^{(2)}f(x_{\pi(2)}, y, z) + \cdots + C^{(2)}f(x_{\pi(n)}, y, z)\\ &\quad = C^{(2)}f(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}, y, z). \end{aligned}$$

This proves (4.1). By a similar procedure, we can also verify (4.2)-(4.3). □

Lemma 4 *Let* τ *be an arbitrary* 2*-cycle in* S_n *and* $f \in \operatorname{Ker} C^{(3)}(S_n, H)$, *then*

$$f(\tau^2) = 0, \tag{4.4}$$

$$Cf(\tau, \tau) = -2f(\tau), \tag{4.5}$$

$$C^{(2)}f(\tau, \tau, \tau) = 4f(\tau), \tag{4.6}$$

$$C^{(3)}f(\tau, \tau, \tau, \tau) = -8f(\tau), \qquad 8f(\tau) = 0. \tag{4.7}$$

Proof It suffices to prove (4.7). The proofs for the rest of the statements are straightforward. Using $\tau^2 = e, f(e) = 0$ and (1.4) we get

$$\begin{aligned} C^{(3)}f(\tau, \tau, \tau, \tau) &= f(\tau^4) - 4f(\tau^3) + 6f(\tau^2) - 4f(\tau)\\ &= -8f(\tau), \end{aligned}$$

which implies that $8f(\tau) = 0$ since $C^{(3)}f = 0$. This completes the proof. □

Lemma 5 *For any* 2*-cycle* σ, τ, υ, *and* $f \in \operatorname{Ker} C^{(3)}(S_n, H)$, *we have*

$$C^{(2)}f(\sigma, \tau, \upsilon) = C^{(2)}f\big((12), (12), (12)\big). \tag{4.8}$$

Proof For any 2-cycle σ, there exists $z \in S_n$ such that $\sigma = z(12)z^{-1}$. Hence, for any $x, y \in S_n$, by (4.3) we have

$$\begin{aligned} C^{(2)}f(x, y, \sigma) &= C^{(2)}f\big(x, y, z(12)z^{-1}\big)\\ &= C^{(2)}f\big(x, y, (12)zz^{-1}\big)\\ &= C^{(2)}f\big(x, y, (12)\big). \end{aligned} \tag{4.9}$$

Similarly,

$$C^{(2)}f(\sigma, x, y) = C^{(2)}f\big((12), x, y\big), \tag{4.10}$$

$$C^{(2)}f(x,\sigma,y) = C^{(2)}f\big(x,(12),y\big). \tag{4.11}$$

In particular, (4.8) follows from (4.9)-(4.11). □

Lemma 6 *Assume that* $Cf(\sigma,\tau) = Cf((12),(12))$ *for every 2-cycle* $\sigma,\tau \in S_n$. *Then for any* $x,y,\beta,\sigma_i \in S_n$, $i = 1,2,\ldots,n$, *rearrangement* π, *where* σ_i, β *are 2-cycles, we have*

$$f(\sigma_1\sigma_2\cdots\sigma_l) = f(\sigma_{\pi(1)}\sigma_{\pi(2)}\cdots\sigma_{\pi(l)}), \tag{4.12}$$

$$f(x\beta y) = f\big(x(12)y\big), \tag{4.13}$$

$$f(\beta) = f\big((12)\big) \tag{4.14}$$

for every $f \in \operatorname{Ker} C^{(3)}(S_n,H)$.

Proof Firstly, for any 2-cycle $\sigma_i \in S_n$, $i = 1,2,\ldots,l$ and rearrangement π, it follows from the assumption $Cf(\sigma,\tau) = Cf((12),(12))$, Proposition 1, and (4.8) that

$$\begin{aligned}
f(\sigma_1\sigma_2\cdots\sigma_l) &= \sum_{i=1}^{l} f(\sigma_i) + \sum_{1\le i<j\le l} Cf(\sigma_i,\sigma_j) + \sum_{1\le i<j<k\le l} C^{(2)}f(\sigma_i,\sigma_j,\sigma_k) \\
&= \sum_{i=1}^{l} f(\sigma_i) + \frac{l(l-1)}{2} Cf\big((12),(12)\big) \\
&\quad + \frac{l(l-1)(l-2)}{6} C^{(2)}f\big((12),(12),(12)\big) \\
&= \sum_{i=1}^{l} f(\sigma_{\pi(i)}) + \frac{l(l-1)}{2} Cf\big((12),(12)\big) \\
&\quad + \frac{l(l-1)(l-2)}{6} C^{(2)}f\big((12),(12),(12)\big) \\
&= f(\sigma_{\pi(1)}\sigma_{\pi(2)}\cdots\sigma_{\pi(l)}),
\end{aligned}$$

which gives (4.12).

On the other hand, for every $x,y,\beta \in S_n$ there exist 2-cycles $\sigma_i,\tau_j,z \in S_n$, $i = 1,\ldots,p$, $j = 1,\ldots,q$, such that $x = \sigma_1\sigma_2\cdots\sigma_p$, $y = \tau_1\tau_2\cdots\tau_q$ and $\beta = z(12)z^{-1}$. Note that $z = \delta_1\delta_2\cdots\delta_r$ for some 2-cycles $\delta_1,\ldots,\delta_r \in S_n$, we have

$$\begin{aligned}
f(x\beta y) &= f\big(\sigma_1\sigma_2\cdots\sigma_p\delta_1\delta_2\cdots\delta_r(12)\delta_r^{-1}\delta_{r-1}^{-1}\cdots\delta_1^{-1}\tau_1\tau_2\cdots\tau_q\big) \\
&= f\big(\sigma_1\sigma_2\cdots\sigma_p(12)\delta_1\delta_2\cdots\delta_r\delta_r^{-1}\delta_{r-1}^{-1}\tau_1\tau_2\cdots\tau_q\big) \\
&= f\big(x(12)y\big)
\end{aligned}$$

by (4.12). In particular, taking $x = y = e$ in (4.13), we obtain (4.14). This completes the proof. □

According to Lemma 6, we give the following main result in this section.

Theorem 2 *Assume that* $Cf(\sigma,\tau) = Cf((12),(12))$ *for every* 2-*cycle* $\sigma,\tau \in S_n$. *Then* $f \in \operatorname{Ker} C^{(3)}(S_n, H)$ *if and only if there is an* $h_0 \in H$ *such that* $8h_0 = 0$ *and*

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is even}, \\ h_0, & \text{if } x \text{ is odd}. \end{cases} \tag{4.15}$$

Proof Necessity. Let $f \in \operatorname{Ker} C^{(3)}(S_n, H)$. Then for any $x \in S_n$, there exist 2-cycles $\alpha_i \in S_n$, $i = 1, 2, \dots, p$, such that $x = \alpha_1\alpha_2 \cdots \alpha_p$. In view of (4.8), (4.14), (4.5)-(4.6), and Proposition 1, we get

$$\begin{aligned} f(x) &= f(\alpha_1\alpha_2\cdots\alpha_p) \\ &= \sum_{i=1}^{p} f(\alpha_i) + \sum_{1\le i<j\le p} Cf(\alpha_i,\alpha_j) + \sum_{1\le i<j<k\le p} C^{(2)}f(\alpha_i,\alpha_j,\alpha_k) \\ &= pf\big((12)\big) + \frac{p(p-1)}{2} Cf\big((12),(12)\big) \\ &\quad + \frac{p(p-1)(p-2)}{6} C^{(2)}f\big((12),(12),(12)\big) \\ &= pf\big((12)\big) + \frac{p^2-p}{2}\big(-2f\big((12)\big)\big) + \frac{p^3-3p^2+2p}{6} \cdot 4f\big((12)\big) \\ &= \left(\frac{2}{3}p^3 - 3p^2 + \frac{10}{3}p\right) f(12). \end{aligned} \tag{4.16}$$

Let $g(p) = \frac{2}{3}p^3 - 3p^2 + \frac{10}{3}p$, we claim that

$$g(p) \in \begin{cases} 8\mathbb{N}, & \text{if } p \text{ is even}, \\ 8\mathbb{N}+1, & \text{if } p \text{ is odd}. \end{cases} \tag{4.17}$$

We first prove the even case. Obviously, (4.17) is true for $p = 2$ since $g(2) = 0$. For an inductive proof, suppose that (4.17) also holds for $p = 2n$, $n \in \mathbf{Z}$. Then we compute that

$$\begin{aligned} g(2n+2) &= \frac{2}{3}(2n+2)^3 - 3(2n+2)^2 + \frac{10}{3}(2n+2) \\ &= \frac{2}{3}\big((2n)^3 + 6(2n)^2 + 24n + 8\big) - 3\big((2n)^2 + 8n + 4\big) + \frac{10}{3}(2n+2) \\ &= \frac{2}{3}(2n)^3 - 3(2n)^2 + \frac{10}{3} \cdot 2n + 16n^2 - 8n \\ &= g(2n) + 16n^2 - 8n, \end{aligned}$$

which yields $g(2n+2) \in 8\mathbb{N}$. This confirms the even case of (4.17). When p is odd, (4.17) is true for $p = 1$ because of $g(1) = 1$. Suppose that (4.17) holds for $p = 2n-1$, and then we get

$$\begin{aligned} g(2n+1) &= \frac{2}{3}(2n+1)^3 - 3(2n+1)^2 + \frac{10}{3}(2n+1) \\ &= \frac{2}{3}\big((2n-1)^3 + 6(2n-1)^2 + 12(2n-1) + 8\big) \\ &\quad - 3\big((2n-1)^2 + 4(2n-1) + 4\big) + \frac{10}{3}(2n-1+2) \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3}(2n-1)^3 - 3(2n-1)^2 + \frac{10}{3}(2n-1) \\
&\quad + 4(2n-1)^2 + 8(2n-1) + \frac{16}{3} - 12(2n-1) - 12 + \frac{20}{3} \\
&= \frac{2}{3}(2n-1)^3 - 3(2n-1)^2 + \frac{10}{3}(2n-1) + 16n^2 - 24n + 8 \\
&= g(2n-1) + 16n^2 - 24n + 8.
\end{aligned}$$

This completes the proof of (4.17). According to (4.17), (4.16) becomes

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is even}, \\ f((12)), & \text{if } x \text{ is odd}. \end{cases}$$

This proves that f must have the form (4.15) with $h_0 = f((12))$.

Sufficiency. Let $f: S_n \to H$ be defined by (4.15), where h_0 is a constant with $8h_0 = 0$. In order to prove the identity of (1.5), by the symmetry of x_1, x_2, x_3, x_4 it suffices to verify the following four cases: case (i) x_1 is odd, and x_2, x_3, x_4 are even; case (ii) x_1, x_2 are odd, and x_3, x_4 are even; case (iii) x_1, x_2, x_3 are odd, and x_4 is even; case (iv) x_1, x_2, x_3 and x_4 are odd.

In fact, for case (i) it is easy to see that $x_1, x_1x_2, x_1x_3, x_1x_4, x_1x_2x_3, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3x_4$ are odd, and $x_2, x_3, x_4, x_2x_3, x_2x_4, x_3x_4, x_2x_3x_4$ are even, which leads to the equality of (1.5). The proofs of the other cases are similar. □

5 Solution on the finite cyclic group C_n

Let $C_n = \langle a \mid a^n = e \rangle$ be a cyclic group of order n with generator a. In this section, we study the general solution on the finite cyclic group C_n.

Theorem 3 *Assume that n is odd and $nCf(a,a) = 0$. Then $f \in \operatorname{Ker} C^{(3)}(C_n, H)$ if and only if it is given by*

$$f(a^p) = pf(a) + \frac{p(p-1)}{2}Cf(a,a) + \frac{p(p-1)(p-2)}{6}C^{(2)}f(a,a,a), \quad \forall p \in \mathbb{Z}, \tag{5.1}$$

where $f(a)$ and $C^{(2)}f(a,a,a)$ satisfy

$$nC^{(2)}f(a,a,a) = 0, \tag{5.2}$$

$$nf(a) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(a,a,a) = 0. \tag{5.3}$$

Proof Necessity. Let $f: C_n \to H$ be a function satisfying (1.5). Then by (2.5), we see that f also satisfies (5.1), *i.e.*,

$$f(a^p) = pf(a) + \frac{p(p-1)}{2}Cf(a,a) + \frac{p(p-1)(p-2)}{6}C^{(2)}f(a,a,a), \quad \forall p \in \mathbb{Z}.$$

Let $p = n$ in (5.1), since $nCf(a,a) = 0$ and $\frac{n(n-1)}{2}$ is an integer, the summand $\frac{n(n-1)}{2}Cf(a,a) = 0$ and by the fact that $a^n = e, f(e) = 0$, we obtain

$$nf(a) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(a,a,a) = 0.$$

Furthermore, by using (2.4), (2.3), and $a^n = e$, we get

$$nC^{(2)}f(a,a,a) = C^{(2)}f\big(a^n,a,a\big) = C^{(2)}f(e,a,a) = 0.$$

This proves (5.2)-(5.3).

Sufficiency. We claim that (5.1)-(5.3) defines a function on C_n. Indeed, for each $p \in \mathbb{Z}$, by (5.1) and (5.3) we have

$$\begin{aligned}
&f\big(a^{p+n}\big) - f(a)\\
&\quad= \left[(p+n)f(a) + \frac{(p+n)(p+n-1)}{2}Cf(a,a)\right.\\
&\qquad\left. + \frac{(p+n)(p+n-1)(p+n-2)}{6}C^{(2)}f(a,a,a)\right]\\
&\qquad - \left[pf(a) + \frac{p(p-1)}{2}Cf(a,a) + \frac{p(p-1)(p-2)}{6}C^{(2)}f(a,a,a)\right]\\
&\quad= \left[nf(a) + \frac{n(n-1)}{2}Cf(a,a) + \frac{n(n-1)(n-2)}{6}C^{(2)}f(a,a,a)\right]\\
&\qquad + pnCf(a,a) + \left(\frac{p(p+n)}{2} - p\right)nC^{(2)}f(a,a,a)\\
&\quad= pnCf(a,a) + \left(\frac{p(p+n)}{2} - p\right)nC^{(2)}f(a,a,a)\\
&\quad= 0,
\end{aligned}$$

where the last identity is obtained because $nCf(a,a) = 0$, (5.2), and n is odd.

Finally, for any $x = a^m$, $y = a^p$, $z = a^q$, and $w = a^l \in C_n$, we have

$$\begin{aligned}
&f(xyzw) - f(xyz) - f(xyw) - f(xzw) - f(yzw)\\
&\qquad + f(xy) + f(xz) + f(xw) + f(yz) + f(yw) + f(zw) - f(x) - f(y) - f(z) - f(w)\\
&\quad= f\big(a^{m+p+q+l}\big) - f\big(a^{m+p+q}\big) - f\big(a^{m+p+l}\big) - f\big(a^{m+q+l}\big) - f\big(a^{p+q+l}\big)\\
&\qquad + f\big(a^{m+p}\big) + f\big(a^{m+q}\big) + f\big(a^{m+l}\big) + f\big(a^{p+q}\big) + f\big(a^{p+l}\big) + f\big(a^{q+l}\big)\\
&\qquad - f\big(a^m\big) - f\big(a^p\big) - f\big(a^q\big) - f\big(a^l\big)\\
&\quad= (m+p+q+l)f(a) + \frac{(m+p+q+l)(m+p+q+l-1)}{2}Cf(a,a)\\
&\qquad + \frac{(m+p+q+l)(m+p+q+l-1)(m+p+q+l-2)}{6}C^{(2)}f(a,a,a)\\
&\qquad - \left[(m+p+q)f(a) + \frac{(m+p+q)(m+p+q-1)}{2}Cf(a,a)\right.\\
&\qquad\left. + \frac{(m+p+q)(m+p+q-1)(m+p+q-2)}{6}C^{(2)}f(a,a,a)\right]\\
&\qquad - \left[(m+p+l)f(a) + \frac{(m+p+l)(m+p+l-1)}{2}Cf(a,a)\right.\\
&\qquad\left. + \frac{(m+p+l)(m+p+l-1)(m+p+l-2)}{6}C^{(2)}f(a,a,a)\right]
\end{aligned}$$

$$
\begin{aligned}
&- \bigg[(m+q+l)f(a) + \frac{(m+q+l)(m+q+l-1)}{2} Cf(a,a) \\
&+ \frac{(m+q+l)(m+q+l-1)(m+q+l-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&- \bigg[(p+q+l)f(a) + \frac{(p+q+l)(p+q+l-1)}{2} Cf(a,a) \\
&+ \frac{(p+q+l)(p+q+l-1)(p+q+l-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(m+p)f(a) + \frac{(m+p)(m+p-1)}{2} Cf(a,a) \\
&+ \frac{(m+p)(m+p-1)(m+p-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(m+q)f(a) + \frac{(m+q)(m+q-1)}{2} Cf(a,a) \\
&+ \frac{(m+q)(m+q-1)(m+q-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(m+l)f(a) + \frac{(m+l)(m+l-1)}{2} Cf(a,a) \\
&+ \frac{(m+l)(m+l-1)(m+l-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(p+q)f(a) + \frac{(p+q)(p+q-1)}{2} Cf(a,a) \\
&+ \frac{(p+q)(p+q-1)(p+q-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(p+l)f(a) + \frac{(p+l)(p+l-1)}{2} Cf(a,a) \\
&+ \frac{(p+l)(p+l-1)(p+l-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&+ \bigg[(q+l)f(a) + \frac{(q+l)(p+l-1)}{2} Cf(a,a) \\
&+ \frac{(q+l)(q+l-1)(q+l-2)}{6} C^{(2)} f(a,a,a) \bigg] \\
&- mf(a) + \frac{m(m-1)}{2} Cf(a,a) - \frac{m(m-1)(m-2)}{6} C^{(2)} f(a,a,a) \\
&- pf(a) + \frac{p(p-1)}{2} Cf(a,a) - \frac{p(p-1)(p-2)}{6} C^{(2)} f(a,a,a) \\
&- qf(a) + \frac{q(q-1)}{2} Cf(a,a) - \frac{q(q-1)(q-2)}{6} C^{(2)} f(a,a,a) \\
&- lf(a) + \frac{l(l-1)}{2} Cf(a,a) - \frac{l(l-1)(l-2)}{6} C^{(2)} f(a,a,a),
\end{aligned}
$$

which, after a long and tedious computation, gives 0. Consequently, $f \in \operatorname{Ker} C^{(3)}(C_n, H)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Aczél, J: Lectures on Functional Equations and Their Applications. Academic Press, New York (1966)
2. Ng, CT: Jensen's functional equation on groups. Aequ. Math. **39**, 85-99 (1990)
3. Ng, CT: Jensen's functional equation on groups II. Aequ. Math. **58**, 311-320 (1999)
4. Baron, K, Kannappan, P: On the Cauchy difference. Aequ. Math. **46**, 112-118 (1993)
5. Brzdęk, J: On the Cauchy difference on normed spaces. Abh. Math. Semin. Univ. Hamb. **66**, 143-150 (1996)
6. Ebanks, B: Generalized Cauchy difference functional equations. Aequ. Math. **70**, 154-176 (2005)
7. Ebanks, B: Generalized Cauchy difference equations, II. Proc. Am. Math. Soc. **136**, 3911-3919 (2008)
8. Ng, CT, Zhao, H: Kernel of the second order Cauchy difference on groups. Aequ. Math. **86**, 155-170 (2013)

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