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Existence and uniqueness of nonlinear deflections of an infinite beam resting on a non-uniform nonlinear elastic foundation

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Abstract

We consider the static deflection of an infinite beam resting on a nonlinear and non-uniform elastic foundation. The governing equation is a fourth-order nonlinear ordinary differential equation. Using the Green's function for the well-analyzed linear version of the equation, we formulate a new integral equation which is equivalent to the original nonlinear equation. We find a function space on which the corresponding nonlinear integral operator is a contraction, and prove the existence and the uniqueness of the deflection in this function space by using Banach fixed point theorem.

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1 Introduction

The topic of the problem of finite or infinite beams which rest on an elastic foundation has received increased attention in a wide range of fields of engineering, because of its practical design applications, say, to highways and railways. The analysis of the problem is thus of interest to many mechanical, civil engineers and, so on: a number of researchers have made their contributions to the problem. For example, from a very early time, the problem of a linear elastic beam resting on a linear elastic foundation and subjected to lateral forces, was investigated by many techniques [1-8].

In contrast to the problem of beams on linear foundation, Beaufait and Hoadley [9] analyzed elastic beams on "nonlinear" foundations. They organized the midpoint difference method for solving the basic differential equation for the elastic deformation of a beam supported on an elastic, nonlinear foundation. Kuo et al. [10] obtained an asymptotic solution depending on a small parameter by applying the perturbation technique to elastic beams on nonlinear foundations.

Recently, Galewski [11] used a variational approach to investigate the nonlinear elastic simply supported beam equation, and Grossinho et al. [12] studied the solvability of an elastic beam equation in presence of a sign-type Nagumo control. With regard to the beam equation, Alves et al. [13] discussed about iterative solutions for a nonlinear fourth-order ordinary differential equation. Jang et al. [14] proposed a new method for the nonlinear deflection analysis of an infinite beam resting on a nonlinear elastic foundation under localized external loads. Although their method appears powerful as a mathematical procedure for beam deflections on nonlinear elastic foundation, in practice, it has a limited applicability: it cannot be applied to a "non-uniform" elastic foundation. Also, their analysis is limited to compact intervals.

Motivated by these limitations, we herein extend the previous study [14] to propose an original method for determining the infinite beam deflection on nonlinear elastic foundation which is no longer uniform in space. In fact, although there are a large number of studies of beams on nonlinear elastic foundation [10,15], most of them are concerned with the uniform foundation; that is, little is known about the non-uniform foundation analysis. This is because the solution procedure for a nonlinear fourth-order ordinary differential equation has not been fully developed. The method proposed in this article does not depend on a small parameter and therefore can overcome the disadvantages and limitations of perturbation expansions with respect to the small parameter. In this article, we derive a new, nonlinear integral equation for the deflection, which is equivalent to the original nonlinear and non-uniform differential equation, and suggest an iterative procedure for its solution: a similar iterative technique was previously proposed to obtain the nonlinear Stokes waves [14,16-19]. Our basic tool is Banach fixed point theorem [20], which has many applications in diverse areas. One difficulty here is that the integral operator concerning the iterative procedure is not a contraction in general for the case of infinite beam. We overcome this by finding out a suitable subspace inside the whole function space, wherein our integral operator becomes a contraction. Inside this subspace, we then prove the existence and the uniqueness of the deflection of an infinite beam resting on a both non-uniform and nonlinear elastic foundation by means of Banach fixed point theorem. In fact, this restriction on the candidate space for solutions is justified by physical considerations.

The rest of the article is organized as follows: in Section 2, we describe our problem in detail, and formulate an integral equation equivalent to the nonlinear and non-uniform beam equation. The properties of the nonlinear, non-uniform elastic foundation are analyzed in Section 3, and a close investigation on the basic integral operator $\mathcal{K}$, which has an important role in both linear and nonlinear beam equations, is performed in Section 4. In Section 5, we define the subspace on which our integral operator Ψ becomes a contraction, and show the existence and the uniqueness of the solution in this space. Finally, Section 6 recapitulates the overall procedure of the article, and explains some of the intuitions behind our formulation for the reader.

2 Definition of the problem

We deal with the question of existence and uniqueness of solutions of nonlinear deflections for an infinitely long beam resting on a nonlinear elastic foundation which is non-uniform in x. Figure 1 shows that the vertical deflection of the beam $u(x)$ results from the net load distribution $p(x)$:

$$p(x) = w(x) - f(u, x) \tag{1}$$

In (1), the two variable function $f(u, x)$ is the nonlinear spring force upward, which depends not only on the beam deflection u but also on the position x, and $w(x)$ denotes the applied loading downward. For simplicity, the weight of the beam is neglected. In fact, the weight of the beam could be incorporated in our static beam

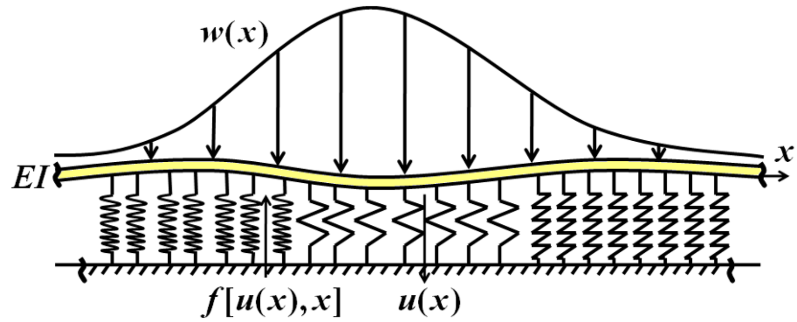

Figure 1 Infinite beam on nonlinear and non-uniform elastic foundation.

deflection problem by adding $m(x)g$ to the loading $w(x)$, where $m(x)$ is the lengthwise mass density of the beam in x-coordinate, and g is the gravitational acceleration. The term $m(x)g$ also plays an important role in the dynamic beam problem, since the second-order time derivative term of deflection must be included as $\mathrm{d}/\mathrm{d}t(m(x)\mathrm{d}u/\mathrm{d}t)$ in the motion equation. Denoting by EI the flexural rigidity of the beam (E and I are Young's modulus and the mass moment of inertia, respectively), the vertical deflection $u(x)$, according to the classical Euler beam theory, is governed by a fourth-order ordinary differential equation

$$EI\frac{\mathrm{d}^4u}{\mathrm{d}x^4} = p(x),$$

which, in turn, becomes the following nonlinear differential equation for the deflection u by (1):

$$EI\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + f(u, x) = w(x). \tag{2}$$

The boundary condition that we consider is

$$\lim_{x\to\pm\infty} u(x) = \lim_{x\to\pm\infty} u'(x) = 0. \tag{3}$$

Note that (2) and (3) together form a well-defined boundary value problem.

We shall attempt to seek a nonlinear integral equation, which is equivalent to the nonlinear differential equation (2). We start with a simple modification made on (2) by introducing an artificial linear spring constant k: (2) is rewritten as

$$EI\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + ku + N(u, x) = w(x), \tag{4}$$

where

$$f(u, x) = ku + N(u, x),$$

or

$$EI\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + ku = w(x) - N(u, x) \equiv \Phi(u, x). \tag{5}$$

The exact determination of k out of the function $f(u, x)$ will be given in Section 3. The modified differential equation (5) is a starting point to the formulation of a nonlinear integral equation equivalent to the original equation (2). For this, we first recall that the linear solution of (2), which corresponds to the case $N(u, x) \equiv 0$ in (4), was derived by Timoshenko [21], Kenney [8], Saito et al. [22], Fryba [23]. They used the Fourier and Laplace transforms to obtain a closed-form solution:

$$u(x) = \int_{-\infty}^{\infty} G(x, \xi) w(\xi) \mathrm{d}\xi, \tag{6}$$

expressed in terms of the following Green's function G:

$$G(x, \xi) = \frac{\alpha}{2k} \exp\left(-\frac{\alpha|\xi - x|}{\sqrt{2}}\right) \sin\left(\frac{\alpha|\xi - x|}{\sqrt{2}} + \frac{\pi}{4}\right), \tag{7}$$

where $\alpha = \sqrt[4]{k/EI}$. A localized loading condition was assumed in the derivation of (6): u, u', u'', and u''' all tend toward zero as $|x| \to \infty$. Green's functions such as (7) play a crucial role in the solution of linear differential equations, and are a key component to the development of integral equation methods. We utilize the Green's function (7) and the solution (6) as a framework for setting up the following nonlinear relations for the case of $N(u, x) \neq 0$:

$$u(x) = \int_{-\infty}^{\infty} G(x, \xi) \Phi(u(\xi), \xi) \, \mathrm{d}\xi. \tag{8}$$

With the substitution of (5), (8) immediately reveals the following nonlinear Fredholm integral equation for u:

$$u(x) = \int_{-\infty}^{\infty} G(x, \xi) w(x) \, \mathrm{d}\xi - \int_{-\infty}^{\infty} G(x, \xi) N(u(\xi), \xi) \, \mathrm{d}\xi. \tag{9}$$

Physically, the term $\int_{-\infty}^{\infty} G(x, \xi) w(x) \, \mathrm{d}\xi$ in (9) amounts to the linear deflection of an infinite beam on a linear elastic foundation having the artificial linear spring constant k, which is uniform in x. The term $-\int_{-\infty}^{\infty} G(x, \xi) N(u(\xi), \xi) \, \mathrm{d}\xi$ in (9) corresponds to the difference between the exact nonlinear solution u and the linear deflection $\int_{-\infty}^{\infty} G(x, \xi) w(x) \, \mathrm{d}\xi$. We define the nonlinear operator Ψ by

$$\Psi[u](x) := \int_{-\infty}^{\infty} G(x, \xi) w(x) \, \mathrm{d}\xi - \int_{-\infty}^{\infty} G(x, \xi) N(u(\xi), \xi) \, \mathrm{d}\xi \tag{10}$$

for functions $u : \mathbb{R} \to \mathbb{R}$. Then the integral equation (9) becomes just $\Psi[u] = u$, which is the equation for fixed points of the operator Ψ. We will show in exact sense the equivalence between (2) and (9) in Lemma 7 in Section 5.

3 Assumptions on *f* and the operator $\mathcal{N}$

Denote $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$ for $u : \mathbb{R} \to \mathbb{R}$, and let $L^\infty(\mathbb{R})$ be the space of all functions $u : \mathbb{R} \to \mathbb{R}$ such that $\|u\|_\infty < \infty$. Let $C_0(\mathbb{R})$ be the space of all continuous functions vanishing at infinity. It is well known [24] that $C_0(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are Banach spaces with the norm $\|\cdot\|_\infty$, and $C_0(\mathbb{R}) \subset L^\infty(\mathbb{R})$. For $q = 0, 1, 2, \ldots$, let $C^q(\mathbb{R})$ be the space of q times differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Here, $C^0(\mathbb{R})$ is just the space of continuous functions $C(\mathbb{R})$.

We have a few assumptions on $f(u, x)$ and $w(x)$. There are four assumptions **F1**, **F2**, **F3**, **F4** on f, and two **W1**, **W2** on w. As one can find out soon, they are general enough, and have natural physical meanings. In this section, we list the assumptions on f. Those on w will appear in Section 5.1.

(**F1**) $f(u, x)$ is sufficiently differentiable, so that $f(u(x), x) \in C^q(\mathbb{R})$ if $u \in C^q(\mathbb{R})$ for $q = 0, 1, 2, \ldots$.

(**F2**) $f(u, x) \cdot u \geq 0$, and $f_u(u, x) \geq 0$ for every $u, x \in \mathbb{R}$.

(**F3**) For every $\upsilon \geq 0, \sup_{x\in\mathbb{R},|u|\leq\upsilon} \left|\dfrac{\partial^q f}{\partial u^q}(u,x)\right| < \infty$ for $q = 0, 1, 2$.

(**F4**) $\inf_{x\in\mathbb{R}} f_u(0, x) > \eta_0 \sup_{x\in\mathbb{R}} f_u(0, x)$, where

$$\eta_0 = \frac{\sqrt{2}\exp\left(-\frac{3\pi}{4}\right)}{1-\exp(-\pi)+\sqrt{2}\exp\left(-\frac{3\pi}{4}\right)} \approx 0.123.$$

Note first that **F1** will free us of any unnecessary consideration for differentiability, and in fact, $f(u, x)$ is usually infinitely differentiable in most applications. **F2** means that the elastic force of the elastic foundation, represented by $f(u, x)$, is *restoring*, and increases in magnitude as does the amount of the deflection u. **F3** also makes sense physically: The case $q = 0$ implies that, within the same amount of deflection $u < |v|$, the restoring force $f(u, x)$, though non-uniform, cannot become arbitrarily large. Note that $f_u(u, x) \geq 0$ is the linear approximation of the spring constant (infinitesimal with respect to x) of the elastic foundation at (u, x). Hence, the case $q = 1$ means that this non-uniform spring constant $f_u(u, x)$ be bounded within a finite deflection $|u| < v$. Although the case $q = 2$ of **F3** does not have obvious physical interpretation, we can check later that it is in fact satisfied in usual situations.

Especially, **F3** enables us to define the constant k:

$$k := \sup_{x\in\mathbb{R}} f_u(0,x). \tag{11}$$

We justifiably rule out the case $k = 0$; hence, we assuming $k > 0$ for the rest of the article. Define

$$N(u,x) := f(u,x) - ku, \tag{12}$$

which is the nonlinear and non-uniform part of the restoring force $f(u, x) = ku + N(u, x)$. Finally, **F4** implies that, for any $x \in \mathbb{R}$, the spring constant $f_u(0, x)$ at $(0, x)$ cannot become smaller than about 12.3% of the maximum spring constant $k = \sup_{x\in\mathbb{R}} f_u(0, x)$. This restriction, which is realistic, comes from the unfortunate fact that the operator $\mathcal{K}$ in Section 4 is not a contraction. The constant η_0 is related to another constant τ, which will be introduced later in (41) in Section 4, by

$$\eta_0 = \frac{\tau - 1}{\tau}. \tag{13}$$

We define a parameter η which measures the non-uniformity of the elastic foundation:

$$\eta := \frac{\inf_{x\in\mathbb{R}} f_u(0,x)}{\sup_{x\in\mathbb{R}} f_u(0,x)} = \frac{\inf_{x\in\mathbb{R}} f_u(0,x)}{k}. \tag{14}$$

Then, by **F4**, we have

$$\eta_0 < \eta \leq 1. \tag{15}$$

A uniform elastic foundation corresponds to the extreme case $\eta = 1$, and the non-uniformity increases as η becomes smaller. In order for our current method to work, the condition **F4** limits the non-uniformity η by $\eta_0 \approx 0.123$.

Using the function N, we define the operator $\mathcal{N}$ by $\mathcal{N}[u](x) := N(u(x), x)$ for functions $u : \mathbb{R} \to \mathbb{R}$. Note that $\mathcal{N}$ is nonlinear in general.

Lemma 1. (a) $\mathcal{N}[u] \in C_0(\mathbb{R})$ *for every* $u \in C_0(\mathbb{R})$.

(b) *For every* $u, v \in L^\infty(\mathbb{R})$, *we have*

$$||\mathcal{N}[u] - \mathcal{N}[v]||_\infty \leq \{(1-\eta)k + \rho(\max\{||u||_\infty, ||v||_\infty\})\} \cdot ||u - v||_\infty$$

for some strictly increasing continuous function $\rho : [0, \infty) \to [0, \infty)$, *such that* $\rho(0) = 0$.

Proof. Suppose $u \in C_0(\mathbb{R})$. $\mathcal{N}[u]$ is continuous by **F1**. Let $\epsilon > 0$. Then there exists $M > 0$ such that $|u(x)| < \epsilon$ if $|x| > M$, since $\lim_{x\to\pm\infty} u(x) = 0$. By the mean value theorem, we have

$$\mathcal{N}[u](x) = N(u(x), x) = f(u(x), x) - ku(x) = f_u(\mu, x) \cdot \{u(x) - 0\} - ku(x),$$

for some μ between 0 and $u(x)$, and hence $|\mu| \leq |u(x)| < \epsilon$ if $|x| > M$. Hence, for $|x| > M$, we have

$$\begin{aligned} |\mathcal{N}[u](x)| = |f_u(\mu, x)u(x) - ku(x)| &\leq \{f_u(\mu, x) + k\} \cdot |u(x)| \\ &\leq \left\{k + \sup_{x\in\mathbb{R}, |\mu|\leq\varepsilon} f_u(\mu, x)\right\} \varepsilon. \end{aligned} \tag{16}$$

Note that (16) can be made arbitrarily small as M gets larger, since $\sup_{x\in\mathbb{R},\ |\mu|\leq\epsilon} f_u(\mu, x) < \infty$ by **F3**. Thus, $\mathcal{N}[u] \in C_0(\mathbb{R})$, which proves (a).

By the mean value theorem, we have

$$N(u, x) - N(v, x) = N_u(\mu, x) \cdot (u - v)$$

for some μ between u and v, and hence $|\mu| \leq \max\{|u|, |v|\}$. Hence,

$$|N(u, x) - N(v, x)| \leq \sup_{|\mu|\leq\max\{|u|,|v|\}} |N_u(\mu, x)| \cdot |u - v|.$$

Now suppose $u, v \in L^\infty(\mathbb{R})$. Then

$$\begin{aligned} ||\mathcal{N}[u] - \mathcal{N}[v]||_\infty &= \sup_{x\in\mathbb{R}} |N(u(x), x) - N(v(x), x)| \\ &\leq \sup_{x\in\mathbb{R}} \left\{ \sup_{|\mu|\leq\max\{|u(x)|,|v(x)|\}} |N_u(\mu, x)| \cdot |u(x) - v(x)| \right\} \\ &\leq \sup_{x\in\mathbb{R}} \left\{ \sup_{|\mu|\leq\max\{|u(x)|,|v(x)|\}} |N_u(\mu, x)| \right\} \cdot \sup_{x\in\mathbb{R}} |u(x) - v(x)| \\ &\leq \left\{ \sup_{x\in\mathbb{R}, |\mu|\leq\max\{||u||_\infty, ||v||_\infty\}} |N_u(\mu, x)| \right\} \cdot ||u - v||_\infty. \end{aligned} \tag{17}$$

Put

$$\rho_1(t) := \sup_{x\in\mathbb{R},|\mu|\le t} |N_u(\mu, x)|, \quad t \ge 0. \tag{18}$$

Note that (18) is well-defined by **F3**, since we have $N_u(\mu, x) = f_u(\mu, x) - k$ from (12). Clearly, ρ_1 is non-negative and non-decreasing.

We want to show ρ_1 is continuous. Fix $t_0 \ge 0$. We first show the left-continuity of ρ_1 at t_0. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0, t_0)$ such that $t_n \nearrow t_0$. Suppose there exists $t' < t_0$ such that $\rho_1(t') = \rho_1(t_0)$. Then, since ρ_1 is non-decreasing, it becomes constant on $[t', t_0]$, and hence ρ_1 is clearly left-continuous at t_0. So we assume that $\rho_1(t') < \rho_1(t_0)$ for every $t' < t_0$. It follows that there exists a sequence $\{(\mu_n, x_n)\}_{n=1}^{\infty}$ in $[-t_0, t_0] \times \mathbb{R}$, such that $|\mu_n| = t_n$ and $|N_u(\mu_n, x_n)| \to \rho_1(t_0)$ as $n \to \infty$, since $|N_u(u, x)|$ is continuous. Thus, we have $\rho_1(t_n) \to \rho_1(t_0)$ as $n \to \infty$, since $|N_u(\mu_n, x_n)| \le \rho_1(t_n) \le \rho_1(t_0)$ for $n = 1, 2, \ldots$. This shows that ρ_1 is left-continuous at t_0.

Suppose ρ_1 is not right-continuous at t_0. Then there exist $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0, ∞), such that $t_n \searrow t_0$ and $\rho_1(t_n) - \rho_1(t_0) \ge \epsilon$ for $n = 1, 2, \ldots$. Suppose there exists $t' > t_0$ such that $\rho_1(t') = \rho_1(t_0)$. Then ρ_1 becomes constant on $[t_0, t']$, so that ρ_1 is right-continuous at t_0. So we assume that $\rho_1(t') > \rho_1(t_0)$ for every $t' > t_0$. It follows that there exists a sequence $\{(\mu_n, x_n)\}_{n=1}^{\infty}$ in $\{(t_0, \infty) \cup (-\infty, -t_0)\} \times \mathbb{R}$, such that $t_0 < |\mu_n| \le t_n$ and $|N_u(\mu_n, x_n)| > \rho_1(t_n) - \frac{\varepsilon}{2^n} > \rho_1(t_0)$ for $n = 1, 2, \ldots$, since $|N_u(u, x)|$ is continuous. With no loss of generality, we can assume $\mu_n > 0$. By the mean value theorem, we have

$$N_u(\mu_n, x_n) - N_u(t_0, x_n) = N_{uu}(\mu, x_n) \cdot (\mu_n - t_0)$$

for some μ between t_0 and μ_n, and so we have

$$\begin{aligned}
\varepsilon &\le \rho_1(t_n) - \rho_1(t_0) \\
&= \{\rho_1(t_n) - |N_u(\mu_n, x_n)|\} + \{|N_u(\mu_n, x_n)| - |N_u(t_0, x_n)|\} + \{|N_u(t_0, x_n)| - \rho_1(t_0)\} \\
&\le \{\rho_1(t_n) - |N_u(\mu_n, x_n)|\} + \{|N_u(\mu_n, x_n)| - |N_u(t_0, x_n)|\} \\
&\le \{\rho_1(t_n) - |N_u(\mu_n, x_n)|\} + |N_u(\mu_n, x_n) - N_u(t_0, x_n)| \\
&< \frac{\varepsilon}{2^n} + |N_{uu}(\mu, x_n)| \cdot |\mu_n - t_0|
\end{aligned} \tag{19}$$

for $n = 1, 2, \ldots$. By **F3**, (19) goes to 0 as $n \to \infty$, since

$$|N_{uu}(\mu, x_n)| \cdot |\mu_n - t_0| \le \sup_{x\in\mathbb{R},|\mu|\le t_1} |N_{uu}(\mu, x)| \cdot |t_n - t_0|.$$

This is a contradiction. It follows that ρ_1 is right-continuous, and thus, is continuous.

By (11) and (14), we have $\eta k \le f_u(0, x) \le k$, and so $-(1 - \eta) k \le f_u(0, x) - k \le 0$ for every $x \in \mathbb{R}$. It follows that

$$\rho_1(0) = \sup_{x\in\mathbb{R},|\mu|\le 0} |N_u(\mu, x)| = \sup_{x\in\mathbb{R}} |N_u(0, x)| = \sup_{x\in\mathbb{R}} |f_u(0, x) - k| \le (1 - \eta)k.$$

Put $\rho_2(t) := \rho_1(t) - \rho_1(0)$. Then ρ_2 is a nondecreasing continuous function such that $\rho_2(0) = 0$. By Lemma 2 below, there exists a strictly increasing continuous function ρ such that $\rho(0) = 0$, and $\rho(t) \ge \rho_2(t)$ for $t \ge 0$. Thus, we have a desired function ρ, since

$$\begin{aligned}||\mathcal{N}[u]-\mathcal{N}[v]||_\infty &\le \rho_1(\max\{||u||_\infty, ||v||_\infty\})\cdot||u-v||_\infty\\ &\le \{\rho_1(0)+\rho_2(\max\{||u||_\infty,||v||_\infty\})\}\cdot||u-v||_\infty\\ &\le \{(1-\eta)k+\rho(\max\{||u||_\infty,||v||_\infty\})\}\cdot||u-v||_\infty,\end{aligned}$$

where the first inequality is from (17) and (18). This proves (b), and the proof is complete.

Lemma 2. *Let* $g:[0,\infty)\to[0,\infty)$ *be a non-decreasing continuous function such that* $g(0)=0$. *Then there exists a strictly increasing continuous function* $\tilde{g}:[0,\infty)\to[0,\infty)$ *such that* $\tilde{g}(0)=0$, *and* $\tilde{g}(t)\ge g(t)$ *for* $t\ge 0$.

Proof. Note that, for every $s\in[0,\infty)$, $g^{-1}(s)$is a compact connected subset of $[0,\infty)$, since g is continuous and non-decreasing. It follows that $g^{-1}(s)$ is either a point or a closed interval in $[0,\infty)$ for every $s\in[0,\infty)$. Let A be the set of all points in $[0,\infty)$ at which g is locally constant, i.e.,

$$A=\{t\in[0,\infty)|g^{-1}(g(t))\text{ is an interval with non-zero length}\}.$$

Define $\tilde{g}:[0,\infty)\to[0,\infty)$ by

$$\tilde{g}(t):=g(t)+l(A\cap[0,t]),\quad t\ge 0,$$

where $l(B)$ is the Lebesque measure, and hence the length in our case, of the set $B\subset[0,\infty)$. From the definition of $\tilde{g}$, it is clear that $\tilde{g}(0)=0$, and $\tilde{g}(t)\ge g(t)$ for $t\ge 0$. We omit the proof that $\tilde{g}$ is continuous and strictly increasing, which is an easy exercise.

Example 1. Let

$$f(u,x)=(1+\varepsilon\cos x)\left(\frac{k}{1+\varepsilon}u+\lambda u^{2n+1}\right),\quad 0\le\varepsilon\le\frac{1}{2},\ \ n\ge 1.$$

Then,

$$f_u(u,x)=\frac{1+\varepsilon\cos x}{1+\varepsilon}k+\lambda(2n+1)(1+\varepsilon\cos x)u^{2n},$$

and hence,

$$\frac{1-\varepsilon}{1+\varepsilon}k\le f_u(0,x)\le k,\quad \eta=\frac{1-\varepsilon}{1+\varepsilon}.$$

We also have

$$\begin{aligned}N(u,x)&=f(u,x)-ku=-\frac{k\varepsilon}{1+\varepsilon}(1-\cos x)u+\lambda(1+\varepsilon\cos x)u^{2n+1},\\ N_u(u,x)&=-\frac{k\varepsilon}{1+\varepsilon}(1-\cos x)+\lambda(2n+1)(1+\varepsilon\cos x)u^{2n},\\ |N_u(u,x)|&\le\frac{2\varepsilon}{1+\varepsilon}k+\lambda(2n+1)|1+\varepsilon\cos x|\cdot|u|^{2n}\\ &\le(1-\eta)k+2(2n+1)\lambda\cdot|u|^{2n}.\end{aligned}$$

Thus, we can take $\rho(t)=\rho_2(t)=2(2n+1)\lambda t^{2n}$.

Example 2. Let

$$f(u,x) = (1 + \varepsilon\cos x)\left[\frac{k}{1+\varepsilon}u + \lambda\{\exp(au) - 1 - au\}\right], \quad 0 \le \varepsilon \le \frac{1}{2}, \quad a > 0.$$

Then,

$$f_u(u,x) = \frac{1+\varepsilon\cos x}{1+\varepsilon}k + a\lambda(1+\varepsilon\cos x)\{\exp(au) - 1\},$$

and hence,

$$\frac{1-\varepsilon}{1+\varepsilon}k \le f_u(0,x) \le k, \qquad \eta = \frac{1-\varepsilon}{1+\varepsilon}.$$

We also have

$$\begin{aligned}
N(u,x) &= f(u,x) - ku = -\frac{k\varepsilon}{1+\varepsilon}(1-\cos x)u + \lambda(1+\varepsilon\cos x)\{\exp(au) - 1 - au\},\\
N_u(u,x) &= -\frac{k\varepsilon}{1+\varepsilon}(1-\cos x) + a\lambda(1+\varepsilon\cos x)\{\exp(au) - 1\},\\
|N_u(u,x)| &\le \frac{2\varepsilon}{1+\varepsilon}k + a\lambda(1+\varepsilon)\cdot\{\exp(au) - 1\}\\
&\le (1-\eta)k + 2a\lambda\cdot\{\exp(au) - 1\}.
\end{aligned}$$

Thus, we can take $\rho(t) = \rho_2(t) = 2a\lambda\ \{\exp(at) - 1\}$.

Example 3. As an extreme case, we take $f(u, x) = ku$, for which the original differential equation (2) becomes linear. Clearly, $\eta = 1$. Since $N(u, x) = N_u(u, x) \equiv 0$, we have $\rho_2(t) \equiv 0$. The function ρ taken according to Lemma 2 would be $\rho(t) = t$. However, a better choice is

$$\rho(t) = \sigma k\left(1 - \frac{1}{\sqrt{1+\sigma^2k^2t}}\right), \tag{20}$$

as we will check in Section 5.1, where the constant σ is defined as well.

4 The Operator $\mathcal{K}$

Let

$$K(y) := \frac{\alpha}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\sin\left(\frac{\alpha}{\sqrt{2}}y + \frac{\pi}{4}\right),$$

so that $G(x, \xi) = K(|\xi - x|)$ for G in (7). Using the function K, we define the linear operator $\mathcal{K}$ by

$$\mathcal{K}[u](x) := \int_{-\infty}^{\infty} K(|x-\xi|)u(\xi)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} G(x,\xi)u(\xi)\,\mathrm{d}\xi$$

for functions $u : \mathbb{R} \to \mathbb{R}$. With this notation, we can rewrite the solution u in (6) of the following linear differential equation:

$$EI\frac{d^4u(x)}{dx^4} + ku(x) = w(x), \tag{21}$$

which is just the linear case of (2), as $u = \mathcal{K}[w]$. In fact, understanding the properties of the operator $\mathcal{K}$ is important not only for the linear case (21), but also for the general nonlinear non-uniform case (2).

Lemma 3.

$$K^{(i)}(y) = \frac{\alpha^{i+1}}{2k} \exp\left(-\frac{\alpha}{\sqrt{2}}y\right) \sin\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)\pi}{4}\right\}, \quad i = 0, 1, 2, \ldots.$$

Proof. We use induction on i. Note first that the case $i = 0$ is trivially true. Suppose that the statement is true for some $i \geq 0$. Using the following trigonometric equality

$$-\sin t + \cos t = \sqrt{2}\left\{\sin t \cos\frac{3\pi}{4} + \cos t \sin\frac{3\pi}{4}\right\} = \sqrt{2}\sin\left(t + \frac{3\pi}{4}\right),$$

we have

$$\begin{aligned}
K^{(i+1)}(y) &= \frac{\mathrm{d}}{\mathrm{d}y}K^{(i)}(y) \\
&= \frac{\alpha^{i+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}\right)\cdot\left(-\frac{\alpha}{\sqrt{2}}\right)\cdot\sin\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)\pi}{4}\right\} \\
&\quad + \frac{\alpha^{i+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\cdot\cos\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)\pi}{4}\right\}\cdot\frac{\alpha}{\sqrt{2}} \\
&= \frac{\alpha^{i+2}}{2\sqrt{2}k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\left\{-\sin\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)}{4}\pi\right\} + \cos\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)}{4}\pi\right\}\right\} \\
&= \frac{\alpha^{i+2}}{2\sqrt{2}k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\cdot\sqrt{2}\sin\left\{\frac{\alpha}{\sqrt{2}}y + \frac{(3i+1)\pi}{4} + \frac{3\pi}{4}\right\} \\
&= \frac{\alpha^{(i+1)+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\sin\left\{\frac{\alpha}{\sqrt{2}}y + \frac{\{3(i+1)+1\}\pi}{4}\right\},
\end{aligned}$$

which shows that the statement is true for $i + 1$. Thus, we have the proof.

Using Lemma 3, we can obtain more detailed information on the derivatives of $\mathcal{K}[u]$. Note that, for every $u \in L^\infty(\mathbb{R})$,

$$\begin{aligned}
\mathcal{K}[u](x) &= \int_{-\infty}^{x} K(x-\xi)u(\xi)\,\mathrm{d}\xi + \int_{x}^{\infty} K(\xi - x)u(\xi)\,\mathrm{d}\xi \\
&= -\int_{\infty}^{0} K(y)u(x-y)\,\mathrm{d}y + \int_{0}^{\infty} K(y)u(x+y)\,\mathrm{d}y \qquad (22)\\
&= \int_{0}^{\infty} K(y)\{u(x-y) + u(x+y)\}\,\mathrm{d}y.
\end{aligned}$$

Lemma 4. (a) *Let* $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then we have*

$$\mathcal{K}[u]^{(i)}(x) = \int_0^\infty K^{(i)}(y)\{u(x-y) + (-1)^i u(x+y)\}\,\mathrm{d}y, \quad i = 1, 2, 3,$$

$$\begin{aligned}
\mathcal{K}[u]^{(4)}(x) &= \int_0^\infty K^{(4)}(y)\{u(x-y) + u(x+y)\}\mathrm{d}y + 2K^{(3)}(0)u(x) \\
&= -\alpha^4\mathcal{K}[u](x) + \frac{\alpha^4}{k}u(x).
\end{aligned}$$

Consequently, $\mathcal{K}[u] \in C^4(\mathbb{R})$ *for every* $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(b) *Let* $q = 0, 1, 2, \ldots$. *Suppose* $u \in C^q(\mathbb{R})$ *and* $u^{(i)} \in L^\infty(\mathbb{R})$ *for* $i = 0, 1, \ldots, q$. *Then we have* $\mathcal{K}[u^{(q)}] = \mathcal{K}[u]^{(q)}$.

Proof. Let $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a function $U \in C^1(\mathbb{R})$ such that $U' = u$. Since $u \in L^\infty(\mathbb{R})$, U has at most linear growth, and hence by Lemma 3,

$$\lim_{y\to\infty} K^{(i)}(y)U(x-y) = \lim_{y\to\infty} K^{(i)}(y)U(x+y) = 0 \tag{23}$$

for $i = 0, 1, 2, \ldots$. Using integration by parts, (22) becomes

$$\begin{aligned}\mathcal{K}[u](x) &= [K(y)\{-U(x-y)+U(x+y)\}]_0^\infty - \int_0^\infty K'(y)\{-U(x-y)+U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K'(y)\{U(x-y)-U(x+y)\}\,\mathrm{d}y,\end{aligned}$$

by (23), and hence we have

$$\begin{aligned}\mathcal{K}[u]'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty K'(y)\{U(x-y)-U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K'(y)\{u(x-y)-u(x+y)\}\,\mathrm{d}y.\end{aligned} \tag{24}$$

By (23) and integration by parts again, (24) becomes

$$\begin{aligned}\mathcal{K}[u]'(x) &= [K'(y)\{-U(x-y)-U(x+y)\}]_0^\infty - \int_0^\infty K''(y)\{-U(x-y)-U(x+y)\}\,\mathrm{d}y\\ &= 2K'(0)U(x) + \int_0^\infty K''(y)\{U(x-y)+U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K''(y)\{U(x-y)+U(x+y)\}\,\mathrm{d}y,\end{aligned}$$

since $K'(0) = 0$ by Lemma 3. Hence,

$$\begin{aligned}\mathcal{K}[u]''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty K''(y)\{U(x-y)+U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K''(y)\{u(x-y)+u(x+y)\}\,\mathrm{d}y.\end{aligned} \tag{25}$$

Again by (23) and integration by parts, (25) becomes

$$\begin{aligned}\mathcal{K}[u]''(x) &= [K''(y)\{-U(x-y)+U(x+y)\}]_0^\infty - \int_0^\infty K^{(3)}(y)\{-U(x-y)+U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K^{(3)}(y)\{U(x-y)-U(x+y)\}\,\mathrm{d}y,\end{aligned}$$

and hence,

$$\begin{aligned}\mathcal{K}[u]^{(3)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty K^{(3)}(y)\{U(x-y)-U(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K^{(3)}(y)\{u(x-y)-u(x+y)\}\,\mathrm{d}y.\end{aligned} \tag{26}$$

Once more by (23) and integration by parts, (26) becomes

$$\begin{aligned}\mathcal{K}[u]^{(3)}(x) &= [K^{(3)}(y)\{-U(x-y)-U(x+y)\}]_0^\infty - \int_0^\infty K^{(4)}(y)\{-U(x-y)-U(x+y)\}\,\mathrm{d}y\\ &= 2K^{(3)}(0)U(x) + \int_0^\infty K^{(4)}(y)\{U(x-y)+U(x+y)\}\,\mathrm{d}y,\end{aligned}$$

and hence, by (22),

$$\begin{aligned}\mathcal{K}[u]^{(4)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^\infty K^{(4)}(y)\{U(x-y)+U(x+y)\}\,\mathrm{d}y + 2K^{(3)}(0)U(x)\right]\\ &= \int_0^\infty K^{(4)}(y)\{u(x-y)+u(x+y)\}\,\mathrm{d}y + 2K^{(3)}(0)u(x)\end{aligned} \tag{27}$$

$$= -\alpha^4\mathcal{K}[u](x) + \frac{\alpha^4}{k}u(x), \tag{28}$$

since $K^{(3)}(0) = \frac{\alpha^4}{2k}$ and $K^{(4)}(y) = -\alpha^4 K(y)$ by Lemma 3. Thus (a) follows from (24), (25), (26), (27), and (28).

From (22), we have

$$\begin{aligned}\mathcal{K}[u]'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty K(y)\{u(x-y)+u(x+y)\}\,\mathrm{d}y\\ &= \int_0^\infty K(y)\{u'(x-y)+u'(x+y)\}\,\mathrm{d}y\end{aligned} \tag{29}$$

for every $u \in C^1(\mathbb{R})$ with $u, u' \in L^\infty(\mathbb{R})$. Suppose now $u \in C^q(\mathbb{R})$ and $u^{(i)} \in L^\infty(\mathbb{R})$ for $i = 0, 1, \ldots, q$. Then, by successively applying (29), we have

$$\mathcal{K}[u]^{(q)}(x) = \int_0^\infty K(y)\{u^{(q)}(x-y)+u^{(q)}(x+y)\}\,\mathrm{d}y,$$

and hence, $\mathcal{K}[u]^{(q)}(x) = \mathcal{K}[u^{(q)}]$ by applying (22) to $u^{(q)}$. This proves (b), and the proof is complete.

Lemma 5. *For every* $u \in C_0(\mathbb{R})$, $\mathcal{K}[u]^{(i)} \in C_0(\mathbb{R})$ *for* $i = 0, 1, 2, 3, 4$.

Proof. Suppose $u \in C_0(\mathbb{R})$. Since $C_0(\mathbb{R}) \subset C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have $\mathcal{K}[u]^{(i)} \in C(\mathbb{R})$ for $i = 0, 1, 2, 3, 4$ by Lemma 4 (a). So it is sufficient to show that $\lim_{x\to\pm\infty}\mathcal{K}[u]^{(i)}(x) = 0$ for $i = 0, 1, 2, 3, 4$. We first consider the case $i = 0, 1, 2, 3$. Let $\epsilon > 0$ be arbitrary. Since $u \in C_0(\mathbb{R})$, there exists $M > 0$ such that

$$|u(x)| < \frac{\sqrt{2}k}{6\alpha^i}\varepsilon, \quad i = 0, 1, 2, 3 \tag{30}$$

for every $|x| \geq M/2$. Moreover, we can assume M is large enough so that it also satisfies

$$\frac{\alpha^i}{\sqrt{2}k}||u||_\infty \exp\left(-\frac{\alpha}{\sqrt{2}}\cdot\frac{M}{2}\right) < \frac{\varepsilon}{2}, \quad i = 0, 1, 2, 3. \tag{31}$$

Suppose $x > M$. By (22) and Lemma 4 (a), we have

$$\begin{aligned}|\mathcal{K}[u]^{(i)}(x)| &\leq \left|\int_0^\infty K^{(i)}(y)u(x-y)\,\mathrm{d}y\right| + \left|(-1)^i\int_0^\infty K^{(i)}(y)u(x+y)\,\mathrm{d}y\right| \\ &\leq \int_0^\infty |K^{(i)}(y)|\cdot|u(x-y)|\,\mathrm{d}y + \int_0^\infty |K^{(i)}(y)|\cdot|u(x+y)|\,\mathrm{d}y\end{aligned} \tag{32}$$

for $i = 0, 1, 2, 3$. Consider the second term in (32). If $y \geq 0$, then $x + y \geq M > M/2$, and so $|u(x+y)| < \frac{\sqrt{2}k}{6\alpha^i}\varepsilon$ by (30). Hence,

$$\int_0^\infty |K^{(i)}(y)|\cdot|u(x+y)|\,\mathrm{d}y \leq \frac{\sqrt{2}k}{6\alpha^i}\varepsilon\int_0^\infty |K^{(i)}(y)|\,\mathrm{d}y \leq \frac{\sqrt{2}k}{6\alpha^i}\varepsilon\cdot\frac{\alpha^i}{\sqrt{2}k} = \frac{\varepsilon}{6}, \tag{33}$$

since

$$|K^{(i)}(y)| \leq \frac{\alpha^{i+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right) \tag{34}$$

by Lemma 3, and hence

$$\int_0^\infty |K^{(i)}(y)|\,\mathrm{d}y \leq \frac{\alpha^{i+1}}{2k}\cdot\left(-\frac{\sqrt{2}}{\alpha}\right)\left[\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\right]_0^\infty = \frac{\alpha^i}{\sqrt{2}k}. \tag{35}$$

Note that the first term in (32) is

$$\begin{aligned}\int_0^\infty |K^{(i)}(y)|\cdot|u(x-y)|\,\mathrm{d}y = &\int_0^{x-M/2} |K^{(i)}(y)|\cdot|u(x-y)|\mathrm{d}y + \int_{x-M/2}^{x+M/2} |K^{(i)}(y)|\cdot|u(x-y)|\mathrm{d}y \\ &+ \int_{x+M/2}^\infty |K^{(i)}(y)|\cdot|u(x-y)|\mathrm{d}y.\end{aligned} \tag{36}$$

If $0 \leq y \leq x - M/2$, then $x - y \geq M/2$, and hence $|u(x-y)| < \frac{\sqrt{2}k}{6\alpha^i}\varepsilon$ by (30). If $y \geq x + M/2$, then $x - y \leq -M/2$, and we also have $|u(x-y)| < \frac{\sqrt{2}k}{6\alpha^i}\varepsilon$ by (30). Thus, by

(35), we have

$$\int_0^{x-M/2} |K^{(i)}(y)| \cdot |u(x-y)| \, \mathrm{d}y \le \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \int_0^{x-M/2} |K^{(i)}(y)| \, \mathrm{d}y \le \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \int_0^{\infty} |K^{(i)}(y)| \, \mathrm{d}y \\ = \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \cdot \frac{\alpha^i}{\sqrt{2}k} = \frac{\varepsilon}{6}, \tag{37}$$

$$\int_{x+M/2}^{\infty} |K^{(i)}(y)| \cdot |u(x-y)| \, \mathrm{d}y \le \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \int_{x+M/2}^{\infty} |K^{(i)}(y)| \, \mathrm{d}y \le \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \int_0^{\infty} |K^{(i)}(y)| \, \mathrm{d}y \\ = \frac{\sqrt{2}k}{6\alpha^i}\varepsilon \cdot \frac{\alpha^i}{\sqrt{2}k} = \frac{\varepsilon}{6}. \tag{38}$$

By (31) and (34), the remaining term in (36) becomes

$$\begin{aligned} &\int_{x-M/2}^{x+M/2} |K^{(i)}(y)| \cdot |u(x-y)| \mathrm{d}y \\ &\le ||u||_\infty \int_{x-M/2}^{x+M/2} |K^{(i)}(y)| \, \mathrm{d}y \le ||u||_\infty \int_{x-M/2}^{\infty} |K^{(i)}(y)| \, \mathrm{d}y \\ &\le ||u||_\infty \int_{x-M/2}^{\infty} \frac{\alpha^{i+1}}{2k} \exp\left(-\frac{\alpha}{\sqrt{2}}y\right) \mathrm{d}y = ||u||_\infty \cdot \frac{\alpha^{i+1}}{2k} \cdot \left(-\frac{\sqrt{2}}{\alpha}\right) \left[\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\right]_{x-M/2}^{\infty} \\ &= ||u||_\infty \frac{\alpha^i}{\sqrt{2}k} \left[\exp\left\{-\frac{\alpha}{\sqrt{2}}(x-M/2)\right\} - 0\right] < ||u||_\infty \frac{\alpha^i}{\sqrt{2}k} \exp\left(-\frac{\alpha}{\sqrt{2}} \cdot \frac{M}{2}\right) \\ &< \frac{\varepsilon}{2}, \end{aligned} \tag{39}$$

since $x > M$. Combining (32), (33), (36), (37), (38), and (39), we have

$$|\mathcal{K}^{(i)}[u](x)| < \frac{\varepsilon}{6} + \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon$$

for every $x > M$. This implies $\lim_{x\to\infty} |\mathcal{K}^{(i)}[u](x)| = 0$ for $i = 0, 1, 2, 3$. We omit the similar proof that $\lim_{x\to\infty} |\mathcal{K}^{(i)}[u](x)| = 0$. Thus we conclude that $\mathcal{K}^{(i)}[u] \in C_0(\mathbb{R})$ for $i = 0, 1, 2, 3$. It follows that $\mathcal{K}^{(4)}[u] \in C_0(\mathbb{R})$, since $\mathcal{K}^{(4)}[u] = -\alpha^4 \mathcal{K}[u] + \frac{\alpha^4}{k} u$ by Lemma 4 (a).

In what follows, we put τ to be the following constant:

$$\tau := 2k \int_0^{\infty} |K(y)| \mathrm{d}y. \tag{40}$$

The exact value of τ can be determined by elementary calculation, which we omit. It turns out that

$$\tau = 1 + \frac{\sqrt{2}\exp\left(-\frac{3\pi}{4}\right)}{1 - \exp(-\pi)} \approx 1.140. \tag{41}$$

Lemma 6. (a) $||\mathcal{K}[u]||_\infty < (\tau/k)\cdot||u||_\infty$ *for every* $u \in L^\infty(\mathbb{R})$. *Thus,* $\mathcal{K}[u] \in L^\infty(\mathbb{R})$ *for every* $u \in L^\infty(\mathbb{R})$.

(b) *For every* $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *we have* $||\mathcal{K}[u]^{(i)}||_\infty \le (\tau\alpha^i/k)\exp(3i\pi/4)\cdot||u||_\infty$ *for* $i = 1, 2, 3$, *and* $||\mathcal{K}[u]^{(4)}||_\infty < ((\tau+1)\alpha^4/k)\cdot||u||_\infty$.

Proof. By (22) and (40), we have

$$\begin{aligned}||\mathcal{K}[u]||_\infty &\le \sup_{x\in\mathbb{R}}\int_0^\infty |K(y)|\cdot|u(x-y)+u(x+y)|\,\mathrm{d}y\\ &\le \int_0^\infty |K(y)|\cdot\sup_{x\in\mathbb{R}}|u(x-y)+u(x+y)|\mathrm{d}y\\ &\le 2||u||_\infty\int_0^\infty |K(y)|\mathrm{d}y = \frac{\tau}{k}||u||_\infty\end{aligned}$$

for every $u \in L^\infty(\mathbb{R})$. This shows (a).

Suppose $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By Lemma 4 (a),

$$\begin{aligned}||\mathcal{K}[u]^{(i)}||_\infty &\le \sup_{x\in\mathbb{R}}\int_0^\infty |K^{(i)}(y)||u(x-y)+(-1)^i u(x+y)|\mathrm{d}y\\ &\le \int_0^\infty |K^{(i)}(y)|\cdot\sup_{x\in\mathbb{R}}|u(x-y)+(-1)^i u(x+y)|\mathrm{d}y\\ &\le \int_0^\infty |K^{(i)}(y)|\cdot\left\{\sup_{x\in\mathbb{R}}|u(x-y)|+\sup_{x\in\mathbb{R}}|u(x+y)|\right\}\mathrm{d}y\\ &\le 2||u||_\infty\int_0^\infty |K^{(i)}(y)|\mathrm{d}y\end{aligned} \tag{42}$$

for $i = 1, 2, 3$. By Lemma 3 and with the substitution $z = y + \dfrac{3i\pi}{2\sqrt{2}\alpha}$, we have

$$\begin{aligned}\int_0^\infty |K^{(i)}(y)|\mathrm{d}y &= \int_0^\infty \frac{\alpha^{i+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\cdot\left|\sin\left\{\frac{\alpha}{\sqrt{2}}y+\frac{\pi}{4}+\frac{3i\pi}{4}\right\}\right|\mathrm{d}y\\ &= \int_{3i\pi/(2\sqrt{2}\alpha)}^\infty \frac{\alpha^{i+1}}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}z+\frac{3i\pi}{4}\right)\cdot\left|\sin\left\{\frac{\alpha}{\sqrt{2}}z+\frac{\pi}{4}\right\}\right|\mathrm{d}z\\ &\le \alpha^i\exp\left(\frac{3i\pi}{4}\right)\int_0^\infty \frac{\alpha}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}z\right)\cdot\left|\sin\left\{\frac{\alpha}{\sqrt{2}}z+\frac{\pi}{4}\right\}\right|\mathrm{d}z\\ &= \alpha^i\exp\left(\frac{3i\pi}{4}\right)\int_0^\infty |K(z)|\mathrm{d}z,\end{aligned}$$

which, together with (40) and (42), gives

$$||\mathcal{K}[u]^{(i)}||_\infty \le 2||u||_\infty\cdot\alpha^i\exp\left(\frac{3i\pi}{4}\right)\cdot\frac{\tau}{2k} = \frac{\tau\alpha^i}{k}\exp\left(\frac{3i\pi}{4}\right)\cdot||u||_\infty$$

for i = 1, 2, 3. This proves (b) for i = 1, 2, 3.

Finally, by Lemma 4 (a) and the above result (a),

$$||\mathcal{K}^{(4)}[u]||_\infty = \left\| -\alpha^4\mathcal{K}[u] + \frac{\alpha^4}{k}u \right\|_\infty \leq \alpha^4||\mathcal{K}[u]||_\infty + \frac{\alpha^4}{k}||u||_\infty$$
$$\leq \alpha^4 \cdot \frac{\tau}{k}||u||_\infty + \frac{\alpha^4}{k}||u||_\infty = \frac{(\tau+1)\alpha^4}{k} \cdot ||u||_\infty,$$

and the proof is therefore complete.

5 Main result

Using the operators $\mathcal{N}$ and $\mathcal{K}$ in Sections 3 and 4, the nonlinear integral operator Ψ defined in (10) can be expressed in abstract notation as

$$\Psi[u] = \mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u]]$$

for $u : \mathbb{R} \to \mathbb{R}$. We will show that Ψ is a contraction when it is restricted to an appropriate function space $X \subset C_0(\mathbb{R})$ which will be defined later in this section.

5.1 Assumptions on *w* and the space *X*

Here, we introduce two assumptions **W1** and **W2** on the function w in (2):

(**W1**) $w \in C_0(\mathbb{R})$.

(**W2**) $\|w\|_\infty < \sup_{0\leq s\leq \sigma k} \{\rho^{-1}(s) \cdot (\sigma k - s)\}$, where σ is defined by

$$\sigma := \frac{1-\tau}{\tau} + \eta. \tag{43}$$

W1 means that the loading w should be sufficiently localized, which was also assumed for the linear solution (6) of (21). Nonetheless, w need not be compactly supported, and it is sufficient that $\lim_{x\to\pm\infty} w(x) = 0$. Note that the constant σ is positive by (13), (14), and (15). The function ρ is taken to satisfy Lemma 1 (b). Since ρ is continuous and strictly increasing, the inverse function $\rho^{-1} : \rho([0, \infty)) \to [0, \infty)$ is well defined, and is also a strictly increasing continuous function with $\rho^{-1}(0) = 0$. It is easy to see that the range $\rho([0, \infty))$ of ρ, which is the domain of ρ^{-1}, is always of the form $[0, \bar{s})$ for some $0 < \bar{s} \leq \infty$. In fact, the supremum in **W2** should be meant to be taken in the range $s \in [0, \sigma k] \cap [0, \bar{s})$. Note that the set $\{\rho^{-1}(s) \cdot (\sigma k - s) | s \in [0, \sigma k] \cap [0, \bar{s})\}$ should be connected, and hence of the form $[0, \bar{c})$ or $[0, \bar{c}]$ for some $0 < \bar{c} \leq \infty$, since $[0, \sigma k] \cap [0, \bar{s})$ is connected and $\rho^{-1}(s) \cdot (\sigma k - s)$ is continuous. In fact, we have $\bar{c} = \sup_{0\leq s\leq \sigma_k}\{\rho^{-1}(s) \cdot (\sigma k - s)\}$. It follows from **W2** that there exists $s_* \in (0, \sigma k) \cap (0, \bar{s}) \subset (0, \sigma k)$ such that

$$||w||_\infty = \rho^{-1}(s_*) \cdot \{\sigma k - s_*\}. \tag{44}$$

We remark that the trivial case $\|w\|_\infty = 0$ is safely excluded in this article. The physical meaning of **W2** is that the size $\|w\|_\infty$ of the loading w cannot be arbitrarily large, and its upper limit is closely related to the nonlinearity and the non-uniformity of the given elastic foundation.

Now define the subset X of $C_0(\mathbb{R})$ by

$$X := \left\{ u \in C_0(\mathbb{R}) \,\middle|\, ||u||_\infty \leq \frac{||w||_\infty}{\sigma k - s_*} \right\}. \tag{45}$$

We view X as a metric space with the metric $\|\cdot - \cdot\|_\infty$. Note that X is a complete metric space, since it is a closed set in $C_0(\mathbb{R})$ which itself is a complete metric space. Note that

$$\frac{1}{\sigma k}\|w\|_\infty < \frac{\|w\|_\infty}{\sigma k - s_*},$$

since $0 < s_* < \sigma k$. It follows that

$$\left\{u \in C_0(\mathbb{R}) \,\middle|\, ||u||_\infty \le \frac{1}{\sigma k}||w||_\infty\right\} \subset X. \tag{46}$$

In our system described by the differential equation (2), it is physically clear that the size $\|u\|_\infty$ of the output deflection u cannot be too large compared to the size $\|w\|_\infty$ of the input loading w. In fact, Lemma 6 (a) describes this relationship quantitatively in the linear case (21). Thus, (46) implies that the space X, though it is not the whole of $C_0(\mathbb{R})$, is big enough in some sense.

Example 4. Consider the case

$$f(u,x) = (1+\varepsilon\cos x)\left(\frac{k}{1+\varepsilon}u + \lambda u^{2n+1}\right), \quad 0 \le \varepsilon \le \frac{1}{2}, \quad n \ge 1,$$

in Example 1. Then we have $\rho(t) = 2(2n+1)\lambda t^{2n}$, and hence $\rho^{-1}(s) = \left(\frac{s}{2(2n+1)\lambda}\right)^{\frac{1}{2n}}$. Put $\varphi(s) = \rho^{-1}(s)\cdot(\sigma k - s)$. Since

$$\begin{aligned}\phi'(s) &= \frac{d}{ds}\left\{\left(\frac{s}{2(2n+1)\lambda}\right)^{\frac{1}{2n}}(\sigma k - s)\right\} = \frac{1}{\sqrt[2n]{2(2n+1)\lambda}}\left\{\frac{1}{2n}s^{\frac{1}{2n}-1}(\sigma k - s) - s^{\frac{1}{2n}}\right\}\\ &= \frac{s^{\frac{1}{2n}-1}}{2n\sqrt[2n]{2(2n+1)\lambda}}\{(\sigma k - s) - 2ns\} = \frac{(2n+1)s^{\frac{1}{2n}-1}}{2n\sqrt[2n]{2(2n+1)\lambda}}\left(\frac{\sigma k}{2n+1} - s\right),\end{aligned}$$

φ is strictly increasing on $\left[0, \frac{\sigma k}{2n+1}\right]$, and strictly decreasing on $\left[\frac{\sigma k}{2n+1}, \sigma k\right]$. Note also that $\varphi(0) = \varphi(\sigma k) = 0$. Thus,

$$\begin{aligned}&\sup_{0\le s\le \sigma_k}\{\rho^{-1}(s)\cdot(\sigma k - s)\}\\ &= \rho^{-1}\left(\frac{\sigma k}{2n+1}\right)\left(\sigma k - \frac{\sigma k}{2n+1}\right) = \left\{\frac{\sigma k}{2(2n+1)^2\lambda}\right\}^{\frac{1}{2n}}\cdot\frac{2n}{2n+1}\sigma k\\ &= \frac{2n}{(2n+1)\{2(2n+1)^2\lambda\}^{\frac{1}{2n}}}\cdot(\sigma k)^{1+\frac{1}{2n}} < \infty.\end{aligned}$$

There are exactly two solutions in $(0, \sigma k)$ of the equation $\rho^{-1}(s)\cdot(\sigma k - s) = \|w\|_\infty$, or equivalently, $s(s-\sigma k)^{2n} - 2(2n+1)\lambda||w||_\infty^{2n} = 0$. Note that we have bigger X, if we take s_* to be the larger among them.

Example 5. Consider the case

$$f(u,x) = (1+\varepsilon\cos x)\left[\frac{k}{1+\varepsilon}u + \lambda\{\exp(au) - 1 - au\}\right], \quad 0 \le \varepsilon \le \frac{1}{2}, \ a > 0,$$

in Example 2. Then we have $\rho(t) = 2a\lambda\ \{\exp(at) - 1\}$, and hence $\rho^{-1}(s) = \frac{1}{a}\ln\left(1+\frac{s}{2a\lambda}\right)$. Putting $\varphi(s) = \rho^{-1}(s)\cdot(\sigma k - s)$, we have

$$\begin{aligned}\phi'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{a}\ln\left(1+\frac{s}{2a\lambda}\right)(\sigma k - s)\right\} = \frac{1}{a}\left\{\frac{\frac{1}{2a\lambda}}{1+\frac{s}{2a\lambda}}(\sigma k - s) - \ln\left(1+\frac{s}{2a\lambda}\right)\right\}\\ &= \frac{1}{2a^2\lambda\left(1+\frac{s}{2a\lambda}\right)}\left\{(\sigma k - s) - 2a\lambda\left(1+\frac{s}{2a\lambda}\right)\ln\left(1+\frac{s}{2a\lambda}\right)\right\}\\ &= \frac{1}{2a^2\lambda\left(1+\frac{s}{2a\lambda}\right)}\left[\sigma k - \left\{s + 2a\lambda\left(1+\frac{s}{2a\lambda}\right)\ln\left(1+\frac{s}{2a\lambda}\right)\right\}\right].\end{aligned}$$

It follows that φ is strictly increasing on $[0,\tilde{s}]$, and strictly decreasing on $[\tilde{s},\sigma k]$, and hence, $\sup_{0\le s\le\sigma_k}\{\rho^{-1}(s)\cdot(\sigma k - s)\} = \phi(\tilde{s}) < \infty$, where $\tilde{s}$ is the unique solution in $(0, \sigma k)$ of the equation

$$\sigma k - \left\{s + 2a\lambda\left(1+\frac{s}{2a\lambda}\right)\ln\left(1+\frac{s}{2a\lambda}\right)\right\} = 0.$$

Again, there are exactly two solutions in $(0, \sigma k)$ of the equation $\rho^{-1}(s)\cdot(\sigma k - s) = ||w||_\infty$. Among them, we take s_* to be preferably the larger.

Example 6. In Example 3, we took ρ as in (20), rather than $\rho(t) = t$, for the case $f(u, x) = ku$. Then we have

$$\rho^{-1}(s) = \frac{1}{(\sigma k - s)^2} - \frac{1}{\sigma^2 k^2}.$$

Let $\varphi(s) = \rho^{-1}(s)\cdot(\sigma k - s)$. We can easily check that φ is strictly increasing on $[0, \sigma k)$, $\varphi(0) = 0$, and $\lim_{s\to\sigma k-}\varphi(s) = \infty$. Thus, we have $\sup_{0\le s\le\sigma k}\{\rho^{-1}(s)\cdot(\sigma k - s)\} = \infty$. This implies that we have *no* restriction on the upper bound of $\|w\|_\infty$, which indeed is expected with the linear equation (21). Note, however, this observation could not have been possible to be made, if we took $\rho(t) = t$. The equation $\varphi(s) = \|w\|_\infty$, which is equivalent to $s^2 - \sigma k(2 + \sigma k\|w\|_\infty)s + \sigma^3 k^3\|w\|_\infty = 0$, has the unique solution

$$s_* = \sigma k\left\{\left(1+\frac{\sigma k||w||_\infty}{2}\right) - \sqrt{1+\left(\frac{\sigma k||w||_\infty}{2}\right)^2}\right\}$$

in $(0, \sigma k)$.

5.2 Contractiveness of the operator Ψ

Suppose $u \in C_0(\mathbb{R})$. Then $\mathcal{N}[u] \in C_0(\mathbb{R})$ by Lemma 1 (a), and again, $\mathcal{K}[\mathcal{N}[u]] \in C_0(\mathbb{R})$ by Lemma 5. We also have $\mathcal{K}[w] \in C_0(\mathbb{R})$ by **W1** and Lemma 5. Thus, we have $\Psi[u] = \mathcal{K}[u] - \mathcal{K}[\mathcal{N}[u]] \in C_0(\mathbb{R})$ for every $u \in C_0(\mathbb{R})$. In short, the operator Ψ is a well-defined map from $C_0(\mathbb{R})$ into $C_0(\mathbb{R})$. The next lemma confirms that the solutions of (2) are the fixed points of Ψ in $C_0(\mathbb{R})$.

Lemma 7. *Suppose $u \in C^4(\mathbb{R}) \cap C_0(\mathbb{R})$ and $u^{(i)} \in L^\infty(\mathbb{R})$ for i = 1, 2, 3, 4. Then u is a solution of the differential equation (2), if and only if $\Psi[u] = u$.*

Proof. Suppose u satisfies $\Psi[u] = u$. By Lemma 4 (a), we have

$$\begin{aligned} u^{(4)} &= \Psi[u]^{(4)} = \{\mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u]]\}^{(4)} = \mathcal{K}[w]^{(4)} - \mathcal{K}[\mathcal{N}[u]]^{(4)} \\ &= \left\{-\alpha^4\mathcal{K}[w] + \frac{\alpha^4}{k}w\right\} - \left\{-\alpha^4\mathcal{K}[\mathcal{N}[u]] + \frac{\alpha^4}{k}\mathcal{N}[u]\right\} \\ &= \frac{\alpha^4}{k}\{w - \mathcal{N}[u]\} - \alpha^4\{\mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u]]\} \\ &= \frac{\alpha^4}{k}\{w - \mathcal{N}[u]\} - \alpha^4\Psi[u] = \frac{\alpha^4}{k}\{w - \mathcal{N}[u]\} - \alpha^4 u, \end{aligned}$$

and hence, u is a solution of (2) by (12).

Conversely, suppose u is a solution of (2), so that $u^{(4)} + \alpha^4 u + \frac{\alpha^4}{k}\mathcal{N}[u] = \frac{\alpha^4}{k}w$ by (12).

Applying the operator $\mathcal{K}$, we get

$$\mathcal{K}[u^{(4)}] + \alpha^4\mathcal{K}[u] + \frac{\alpha^4}{k}\mathcal{K}[\mathcal{N}[u]] = \frac{\alpha^4}{k}\mathcal{K}[w],$$

and hence

$$\begin{aligned} \Psi[u] &= \mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u]] = \frac{k}{\alpha^4}\mathcal{K}[u^{(4)}] + k\mathcal{K}[u] = \frac{k}{\alpha^4}\mathcal{K}[u]^{(4)} + k\mathcal{K}[u] \\ &= \frac{k}{\alpha^4}\left\{-\alpha^4\mathcal{K}[u] + \frac{\alpha^4}{k}u\right\} + k\mathcal{K}[u] = u \end{aligned}$$

by Lemma 4 (a), and (b), and the proof is complete.

Unfortunately, Ψ is not a contraction on the whole of $C_0(\mathbb{R})$. Nevertheless, if we restrict Ψ to the subset X of $C_0(\mathbb{R})$ defined in (45), then we can show that Ψ is a contraction from X *into* X. This enables us to use the usual argument of the Banach fixed point theorem, and to prove the existence and the uniqueness of the fixed point of Ψ, which is the solution of the differential equation (2), at least in X.

Lemma 8. *$\Psi[u] \in X$ for every $u \in X$. Moreover, $\Psi: X \to X$ is a contraction, i.e., $\|\Psi[u] - \Psi[v]\|_\infty \le L \cdot \|u - v\|_\infty$ for every $u, v \in X$ for some constant $L < 1$.*

Proof. Suppose $u \in X$. Note that $\mathcal{N}[\mathbf{0}] = \mathbf{0}$ by **F2**, if we denote the zero function by $0(x) \equiv 0$. Hence, by Lemma 1 (b) and Lemma 6 (a), we have

$$\begin{aligned} ||\Psi[u]||_\infty &= ||\mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u]]||_\infty \le ||\mathcal{K}[w]||_\infty + ||\mathcal{K}[\mathcal{N}[u]]||_\infty \\ &\le \frac{\tau}{k}||w||_\infty + \frac{\tau}{k}||\mathcal{N}[u]||_\infty \le \frac{\tau}{k}||w||_\infty + \frac{\tau}{k}||\mathcal{N}[u] - \mathcal{N}[\mathbf{0}]||_\infty \\ &\le \frac{\tau}{k}||w||_\infty + \frac{\tau}{k}\cdot\{(1-\eta)k + \rho(||u||_\infty)\}\cdot||u||_\infty, \end{aligned}$$

where ρ is taken as in Lemma 1 (b). Hence, by (44) and (43), we have

$$\begin{aligned} ||\Psi[u]||_\infty &\le \frac{\tau}{k}||w||_\infty + \frac{\tau}{k}\cdot\left\{(1-\eta)k + \rho\left(\frac{||w||_\infty}{\sigma k - s_*}\right)\right\}\cdot\frac{||w||_\infty}{\sigma k - s_*} \\ &= \frac{\tau}{k}||w||_\infty + \frac{\tau}{k}\cdot\{(1-\eta)k + \rho(\rho^{-1}(s_*))\}\cdot\frac{||w||_\infty}{\sigma k - s_*} \\ &= \frac{\tau}{k}\left\{1 + \frac{(1-\eta)k + s_*}{\sigma k - s_*}\right\}||w||_\infty = \frac{\tau(\sigma + 1 - \eta)}{\sigma k - s_*}||w||_\infty \\ &= \frac{\tau\left(\dfrac{1-\tau}{\tau} + \eta + 1 - \eta\right)}{\sigma k - s_*}||w||_\infty = \frac{||w||_\infty}{\sigma k - s_*}, \end{aligned}$$

which shows $\Psi[u] \in X$.

Now suppose $u, v \in X$. Again by Lemma 6 (a) and Lemma 1 (b), we have

$$\begin{aligned}||\Psi[u]-\Psi[v]||_\infty &= ||\{\mathcal{K}[w]-\mathcal{K}[\mathcal{N}[u]]\}-\{\mathcal{K}[w]-\mathcal{K}[\mathcal{N}[v]]\}||_\infty = ||\mathcal{K}[\mathcal{N}[u]]-\mathcal{K}[\mathcal{N}[v]]||_\infty \\ &= ||\mathcal{K}[\mathcal{N}[u]-\mathcal{N}[v]]||_\infty \le \frac{2\tau}{k}||\mathcal{N}[u]-\mathcal{N}[v]||_\infty \\ &\le \frac{\tau}{k}\{(1-\eta)k+\rho(\max\{||u||_\infty, ||v||_\infty\})\}\cdot ||u-v||_\infty \\ &\le \frac{\tau}{k}\left\{(1-\eta)k+\rho\left(\frac{||w||_\infty}{\sigma k - s_*}\right)\right\}\cdot ||u-v||_\infty \\ &= \frac{\tau}{k}\{(1-\eta)k+\rho(\rho^{-1}(s_*))\}\cdot ||u-v||_\infty = \tau\left(1+\eta+\frac{s_*}{k}\right)\cdot ||u-v||_\infty.\end{aligned}$$

Since $0 < s_* < \sigma k$, we have

$$\tau\left(1-\eta+\frac{s_*}{k}\right) < \tau\left(1-\eta+\frac{\sigma k}{k}\right) = \tau(1-\eta+\sigma) = \tau\left(1-\eta+\frac{1-\tau}{\tau}+\eta\right) = 1$$

by (43). Thus, we have the desired inequality by taking $L = \tau\left(1-\eta+\frac{s_*}{k}\right)$, and the proof is complete.

Proposition 1. (Banach Fixed Point Theorem [20]) *Let Y be a complete metric space with the metric $d(\cdot,\cdot)$, and suppose the map Φ: $Y \to Y$ satisfies $d(\Phi(y_1), \Phi(y_2)) \le L \cdot d(y_1, y_2)$ for every $y_1, y_2 \in Y$ for some constant $L < 1$. Then Φ has a unique fixed point in Y. Moreover, for any $y_0 \in Y$, the sequence $\{y_n\}_{n=0}^{\infty}$, defined by $y_n = \Phi(y_{n-1})$, n = 1, 2, ..., converges to this unique fixed point.*

Lemma 9. Ψ *has a unique fixed point in X. Moreover, this fixed point, denoted by u^*, is in $C^4(\mathbb{R})$, and $u_*^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4.*

Proof. The fact that Ψ has a unique fixed point in X is immediate from Proposition 1 and Lemma 8, since X is a complete metric space with the metric $\|\cdot - \cdot\|_\infty$. Let u_* be this unique fixed point.

Take any u_0 in X, and define $u_n = \Psi[u_{n-1}]$, n = 1, 2, ... By Proposition 1, the sequence of functions $\{u_n\}_{n=0}^{\infty}$ in X converges uniformly to the fixed point $u_* \in X$. We assume $u_0 \in C^4(\mathbb{R})$ and $u_0^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4, which can always be achieved: For example, we could take u_0 to be the zero function. Suppose, for some n, $u_{n-1} \in C^4(\mathbb{R})$ and $u_{n-1}^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4. Then $\mathcal{N}[u_{n-1}] \in C_0(\mathbb{R})$ by Lemma 1 (a), since $u_{n-1} \in C_0(\mathbb{R})$. Hence, $\mathcal{K}[\mathcal{N}[u_{n-1}]] \in C^4(\mathbb{R})$ by Lemma 4 (a), and $\mathcal{K}[\mathcal{N}[u_{n-1}]]^{(i)} \in C_0(\mathbb{R})$, i = 1, 2, 3, 4 by Lemma 5. Since $w \in C_0(\mathbb{R})$ by **W1**, we also have $\mathcal{K}[w] \in C^4(\mathbb{R})$ by Lemma 4 (a), and $\mathcal{K}[w]^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4 by Lemma 5. Hence, we have $u_n \in C^4(\mathbb{R})$ and $u_n^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4, since $u_n = \Psi[u_{n-1}] = \mathcal{K}[w] - \mathcal{K}[\mathcal{N}[u_{n-1}]]$. Thus, by induction on n, we have $u_n \in C^4(\mathbb{R})$ and $u_n^{(i)} \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4, for every n = 0, 1, 2, ...

By Lemma 6 (b), we have $||\mathcal{K}[u]^{(i)}||_\infty \le A \cdot ||u||_\infty$, i = 1, 2, 3, 4, for every $u \in C_0(\mathbb{R}) \subset C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where we put

$$A = \max\left\{\max_{i=1,2,3}\frac{\tau\alpha^i}{k}\exp\left(\frac{3i\pi}{4}\right), \frac{(\tau+1)\alpha^4}{k}\right\}.$$

Hence, we have

$$\begin{aligned}\left\|u_{n+1}^{(i)}-u_n^{(i)}\right\|_\infty &= \left\|\Psi[u_n]^{(i)}-\Psi[u_{n-1}]^{(i)}\right\|_\infty\\ &= \left\|\{\mathcal{K}[w]-\mathcal{K}[\mathcal{N}[u_n]]\}^{(i)}-\{\mathcal{K}[w]-\mathcal{K}[\mathcal{N}[u_{n-1}]]\}^{(i)}\right\|_\infty\\ &= \left\|\mathcal{K}[\mathcal{N}[u_n]]^{(i)}-\mathcal{K}[\mathcal{N}[u_{n-1}]]^{(i)}\right\|_\infty = \left\|\mathcal{K}[\mathcal{N}[u_n]-\mathcal{N}[u_{n-1}]]^{(i)}\right\|_\infty\\ &\le A\cdot\|\mathcal{N}[u_n]-\mathcal{N}[u_{n-1}]\|_\infty, \quad i=1,2,3,4,\end{aligned} \tag{47}$$

for every n = 0, 1, 2, ..., since $\mathcal{N}[u_n]-\mathcal{N}[u_{n-1}]\in C_0(\mathbb{R})$. Since u_{n-1}, $u_n\in X\subset L^\infty(\mathbb{R})$, we have

$$\begin{aligned}\left\|u_{n+1}^{(i)}-u_n^{(i)}\right\|_\infty &\le A\{k+\rho(\max\{\|u_n\|_\infty,\|u_{n-1}\|_\infty\})\}\cdot\|u_n-u_{n-1}\|_\infty\\ &\le A\left\{k+\rho\left(\frac{\|w\|_\infty}{\sigma k-s_*}\right)\right\}\cdot\|u_n-u_{n-1}\|_\infty\\ &= A\{k+\rho(\rho^{-1}(s_*))\}.\|u_n-u_{n-1}\|_\infty\\ &= A(k+s_*)\cdot\|u_n-u_{n-1}\|_\infty, \quad i=1,2,3,4\end{aligned} \tag{48}$$

by Lemma 1 (b) and (44), (47). Since

$$\|u_{n+1}-u_n\|_\infty = \|\Psi[u_n]-\Psi[u_{n-1}]\|_\infty \le L\cdot\|u_n-u_{n-1}\|_\infty, \quad n=1,2,\dots$$

by Lemma 8, we have

$$\|u_n-u_{n-1}\|_\infty \le L^{n-1}\cdot\|u_1-u_0\|_\infty, \quad n=1,2,\dots. \tag{49}$$

Combining (48) and (49), we have

$$\left\|u_{n+1}^{(i)}-u_n^{(i)}\right\|_\infty \le A(k+s_*)\|u_1-u_0\|_\infty\cdot L^{n-1}, \quad n=1,2,\dots,i=1,2,3,4. \tag{50}$$

Let $\epsilon > 0$. Since $0 \le L < 1$, we can take N large enough so that

$$\frac{L^N}{1-L}\cdot A(k+s_*)\|u_1-u_0\|_\infty < \varepsilon. \tag{51}$$

Let m, $n >N$. Assume $m >n$ with no loss of generality. Then by (50) and (51), we have

$$\begin{aligned}\left\|u_m^{(i)}-u_n^{(i)}\right\|_\infty &= \left\|\sum_{j=0}^{m-n-1}\left(u_{n+j+1}^{(i)}-u_{n+j}^{(i)}\right)\right\|_\infty \le \sum_{j=0}^{m-n-1}\left\|u_{n+j+1}^{(i)}-u_{n+j}^{(i)}\right\|_\infty\\ &\le \sum_{j=0}^{m-n-1}L^{n+j-1}\cdot A(k+s_*)\|u_1-u_0\|_\infty \le \frac{L^{n-1}}{1-L}\cdot A(k+s_*)\|u_1-u_0\|_\infty\\ &\le \frac{L^N}{1-L}\cdot A(k+s_*)\|u_1-u_2\|_\infty < \varepsilon, \quad i=1,2,3,4.\end{aligned}$$

This implies that, for every i = 1, 2, 3, 4, the sequence $\{u_n^{(i)}\}_{n=0}^\infty$ is Cauchy in $C_0(\mathbb{R})$ with respect to the metric $\|\cdot - \cdot\|_\infty$, and hence, converges uniformly to a function $v_i \in C_0(\mathrm{R})$. So by Lemma 10 below, $u_* \in C^1(\mathbb{R})$ and $u_*' = v_1$, since u_n converges uniformly to u_* and u_n' converges uniformly to v_1. Applying Lemma 10 again to u_n', we see that $v_1 \in C^1(\mathbb{R})$ and $v_i' = v_2$. By repeating the same argument, we see that $v_2 \in C^1(\mathbb{R})$,

$u_*^{(i)} = v_1 \in C_0(\mathbb{R})$, and $v_3 \in C^1(\mathbb{R})$, $v_3' = v_4$. Thus, we have $u_* \in C^4(\mathbb{R})$ and $u_*^{(i)} = v_1 \in C_0(\mathbb{R})$ for i = 1, 2, 3, 4. Hence, the proof is complete.

Lemma 10. *Suppose a sequence of functions* $\{g_n\}_{n=1}^{\infty}$ *in* $C^1(\mathbb{R})$ *converges uniformly to a function g. Suppose also* g_n' *converges uniformly to a function h. Then* $g \in C^1(\mathbb{R})$ *and* $g' = h$.

Proof. Fix $x_0 \in \mathbb{R}$, and define $h_n : \mathbb{R} \to \mathbb{R}$, n = 1, 2, ... by

$$h_n(x) = \begin{cases} \dfrac{g_n(x) - g_n(x_0)}{x - x_0}, & x \neq x_0, \\ g'_n(x_0), & x = x_0. \end{cases},$$

which is continuous since $g_n \in C^1(\mathbb{R})$. Note that $h_n(x_0) = g_n'(x_0) \to h(x_0)$ as $n \to \infty$. For $x \neq x_0$, we have

$$\begin{aligned} h_m(x) - h_n(x) &= \frac{g_m(x) - g_m(x_0)}{x - x_0} - \frac{g_n(x) - g_n(x_0)}{x - x_0} \\ &= \frac{1}{x - x_0}[\{g_m(x) - g_n(x)\} - \{g_m(x_0) - g_n(x_0)\}] \\ &= \frac{1}{x - x_0} \cdot \{g'_m(\xi) - g'_n(\xi)\}(x - x_0) = g'_m(\xi) - g'_n(\xi) \end{aligned}$$

for some ξ between x_0 and x by the mean value theorem for g_m - g_n. Thus, we have $||h_m - h_n||_\infty \leq ||g_m' - g_n'||_\infty$ for any m, n. It follows that h_n converges uniformly to a continuous function, since g_n' converges uniformly. Note that

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \lim_{n \to \infty} \frac{g_n(x) - g_n(x_0)}{x - x_0} = \lim_{x \to x_0} \lim_{n \to \infty} h_n(x). \tag{52}$$

Since h_n converges uniformly, we can change the order of the limit in (52), so that

$$\lim_{x \to x_0} \lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} h_n(x) = \lim_{n \to \infty} g_n'(x_0) = h(x_0).$$

Hence, $g'(x_0) = \lim_{x \to x_0} \dfrac{g(x) - g(x_0)}{x - x_0}$ exists, and is equal to $h(x_0)$. Thus, the proof is complete, since x_0 is arbitrary.

Now the following main result of the article is immediate from Lemmas 9 and 7.

Theorem *Suppose the functions f*(u, x) *and w*(x) *satisfy the conditions* **F1**, **F2**, **F3**, **F4**, *and* **W1**, **W2**. *Then the differential equation (2) has a unique solution in*

$$X = \left\{ u \in C_0(\mathbb{R}) \,\middle|\, ||u||_\infty \leq \frac{||w||_\infty}{\sigma k - s_*} \right\},$$

where k, σ, s_* are as defined in (11), (43), (44) respectively. Moreover, the unique solution, denoted by u, satisfies* $\lim_{x \to \pm\infty} u_*^{(i)}(x) = 0$ *for* i = 1, 2, 3, 4.

6 Concluding remarks

It is intuitively clear that the nature of the resulting beam deflection depends on both the nonlinearity and the non-uniformity of the given elastic foundation. In this study, we introduced a physical parameter η in (14) measuring the non-uniformity, and a function ρ in Lemma 1 which mainly measures the nonlinearity. Accordingly, the pair (η, ρ) may be considered as a systematic encoding of the non-uniformity and the nonlinearity of the given foundation. Together with the maximal linear spring constant k in (11) at the equilibrium state $u \equiv 0$, η and ρ capture the dominating mechanical properties of the present beam problem represented by the differential equation (2).

We transformed the original nonlinear differential equation into an equivalent nonlinear integral equation $\Psi[u] = u$, thereby positioning our problem into the realm of the fixed point theory. However, the integral operator Ψ is not a contraction in the whole function space $C_0(\mathbb{R})$ equipped with the usual sup-norm $\|\cdot\|_\infty$. The reason for this is two-fold: first, the nonlinearity of the elastic foundation, encoded in the function ρ, makes Ψ expansive for functions with large norms, which can be seen from Lemma 1 (b). Second, the value of the constant τ which gives the L^∞-norm of the operator $\mathcal{K}$ in Lemma 6 (a), is greater than 1. Because of this, too much non-uniformity of the elastic foundation, encoded in the parameter η, can also contribute to the non-contractiveness of Ψ.

Thus to resort to the Banach fixed point theorem, it is necessary to find a subspace smaller than $C_0(\mathbb{R})$, where the operator Ψ is contractive. This “shrinking the space” idea also conforms with the physical intuition that the norm of the resulting beam deflection cannot be too large compared to that of the input loading w. Meanwhile, the nonlinearity and the non-uniformity of the system suggest that the norm of the loading w itself should also be bounded. All these heuristic ideas were materialized into the actual construction of the upper-bound in **W2** and the subspace X, which, besides the input loading w, depend only on the three main attributes k, η, and ρ of the given mechanical system.

The subspace we are looking for should satisfy two conditions other than completeness: first, it should be invariant under the operator Ψ. Second, the restriction of Ψ to it should be contractive. Once we proved in Lemma 8 that the function space X actually satisfies these conditions, the existence and the uniqueness of the solution in Lemma 9 follows immediately from the Banach fixed point theorem. Note carefully that Lemma 7 establishes the equivalence between the original differential equation (2) and our integral equation (9), *only for* solutions satisfying the regularity condition to be in $C^4(\mathbb{R})$. In this respect, what Lemma 9 is really up to are the regularity of the unique solution thus found, and its behavior at infinity. Consequently, the main theorem in Section 5.2 states that the unique solution u_* has enough regularity for the differential equation (2), and satisfies our boundary condition (3), and hence, is the solution of the present nonlinear boundary value problem.

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Authors' contributions
TSJ formulated the integral equation (9) which is equivalent to the original nonlinear beam equation (2), and introduced the overall problem to SWC. SWC found the subspace (45) and proved that the integral operator (10) is a contraction on that subspace. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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