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On stability of delay difference equations with variable coefficients: successive products tests

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Abstract

In this paper, we report an error in the paper of the first author in *Advances in Difference Equations*, 2009, article 104310, present the revised versions of the theorem with several examples, and outline the cases when the previous result is valid.

1 Introduction

The purpose of this short note is to indicate an error in the previous paper [1] published in 'Advances in Difference Equations' and an inaccuracy in the recent paper [2]; to present a corrected result for [1] and clarification for [2]; and to outline the cases when the analogue of the result of [1] is still correct.

Consider the equation

$$x(n+1) = \sum_{\ell=1}^{m} a_\ell(n) x\big(h_\ell(n)\big) \quad \text{for } n \geq n_0, \tag{1}$$

where $\{a_\ell(n)\}$ are sequences of real numbers, and $\{h_\ell(n)\}$ are sequences of integers such that there exists a nonnegative integer τ satisfying $n - \tau \leq h_\ell(n) \leq n$ for all $n \geq n_0$ and $\ell = 1, 2, \dots, m$.

Theorem A *Suppose that $n - h_\ell(n) < d$ for some $d \in \mathbb{N}$, $\ell = 1, 2, \dots, m$, and there exists $r \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \prod_{j=0}^{r} \sum_{\ell=1}^{m} \big|a_\ell(n-j)\big| < 1. \tag{2}$$

Then (1) *is exponentially stable.*

Example 1 (Counterexample to Theorem A) Consider the delay difference equation

$$x(n+1) = a(n) x\big(h(n)\big) \quad \text{for } n \geq 0, \tag{3}$$

where

$$a(n) = \begin{cases} p, & n = 2l, \\ q, & n = 2l + 1 \end{cases} \quad \text{and} \quad h(n) = \begin{cases} n, & n = 2l, \\ n-1, & n = 2l + 1 \end{cases} \tag{4}$$

for some $p, q \in \mathbb{R}$. Simple computation gives us that the solution of (3) is

$$x(n) = \begin{cases} q^l x(0), & n = 2l, \\ pq^l x(0), & n = 2l + 1, \end{cases} \tag{5}$$

which is stable if and only if $|q| < 1$. More precisely, we have $\lim_{n\to\infty} |x(n)| = \infty$ for any $|q| > 1$ provided that $x(0) \neq 0$. If we compute (2) with $r = 1$ for (3), we get

$$\limsup_{n\to\infty} \bigl|a(n)a(n-1)\bigr| = |pq| \tag{6}$$

showing that the assumption of Theorem A is fulfilled if $|pq| < 1$. However, we can find $p, q \in \mathbb{R}$ such that $|q| \geq 1$ and $|pq| < 1$, for instance, $p = 1/2$ and $q = 6/5$. In this case, the right-hand side in (6) is $3/5 < 1$, but by (5) $x(2l) = 1.2^l x(0)$, $x(2l + 1) = 0.5 \cdot 1.2^l$, which is a divergent sequence, the solution is unstable.

Hence, in general, Theorem A is incorrect.

Let us note that in [2] and further in this paper, we apply the idea of reduction of higher (but bounded) order equations to first-order matrix equations. This method was widely used in [3–5] and in the earlier paper [6].

Also, in the discussion section of [2], the inequality

$$\rho(A_{k(n)}A_{k(n)-1} \cdots A_n) \leq \lambda \tag{7}$$

is considered as a sufficient asymptotic stability condition for the trivial solution of the first-order matrix equation

$$X_{n+1} = A_n X_n. \tag{8}$$

Here A_n are $d \times d$ matrices, $\rho(A)$ is the spectral radius of the matrix A, $\lambda \in (0, 1)$, $n_0 \in \mathbb{N}$, and $k(n) \geq n$ is a certain number which exists for any $n \geq n_0$. Similarly, the condition

$$\limsup_{n\to\infty} \rho(A_{n+k-1}A_{n+k-2} \cdots A_n) < 1 \tag{9}$$

is treated as a sufficient exponential stability condition for the trivial solution. This is not true, as the example from [7, Example 4.17, pp.190-191] illustrates (here $k = 1$, $k(n) = n$); see also the recent review [8] and Example 2 below.

Example 2 Equation (8), with

$$A_{2m} = \begin{pmatrix} 0 & 1.2 \\ 0.6 & 0 \end{pmatrix}, \qquad A_{2m+1} = \begin{pmatrix} 0 & 0.6 \\ 1.2 & 0 \end{pmatrix}, \quad m = 0, 1, 2, \ldots \tag{10}$$

satisfies $\rho(A_n) = 0.6\sqrt{2} < 1$ since both A_{2m} and A_{2m+1} have eigenvalues $\pm 0.6\sqrt{2}$. However, if we assume $X_0 = (0,1)^T$, then simple calculations lead to $X_{2m} = (0, 1.44^m)^T$, thus the system is unstable.

On the other hand, if we use in Example 2 the norm

$$\|A\| = \sup_{\|X\|=1} \|AX\|,$$

where $\|\cdot\|$ is the Euclidean vector norm, instead of the spectral radius, then $\|A_{mn}\| = \|A_{2m+1}\| = 1.2 > 1$ since $\|A_{2m}(0,1)^T\| = \|A_{2m+1}(1,0)^T\| = 1.2$, where B^T is the transpose of B.

We will use the following result in the recovery of Theorem A, which was obtained in [9]; see also [10, 11].

Theorem B ([9, Theorem 2]) *Let $m \in \mathbb{N}$ and $f : \mathbb{Z} \times \mathbb{R}^m \to \mathbb{R}$. If there exists $\lambda \in (0,1)$ such that*

$$\left|f(n, u_1, u_2, \dots, u_m)\right| \leq \lambda \max_{1 \leq j \leq m} \left\{|u_j|\right\} \quad \text{for all } n \geq n_0,$$

then

$$x_{n+1} = f(n, x_n, x_{n-1}, \dots, x_{n-m+1}) \quad \text{for } n \geq n_0$$

is globally exponentially stable. More precisely, any solution satisfies

$$|x_n| \leq \lambda^{(n-n_0)/m} \max_{n_0 - m + 1 \leq j \leq n_0} \left\{|x_j|\right\} \quad \text{for all } n \geq n_0.$$

2 Main results

For $k \in \mathbb{N}$, define a sequence

$$b_k(n) := \begin{cases} 1, & k = 0, \\ \sum_{\ell=1}^{m} |a_\ell(n)| b_{k-1}(h_\ell(n) - 1), & k \geq 1 \end{cases} \tag{11}$$

for $n \geq n_0 + k(\tau + 1)$.

Theorem 1 (Correction of Theorem A) *Suppose that there exists $r \in \mathbb{N}$ such that*

$$\limsup_{n \to \infty} b_r(n) < 1.$$

Then (1) *is exponentially stable.*

Proof Let us prove for all $k \in \mathbb{N}$ that

$$\left|x(n+1)\right| \leq b_k(n) \max_{n - k(\tau+1) \leq j \leq n} \left\{\left|x(j)\right|\right\} \quad \text{for all } n \geq n_0 + k(\tau + 1). \tag{12}$$

We proceed by induction in k. From (1), for $k = 1$, we have

$$\begin{aligned} |x(n+1)| &\le \sum_{\ell=1}^{m} |a_\ell(n)| |x(h_\ell(n))| \\ &\le \sum_{\ell=1}^{m} |a_\ell(n)| \max_{n-(\tau+1)\le j\le n} \{|x(j)|\} \\ &= b_1(n) \max_{n-(\tau+1)\le j\le n} \{|x(j)|\} \end{aligned} \tag{13}$$

for all $n \ge n_0 + \tau + 1$. Thus, the claim is true for $k = 1$. Assume now that the claim is true for some $k \ge 1$. From (12) and (13), for all $n \ge n_0 + (k+1)(\tau + 1)$, we have

$$\begin{aligned} |x(n+1)| &\le \sum_{\ell=1}^{m} |a_\ell(n)| b_k(h_\ell(n)-1) \max_{h_\ell(n)-1-k(\tau+1)\le j\le h_\ell(n)-1} \{|x(j)|\} \\ &\le \sum_{\ell=1}^{m} |a_\ell(n)| b_k(h_\ell(n)-1) \max_{n-(k+1)(\tau+1)\le j\le n} \{|x(j)|\} \\ &= b_{k+1}(n) \max_{n-(k+1)(\tau+1)\le j\le n} \{|x(j)|\}, \end{aligned}$$

which shows that (12) is true when k is replaced with $(k+1)$. Using (12) with $k = r$, we see that the solution is exponentially stable by Theorem B. □

Theorem 1 with $r = 1$ immediately yields the following result.

Corollary 1 *Assume that*

$$\limsup_{n\to\infty} \sum_{\ell=1}^{m} |a_\ell(n)| < 1.$$

Then (1) *is exponentially stable.*

Remark 1 The claim of Theorem A for $r = 0$ is correct.

Setting $r = 2$ in Theorem 1, we obtain the following corollary, which is also proved in [1, Theorem 2.17].

Corollary 2 *Assume that*

$$\limsup_{n\to\infty} \sum_{\ell_1=1}^{m} |a_{\ell_1}(n)| \sum_{\ell_2=1}^{m} |a_{\ell_2}(h_{\ell_1}(n)-1)| < 1.$$

Then (1) *is exponentially stable.*

Remark 2 Theorem A for the nondelay equation

$$x(n+1) = a(n)x(n) \quad \text{for } n \ge n_0$$

is correct. Indeed, Theorem 1 reduces to Theorem A since for $k \geq 1$, we get

$$b_k(n) = \prod_{j=0}^{k-1} |a(n-j)| \quad \text{for } n \geq n_0 + k.$$

Setting $r = 3$ in Theorem 1 gives us the following corollary.

Corollary 3 *Assume that*

$$\limsup_{n\to\infty} \sum_{\ell_1=1}^{m} |a_{\ell_1}(n)| \sum_{\ell_2=1}^{m} |a_{\ell_2}(h_{\ell_1}(n)-1)| \sum_{\ell_3=1}^{m} |a_{\ell_3}(h_{\ell_2}(h_{\ell_1}(n)-1)-1)| < 1.$$

Then (1) *is exponentially stable.*

Example 3 Consider the delay difference equation (3) with (4), where $p, q \in \mathbb{R}$, which can be written in the two equivalent forms:

$$x(n+1) = -a_1(n)x(n) - a_2(n)x(n-1) \quad \text{for } n \geq 0 \tag{14}$$

and

$$x(n+1) - x(n) = -\big(a_1(n)+1\big)x(n) - a_2(n)x(n-1) \quad \text{for } n \geq 0, \tag{15}$$

where

$$a_1(n) = \begin{cases} p, & n = 2l, \\ 0, & n = 2l+1 \end{cases} \quad \text{and} \quad a_2(n) = \begin{cases} 0, & n = 2l, \\ q, & n = 2l+1. \end{cases}$$

Computing $\{b_k(n)\}$ defined by (11), we see that

$$b_k(n) = \begin{cases} |pq^{k-1}|, & n = 2l, \\ |q^k|, & n = 2l+1 \end{cases} \quad \text{for } k \in \mathbb{N}, n \geq 2k.$$

Equation (3) is exponentially stable by Theorem 1 if $|q| < 1$ because there always exists $r \in \mathbb{N}$ such that $|pq^{r-1}| < 1$ and $|q^r| < 1$. From (5), we see that $|q| < 1$ is the best possible condition for the global exponential stability of (3) with (4).

Application of a recent result [12, Theorem 6] to (14) gives us $(1 + |p|)|q| < 1$, which implies $|q| < 1$.

The so-called '3/2-test' (see [13] and [1, Theorem A]) can be applied to (15) if $p > -1$ and $q > 0$, and ensures global exponential stability when

$$p + q + 2 < \frac{3}{2} + \frac{1}{2 \cdot 2} = \frac{7}{4} \quad \text{or equivalently} \quad p + q < -\frac{1}{4}$$

for which $0 < q < 3/4$ is necessary.

It is obvious that these two results and Corollary 1 cannot deliver any answer for the exponential stability when $p = 1$ and $q = 1/2$.

As mentioned in Remark 2, Theorem A is valid for a nondelay scalar equation. Next, any higher-order (of the order not exceeding d) equation (1), with $n-d < h_\ell(n) \le n$, can be rewritten as the first-order system

$$X(n+1) = A(n)X(n), \quad n = 0,1,2,\dots, \tag{16}$$

where $X(n) \in \mathbb{R}^d$, $A(n)$ are $d \times d$ matrices. Indeed, denote $X(0) = (x(-d+1), x(-d+2),\dots,x(0))^T$, $X(n) = (x(nd-d+1), x(nd-d+2),\dots,x(nd))^T$ and rewrite (1) as

$$x(n) = \sum_{j=1}^{d} b(n,j)x(n-j), \quad n \in \mathbb{N}, \tag{17}$$

where

$$b(n,j) = \sum_{l\in\{1,\dots,m|h_l(n)=n-j\}} a_l(n).$$

Then we can define the matrix $A(0) = (c(i,j))_{i=1}^{d}$ as follows:

$$\begin{aligned}
x(1) &= \sum_{j=1}^{d} b(1,j)x(1-j) = \sum_{j=1}^{d} c(1,j)x(1-j),\\
x(2) &= \sum_{j=1}^{d} b(2,j)x(2-j) = \sum_{j=2}^{d} b(2,j)x(2-j) + b(2,1)\sum_{j=1}^{d} b(1,j)x(1-j)\\
&= \sum_{j=1}^{d-1} b(2,j+1)x(1-j) + \sum_{j=1}^{d} b(2,1)b(1,j)x(1-j) = \sum_{j=1}^{d} c(2,j)x(1-j),\\
&\dots\\
x(d) &= \sum_{j=1}^{d} c(d,j)x(1-j),
\end{aligned}$$

and $X(1) = A(0)X(0)$. Similarly, we construct $A(n)$, $n \in \mathbb{N}$ and obtain system (16). Since $\|X(n)\| \ge |x(j)|$, $j = nd-d+1, nd-d+2,\dots,nd$, exponential (asymptotical) stability of (16) implies the relevant stability of (1). We recall that (16) is exponentially stable if there exist $n_0 \in \mathbb{N}$, $L > 0$, and $\mu \in (0,1)$ such that $\|X(n)\| \le L\mu^n\|X(0)\|$, $n \ge n_0$.

Theorem 2 *If there exist $\lambda \in (0,1)$, $M > 0$, and $n_0 \in \mathbb{N}$ such that $\|A(n)\| \le M$ and $\|\prod_{j=n}^{n+k-1} A(j)\| \le \lambda$ for every $n \ge n_0$ and for some positive integer k, then* (16) *is exponentially stable.*

Proof Without loss of generality, we can assume $M > 1$ and $\prod_{j=0}^{n_0-1}\|A(j)\| \le M$. Further, for any $n \ge n_0$ denote $m = [\frac{n-n_0-1}{k}]$, where $[t]$ is the integer part of t, and obtain the estimate

$$\begin{aligned}
\|X(n)\| &= \|A(n-1)\cdots A(0)X(0)\| \le \|A(n-1)\cdots A(0)\|\,\|X(0)\|\\
&= \left\|\prod_{j=n_0+km}^{n-1} A(j)\cdots \prod_{j=n_0}^{n_0+k-1} A(j)\prod_{j=0}^{n_0-1} A(j)\right\|\,\|X(0)\|
\end{aligned}$$

$$\leq \left\| \prod_{j=n_0+km}^{n-1} A(j) \right\| \cdots \left\| \prod_{j=n_0}^{n_0+k-1} A(j) \right\| \left\| \prod_{j=0}^{n_0-1} A(j) \right\| \|X(0)\|$$

$$\leq M\lambda^m M^k \leq M^{k+1} \left(\frac{1}{\lambda}\right)^{1+n_0/k} \left(\lambda^{1/k}\right)^n \|X(0)\| = L\mu^n \|X(0)\|$$

for $n \geq n_0$, where $L = M^{k+1}\lambda^{-1-n_0/k}$, $\mu = \lambda^{1/k}$. □

Example 4 (Example 6 in [2]) If in (16)

$$A_{2m} = \frac{1}{2}\begin{pmatrix} 5.1 & 4.9 \\ 4.9 & 5.1 \end{pmatrix}, \qquad A_{2m+1} = \frac{1}{2}\begin{pmatrix} 7.1 & -6.9 \\ -6.9 & 7.1 \end{pmatrix}, \quad m = 0, 1, 2, \ldots$$

then both A_{2m} and A_{2m+1} have the norms exceeding one (they have eigenvalues of 5 and 7, respectively), but the product

$$A_{2m}A_{2m+1} = A_{2m+1}A_{2m} = \begin{pmatrix} 0.6 & -0.1 \\ -0.1 & 0.6 \end{pmatrix}$$

has the norm $\|A_{2m}A_{2m+1}\| = \|A_{2m+1}A_{2m}\| = 0.7 < 1$, thus (16) is exponentially stable.

Example 5 Consider (16) with a 4-periodic matrix $A(n)$, where $A_{4m+j} = \left(\begin{smallmatrix} 0 & -1.1 \\ 1.1 & 0 \end{smallmatrix}\right)$, $m = 0, 1, 2, \ldots$, $j = 1, 2, 3$, $A_{4m} = \left(\begin{smallmatrix} 0 & -0.7 \\ 0.7 & 0 \end{smallmatrix}\right)$. Then $\|A_{4m+j}\| = 1.1 > 1$, $j = 1, 2, 3$, but (16) is exponentially stable since $A_{n+3}A_{n+2}A_{n+1}A_n = \left(\begin{smallmatrix} 0.9317 & 0 \\ 0 & 0.9317 \end{smallmatrix}\right)$ for any $n = 0, 1, 2, \ldots$, and $\lambda = \|A_{n+3}A_{n+2}A_{n+1}A_n\| = 0.9317 < 1$.

3 Discussion

The dynamics of higher-order difference equations with variable coefficients, as well as of non-autonomous systems of difference equations, is much more complicated than that of the relevant autonomous models; see, for example, [8, 14]. For example, the fact that the spectral radius of each matrix is less than one does not imply exponential stability of the system. On the other hand, as demonstrated in Example 5, non-autonomous systems, where some matrices have norms exceeding one, can still be exponentially stable. The challenge is to extend recursive results to other type of stability, for example, asymptotic and l^p stability; see, for example, [3].

Regarding generalizations to some types of nonlinear models, the analogue of Theorem 1 can be found in [2], while Theorem 2 can be reformulated for the nonlinear first-order system

$$X(n+1) = F_n\big(X(n)\big), \quad n = 0, 1, 2, \ldots, \tag{18}$$

in the following way, with the same proof repeated.

Theorem 3 *If there exist* $\lambda \in (0,1)$, $k \in \mathbb{N}$, $M > 0$, *and* $n_0 \in \mathbb{N}$ *such that* $\|F_n(X)\| \leq M\|X\|$ *and* $\|F_{n+k-1}(\cdots F_{n+1}(F_n(X))\cdots)\| \leq \lambda\|X\|$ *for any* $X \in \mathbb{R}^d$ *and* $n \geq n_0$, *then* (18) *is uniformly exponentially stable*.

Again, the case of possible asymptotic stability when

$$\left\| F_{n+k-1}\left(\cdots F_{n+1}\left(F_n(X)\right)\cdots\right)\right\| \le \lambda_n \|X\|$$

and $\limsup_{n\to\infty} \lambda_n = 1$ is still to be considered.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

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