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Matrix operator inequalities based on Mond, Pecaric, Jensen and Ando's

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Abstract

This work is concerned with discussing matrix operator's inequalities based on Mond and Pecaric's matrix inequalities and matrix operator inequalities for convex and concave functions that occur in inequalities of Jensen and Ando's for positive linear maps.

Keywords: positive definite, convex (concave) function, operator mean, Hermitian matrix

1. Introduction

Operator inequalities play important roles in various mathematical and statistical contexts. In early of study operator inequalities, there have been many research results, where useful references centered mainly literatures [1-4]. In nearly 20 years, some new research results of operator inequalities continuously appears, readers may refer to Mond and Pecaric [5], Beesack [1], Hansen [6] and Kubo and Ando [2] and so on. In [5], Mond Pecaric gave several matrix operator inequalities associated with positive linear maps by means of concavity and convexity theorems. Beesack and Pěcarić in [1] showed the Jensen's inequality for real valued convex functions. Hansen in [6] also gave operator inequalities associated with Jensen's inequality. In this article, we will give some new operator inequalities of matrix version based on Mond, Pecaric, Jensen and Ando's.

This article is organized as follows: In Section 2, based on Mond and Pecaric's operator inequalities we give matrix operator inequalities of real valued convex functions. In Section 3, we first present some new matrix operator inequalities by means of Jensen's and Ando's matrix inequalities, then we give Jensen inequality for convex functions in matrix form and conduct further research on the properties of Jensen inequality.

Throughout the article, $A_j(j = 1,2,...,k)$ denote positive definite Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$ $(0 <m <M)$, $U_j(j = 1,2,...,k)$ denote $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. If A is a $n \times n$ Hermitian matrix, then there exists a unitary matrix U such that $A = U^*[\lambda_1,...,\lambda_n]U$, where $[\lambda_1,...,\lambda_n]$ is diagonal matrix and λ_i $(i = 1,2,...,n)$ are eigenvalues of A, respectively. $f(A)$ is then defined by $f(A) = U^*[f(\lambda_1),...,f(\lambda_n)]U$. $A \geq B$ means $A-B$ is positive semi-definite.

If $F(t) = F(f(t),g(t))$, then we use $F(A)$ to denote matrix operator of $F(f(A), g(A))$, it means that $F(A) = F(f(A),g(A))$, while $F(A,B)$ denotes the matrix function of two variables as it was well defined in some known articles.

2. Matrix operator inequalities based on Mond and Pecaric's

In [5], Mond and Pecaric once gave two important matrix inequalities which are the following Theorems A and B:

Theorem A. Let $A_j(j = 1,2,...,k)$ be Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$, and let $U_j(j = 1,2,...,k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$.
If f is a real valued continuous convex function on $[m,M]$, then

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \leq \frac{MI - \sum_{j=1}^{k} U_j A_j U^*}{M - m} f(m) + \frac{\sum_{j=1}^{k} U_j A_j U^* - mI}{M - m} f(M).$$

Theorem B. Let $A_j(j = 1,2,...,k)$ be Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$, and let $U_j(j = 1,2,...,k)$ be $r \times n$ matrices those such that $\sum_{j=1}^{k} U_j U_j^* = I$. If f is a continuous convex function on $[m,M]$, J is an interval such that $J \supset [m,M]$ and $F(u,v)$ is a real valued continuous function defined on $J \times J$ and it is a matrix increasing function to its first variable, then

$$F\left(\sum_{j=1}^{k} U_j f(A_j) U_j^*, f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right)\right)$$

$$\leq \left\{\max_{x \in [m,M]} F\left[\frac{M - x}{M - m} f(m) + \frac{x - m}{M - m} f(M), f(x)\right]\right\} I$$

$$= \left\{\max_{\theta \in [0,1]} F\left[\theta f(m) + (1 - \theta) f(M), f(\theta m + (1 - \theta) M)\right]\right\} I.$$

From Theorems A and B, we can make a more profound study and get more results which have the extensive meaning.

Theorem 2.1. Let $A_j(j = 1,2,...,k)$ be Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$, and $U_j(j = 1,2,...,k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. If f is a real valued continuous convex function on $[m,M]$, then

$$\sum_{j=1}^{k} U_j f(mI + MI - A_j) U_j^* \leq \frac{\sum_{j=1}^{k} U_j A_j U_j^* - mI}{M - m} f(m) + \frac{MI - \sum_{j=1}^{k} U_j A_j U_j^*}{M - m} f(M)$$

$$\leq f(m) I + f(M) I - \sum_{j=1}^{k} U_j A_j U_j^*.$$

Proof. By the proof of Theorem 1 of [5], we know that for any real valued convex function, the following inequality holds

$$f(m + M - z) \leq \frac{M - (m + M - z)}{M - m} f(m) + \frac{(m + M - z) - m}{M - m} f(M)$$

$$= \frac{z - m}{M - m} f(m) + \frac{M - z}{M - m} f(M)$$

$$= f(m) + f(M) - \left(\frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) \right)$$

$$\leq f(m) + f(M) - f(z) \qquad (m + M - z \in [m, M]).$$

It follows at once from the above inequality that

$$f(mI + MI - A_j) \leq f(m) I + f(M) I - f(A_j),$$

where $mI \leq A_j \leq MI (j = 1, 2, \dots, k)$. The following matrix inequality can be deduced from above one

$$\sum_{j=1}^{k} U_j f(mI + MI - A_j) U_j^* \leq f(m) I + f(M) I - \sum_{j=1}^{k} U_j f(A_j) U_j^*.$$

□

Similarly, we can get the following theorem:

Theorem 2.2. Let $A_j (j = 1, 2, \dots, k)$ be Hermitian matrices of order $n \times n$ with eigenvalues in the interval $[m, M]$, and let $U_j (j = 1, 2, \dots, k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. If f is a real valued continuous convex function on $[m, M]$, then

$$\sum_{j=1}^{k} U_j f(mI + MI - A_j) U_j^* \geq f\left(\frac{m + M}{2}\right) I - \sum_{j=1}^{k} U_j f(A_j) U_j^*$$

$$\geq 2f\left(\frac{m + M}{2}\right) I - \left(\frac{MI - \sum_{j=1}^{k} U_j A_j U_j^*}{M - m} f(m) + \frac{\sum_{j=1}^{k} U_j A_j U_j^* - mI}{M - m} f(M) \right).$$

Proof. By the proof of Theorem 2.1, we know that for any a real valued convex function, the following inequality holds:

$$f(m + M - z) \geq 2f\left(\frac{m + M}{2}\right) - f(z)$$

$$\geq 2f\left(\frac{m + M}{2}\right) - \left(\frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) \right) \qquad (m + M - z \in [m, M]).$$

It follows from the above inequality that

$$f(mI + MI - A_j) \geq f\left(\frac{m + M}{2}\right) I - f(A_j)$$

$$\geq 2f\left(\frac{m + M}{2}\right) I - \left(\frac{MI - A_j}{M - m} f(m) + \frac{A_j - mI}{M - m} f(M) \right),$$

where $mI \leq A_j \leq MI (j = 1, 2, \dots, k)$. Further, the following matrix inequality can be deduced from above one

$$\sum_{j=1}^{k} U_j f\left(mI + MI - A_j\right) U_j^* \geq f\left(\frac{m+M}{2}\right) I - \sum_{j=1}^{k} U_j f\left(A_j\right) U_j^*$$

$$\geq 2f\left(\frac{m+M}{2}\right) I - \left(\frac{MI - \sum_{j=1}^{k} U_j A_j U_j^*}{M-m} f(m) + \frac{\sum_{j=1}^{k} U_j A_j U_j^* - mI}{M-m} f(M)\right).$$

□

As a special case of Theorem B, we give the following:

Theorem 2.3. Let A_j(j = 1,2,...,k) be Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$ and U_j(j = 1,2,...,k) be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. If f is a continuous convex function on $[m,M]$, J is an interval which satisfy $J \supset [m,M]$, $F(u,v) = u\text{-}v$ is matrix increasing in its first variable, then the following inequality holds:

$$\sum_{j=1}^{k} U_j f\left(A_j\right) U_j^* - f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right) \leq \max_{\theta \in [0,1]}\left[\theta f(m) + (1-\theta) f(M) - f(\theta m + (1-\theta) M)\right] I.$$

We can also get $\sum_{j=1}^{k} U_j f\left(A_j\right) U_j^* \leq \max_{\theta \in [0,1]}\left[\frac{\theta f(m) + (1-\theta) f(M)}{f(\theta m + (1-\theta) M)}\right] f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right).$

3. Matrix operator inequalities based on Jensen and Ando's

In this section, we discuss two issues, the first is matrix operator inequalities that get based on an important lemma, and the second is the Jensen inequality of matrix form and further research on the properties of the inequality.

In the first part, we suppose A is a positive definite Hermitian matrix of order $n \times n$ with $mI \leq A \leq MI$, $f(t)$ is defined as a nonnegative concave function on $[m,M]$ with $0 < m < M$, then $f(A)$ is defined as a positive operator by the usual functional calculus. A real valued function f is said to be operator monotone on interval $[m,M]$ if $A \geq B \Rightarrow f(A) \geq f(B)$ for every positive definite Hermitian matrices A and B whose eigenvalues are contained on $[m,M]$, and f is said to be operator concave on $[m,M]$ if $f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B)$ for all real numbers $0 \leq \lambda \leq 1$ and every positive definite Hermitian matrices A and B whose eigenvalues are contained in $[m,M]$.

We begin with the following lemma.

Lemma 3.1. Let A_j(j = 1,2,...,k) be positive definite Hermitian matrices of order $n \times n$ with eigenvalues in $[m,M]$, $f(t)$ be a nonnegative real value continuous strictly concave twice differentiable function on $[m,M]$ with $0 < m < M$, and U_j(j = 1,2,...,k) be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then for any given $\alpha > 0$,

$$\sum_{j=1}^{k} U_j f\left(A_j\right) U_j^* \geq \alpha f\left(\sum_{j=1}^{k} U_j A_j U_j^*\right) + \beta I \quad \text{holds for}$$

$$\beta = \beta\left(m, M, f, \alpha\right) = at + b - \alpha f\left(t_0\right),$$

where t_0 is defined as the unique solution of $f'(t) = \dfrac{a}{\alpha}$ when

$f'(M) \leq \dfrac{a}{\alpha} \leq f'(m) \left(a = \dfrac{f(M) - f(m)}{M - m}, b = \dfrac{Mf(m) - mf(M)}{M - m} \right)$; Otherwise t_0 is

defined as M or m according as $\dfrac{a}{\alpha} \leq f'(M)$ or $f'(m) \leq \dfrac{a}{\alpha}$.

Proof. We take $h(t) = at + b - \alpha f(t_0)$. Since $f(t)$ is strictly concave, its derivative $f'(t)$ is

strictly decreasing. Hence, if $f'(M) \leq \dfrac{a}{\alpha} \leq f'(m)$, then $h'(t) = 0$ occurs in interval $[m,$

$M]$ only at point t_0. Since $h''(t) = -\alpha f''(t) > 0$, this means that $\min\limits_{m \leq t \leq M} h(t) = h(t_0) \equiv \beta$.

Next, we consider $\dfrac{a}{\alpha} \leq f'(M)$. Since, $h'(t) < 0$, $h(t)$ is decreasing on $[m,M]$. Therefore,

we can deduce that $\min\limits_{m \leq t \leq M} h(t) = h(t_0) \equiv \beta$ as $t_0 = M$. Similarly, if $f'(m) \leq \dfrac{a}{\alpha}$, then

we deduce that $\min\limits_{m \leq t \leq M} h(t) = h(t_0) \equiv \beta$ as $t_0 = m$. It follows that $at + b \geq \alpha f(t) + \beta$ for $t \in$

$[m,M]$.

If applying the inequality above to $\sum\limits_{j=1}^{k} U_j A_j U_j^*$, we have

$$a \sum_{j=1}^{k} U_j A_j U_j^* + bI \geq \alpha f \left(\sum_{j=1}^{k} U_j A_j U_j^* \right) + \beta I,$$

where $mI \leq \sum\limits_{j=1}^{k} U_j A_j U_j^* \leq MI$.

On the other hand, since $f(t)$ is concave, we have $f(t) \geq at + b$ for $t \in [m,M]$, applying
this inequality to A_j, we have $f(A_j) \geq aA_j + bI$, and then

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \geq a \sum_{j=1}^{k} U_j A_j U_j^* + bI.$$

Combining these two inequalities we obtain

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \geq \alpha f \left(\sum_{j=1}^{k} U_j A_j U_j^* \right) + \beta I.$$

☐

Use a similar method, the following Theorem can be proved easily.

Remark 3.1. If we put $\alpha = 1$ in Lemma 3.1, then

$$-\beta I \geq f \left(\sum_{j=1}^{k} U_j A_j U_j^* \right) - \sum_{j=1}^{k} U_j f(A_j) U_j^*$$

holds for $\beta = at + b - f(t_0)$ and t_0 satisfy $f'(t) = a$. Since f is strictly concave, it can
deduce that $f'(M) \leq a \leq f'(m)$.

The following corollaries are the special case of Lemma 3.1.

Corollary 3.1. Let, $A_j(j = 1,2,...,k)$ be positive definite Hermitian matrices of order $n \times n$ with eigenvalues in interval $[m,M]$ that $0 <mI \leq A_j \leq MI$ and $0 <p < 1$

(*resp.* $p < 0$, $p > 1$), and let $U_j(j = 1,2,...,k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$.

Then for any given $\alpha > 0$

$$\sum_{j=1}^{k} U_j A_j^p U_j^* \geq \alpha \left(\sum_{j=1}^{k} U_j A_j U_j^* \right)^p + \beta I \quad \left(resp. \sum_{j=1}^{k} U_j A_j^p U_j^* \leq \alpha \left(\sum_{j=1}^{k} U_j A_j U_j^* \right)^p + \beta I \right)$$

holds for

$$\beta = \beta \left(m, M, t^p, \alpha \right) = \begin{cases} \alpha \left(p - 1 \right) \left(\dfrac{1}{\alpha p} \dfrac{M^p - m^p}{M - m} \right)^{\frac{p}{p-1}} + \dfrac{Mm^p - mM^p}{M - m} \\ \quad \text{if} \, pM^{p-1} \leq \dfrac{1}{\alpha} \dfrac{M^p - m^p}{M - m} \leq pm^{p-1}, \\ \min \left\{ (1 - \alpha) M^p, (1 - \alpha) m^p \right\} \quad \text{otherwise,} \end{cases}$$

resp.

$$\beta = \beta \left(m, M, t^p, \alpha \right) = \begin{cases} \alpha \left(p - 1 \right) \left(\dfrac{1}{\alpha p} \dfrac{M^p - m^p}{M - m} \right)^{\frac{p}{p-1}} + \dfrac{Mm^p - mM^p}{M - m} \\ \quad \text{if} \, pm^{p-1} \leq \dfrac{1}{\alpha} \dfrac{M^p - m^p}{M - m} \leq pM^{p-1}, \\ \max \left\{ (1 - \alpha) M^p, (1 - \alpha) m^p \right\} \quad \text{otherwise.} \end{cases}$$

Proof. We only prove the former, as in proof of Theorem 3.1, let $f(t) = t^p$, since t_0 is defined as the unique solution of $f'(t) = \dfrac{a}{\alpha}$ when $f'(m) \leq \dfrac{a}{\alpha} \leq f'(M)$, so we get

$$t_0 = \left(\frac{a}{p\alpha} \right)^{\frac{1}{p-1}} \quad \text{and} \quad \beta = \alpha (p - 1) \left(\frac{1}{\alpha p} \frac{M^p - m^p}{M - m} \right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M - m} \quad \text{by}$$

$f'(t_0) = \dfrac{a}{\alpha} = pt^{p-1}$, when $pM^{p-1} \leq \dfrac{1}{\alpha} \dfrac{M^p - m^p}{M - m} \leq pm^{p-1}$, where $a = \dfrac{f(M) - f(m)}{M - m}$ and

$b = \dfrac{Mf(m) - mf(M)}{M - m}$. Otherwise $\beta = \min\{(1-\alpha)M^p, (1-\alpha)m^p\}$. Hence the results obtained by the conclusion of Lemma 3.1 since $f(t) = t^p$.

As a special case of Corollary3.1, we get the following corollary.

Corollary 3.2. Let $A_j(j = 1,2,...,k)$ be positive definite Hermitian matrices of order $n \times n$ with eigenvalues in the interval $[m,M]$ that $0 <mI \leq A_j \leq MI$, let $U_j(j = 1,2,...,k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then for any given $\alpha > 0$

$$\sum_{j=1}^{k} U_j A_j^{-1} U_j^* \leq \alpha \left(\sum_{j=1}^{k} U_j A_j U_j^* \right)^{-1} + \beta I$$

holds for

$$
\beta = \beta\left(m, M, t^{-1}, \alpha\right) = \begin{cases} \dfrac{M+m}{mM} - 2\sqrt{\dfrac{\alpha}{Mm}} & \text{if } \dfrac{m}{M} \le \alpha \le \dfrac{M}{m}, \\ \max\left\{\dfrac{1-\alpha}{m}, \dfrac{1-a}{M}\right\} & \text{if either } 0 < \alpha < \dfrac{m}{M} \text{ or } \dfrac{M}{m} < \alpha. \end{cases}
$$

In particular,

$$
\sum_{j=1}^{k} U_j A_j^{-1} U_j^* - \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{-1} \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{Mm} I.
$$

$$
\sum_{j=1}^{k} U_j A_j^{-1} U_j^* \le \frac{(M+m)^2}{4Mm} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{-1}.
$$

Proof. The results obtain by the conclusion of Corollary 3.1 as $f(t) = t^{-1}$.

Remark 3.2. If we put $\alpha = 1$ and $p = 2$ in Corollary 3.1, then

$$
\sum_{j=1}^{k} U_j A_j^2 U_j^* - \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^2 \le \frac{(M-m)^2}{4} I.
$$

The results were obtained by Liu and Neudecker [7].

The theory of operator means for positive linear operators on a Hilbert space which in connection with Löwner's theory for operator monotone functions was established by Kubo and Ando [8]. In the following discussion, as matrix versions, we will give the generalization of matrix inequalities through the theory of operator means which we apply it to derive Jensen's and Ando's matrix inequalities. For that we give the following definition:

A map $(A,B) \rightarrow A\sigma B$ is called an operator mean if the following conditions are satisfied:

   (M1) Monotonicity: $A \le C$ and $B \le D$ imply $A\sigma B \le C\sigma D$;
   (M2) upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \sigma B_n \downarrow A\sigma B$;
   (M3) transformer inequality: $U(A\sigma B)U^* \le (UAU^*)\sigma(UBU^*)$ for any operator U.

If f is nonnegative operator concave function, we define an operator mean σ through the formula

$$
A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}
$$

for all positive definite Hermitian matrices A and B of order $n \times n$. Simple example of operator means is the geometric mean # defined by

$$
A \#_{\frac{1}{2}} B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}.
$$

Before proceeding further, we present a useful lemma.

Lemma 3.2. Let f be a nonnegative operator concave function on $[m,M]$ with $0 <m <M$, for all positive definite Hermitian matrices A and B, let U be any $r \times n$ matrix such that $UU^* = I$. Then the following statements are mutually equivalent:

  (i) $U(A\sigma B)U^* \leq (UAU^*)\sigma(UBU^*)$,
  (ii) $Uf(A)U^* \leq f(UAU^*)$.

Proof. By (i), (ii) is obvious as f is concave. Then we will show that (ii) implies (i). In fact,

$$U(A\sigma B)U^* = UA^{\frac{1}{2}}f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}U^*$$

$$= (UAU^*)^{\frac{1}{2}}(UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}}(UAU^*)^{\frac{1}{2}}$$

$$= (UAU^*)^{\frac{1}{2}}\left[\left((UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}\right)f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\left(A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}}\right)\right](UAU^*)^{\frac{1}{2}}$$

$$\leq (UAU^*)^{\frac{1}{2}}\left[f\left((UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}}\right)\right](UAU^*)^{\frac{1}{2}}$$

$$= (UAU^*)^{\frac{1}{2}}f\left((UAU^*)^{-\frac{1}{2}}UBU^*(UAU^*)^{-\frac{1}{2}}\right)(UAU^*)^{\frac{1}{2}}$$

$$= (UAU^*)\sigma(UBU^*)$$

holds for $VV^* = (UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}} = I$, where $V = (UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}$.

We now introduce some new theorems of operator means for concave matrix functions.

Theorem 3.1. Let A and B be positive definite Hermitian matrices such that $0 <m_1I \leq A \leq M_1I$, $0 <m_2I \leq B \leq M_2I$ and $f(t)$ be a nonnegative real valued continuous twice differentiable function on $[m,M]$ with $0 <m <M$, let U be $r \times n$ matrix such that $UU^* = I$. Then $U(A\sigma B)U^* \geq \alpha(UAU^*)\sigma(UBU^*)+\beta UAU^*$ holds for any given $\alpha > 0$ and $\beta = \beta(m,M,f,\alpha) = at+b-\alpha f(t_0)$ where t_0 is defined as the unique solution of $f'(t) = \dfrac{a}{\alpha}$

when $f'(M) \leq \dfrac{a}{\alpha} \leq f'(m)$, here $a = \dfrac{f(M) - f(m)}{M - m}$ and $b = \dfrac{Mf(m) - mf(M)}{M - m}$;

Otherwise t_0 is defined as M or m according as $\dfrac{a}{\alpha} \leq f'(M)$ or $f'(m) \leq \dfrac{a}{\alpha}$, where $m = \dfrac{m_2}{M_1}$ and $M = \dfrac{M_2}{m_1}$.

$U(B\sigma A)U^* \geq \alpha(UBU^*)\sigma(UAU^*)+\beta UBU^*$ holds for which is defined just as above with $m = \dfrac{m_1}{M_2}, M = \dfrac{M_1}{m_2}$.

Proof. We only prove the former. It follows from Lemma 3.1 that for a given $\alpha > 0$

$$\left((UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}\right)f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\left(A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}}\right)$$

$$\geq \alpha f\left((UAU^*)^{-\frac{1}{2}}UA^{\frac{1}{2}}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{\frac{1}{2}}U^*(UAU^*)^{-\frac{1}{2}}\right) + \beta I$$

holds for $VV^* = (UAU^*)^{-\frac{1}{2}} UA^{\frac{1}{2}} A^{\frac{1}{2}} U^* (UAU^*)^{-\frac{1}{2}} = I$ and $\beta = \beta \left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, \alpha \right)$,

where $V = (UAU^*)^{-\frac{1}{2}} UA^{\frac{1}{2}}$ and

$$U (A\sigma B) U^* = UA^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) A^{\frac{1}{2}} U^*$$

$$= (UAU^*)^{\frac{1}{2}} (UAU^*)^{-\frac{1}{2}} UA^{\frac{1}{2}} f \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) A^{\frac{1}{2}} U^* (UAU^*)^{-\frac{1}{2}} (UAU^*)^{\frac{1}{2}}$$

$$= (UAU^*)^{\frac{1}{2}} \left[\left((UAU^*)^{-\frac{1}{2}} UA^{\frac{1}{2}} \right) f \left(A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) \left(A^{\frac{1}{2}} U^* (UAU^*)^{-\frac{1}{2}} \right) \right] (UAU^*)^{\frac{1}{2}}$$

$$\leq (UAU^*)^{\frac{1}{2}} \left[\alpha f \left((UAU^*)^{-\frac{1}{2}} UA^{\frac{1}{2}} A^{-\frac{1}{2}} BA^{-\frac{1}{2}} A^{\frac{1}{2}} U^* (UAU^*)^{-\frac{1}{2}} \right) + \beta I \right] (UAU^*)^{\frac{1}{2}}$$

$$= (UAU^*)^{\frac{1}{2}} \left[\alpha f \left((UAU^*)^{-\frac{1}{2}} UBU^* (UAU^*)^{-\frac{1}{2}} \right) + \beta I \right] (UAU^*)^{\frac{1}{2}}$$

$$= \alpha (UAU^*) \sigma (UBU^*) + \beta UAU^*.$$

Remark 3.3. If we put $\alpha = 1$ in Theorem 3.1, then we have the following:

$$-\beta UAU^* \geq (UAU^*) \sigma (UBU^*) - U (A\sigma B) U^*$$
$$\text{resp.} \left(-\beta UBU^* \geq (UBU^*) \sigma (UAU^*) - U (B\sigma A) U^* \right)$$

holds for $\beta = at + b - f(t_0)$ and t_0 such that $f'(t) = a$, where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$

$$\left(\text{resp.} m = \frac{m_1}{M_2}, M = \frac{M_1}{m_2} \right)$$

.

If we put $f(t) = t^{\frac{1}{2}}$ in Theorem 3.1, the following Corollary 3.3 follows from Theorem 3.1.

Corollary 3.3. Let A and B be positive definite Hermitian matrices such that $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_2 I \leq B \leq M_2 I$. Let U be $r \times n$ matrix such that $UU^* = I$. Then for any given $\alpha > 0$

$$U \left(A \#_{\frac{1}{2}} B \right) U^* \geq \alpha (UAU^*) \#_{\frac{1}{2}} (UBU^*) + \beta UAU^*$$

holds for

$$\beta = \beta \left(m, M, t^{-1}, \alpha \right) = \begin{cases} \dfrac{Mm^{\frac{1}{2}} - mM^{\frac{1}{2}}}{M - m} - \dfrac{\alpha^2}{4} \left(\sqrt{M} + \sqrt{m} \right) & \text{if } \dfrac{2\sqrt{m}}{\sqrt{M} + \sqrt{m}} \leq \alpha \leq \dfrac{2\sqrt{M}}{\sqrt{M} + \sqrt{m}}, \\ \min \left\{ (1 - \alpha) M^{\frac{1}{2}}, (1 - \alpha) m^{\frac{1}{2}} \right\} & \text{otherwise,} \end{cases}$$

where $m = \dfrac{m_2}{M_1}$ and $M = \dfrac{M_2}{m_1}$, and

$$U\left(B\#_{\frac{1}{2}} A\right) U^* \geq \alpha \left(UBU^*\right) \#_{\frac{1}{2}} \left(UAU^*\right) + \beta UBU^*$$

holds for β which is defined just as above with $m = \dfrac{m_1}{M_2}, M = \dfrac{M_1}{m_2}$.

In the following part, according to the traditional Jensen inequality of convex function, we discuss Jensen inequality in matrix form that based on Hermitian matrix and conduct further research on the properties of Jensen inequality.

An inequality be said to be a Jensen inequality of Hermitian matrix is means that a convex function f defined on $I = [m,M]$ satisfies $f\left(\sum\limits_{i=1}^{k} P_i A_i\right) \leq \sum\limits_{i=1}^{k} P_i f(A_i)$, where A_i $(i = 1,...,k)$ are $n \times n$ Hermitian and commutative matrices with eigenvalues in $[m,M]$, $P_i > 0 (i = 1,...,k)$ with $\sum\limits_{i=1}^{k} P_i = 1$.

Lemma 3.3. Let f be convex function on $I = [m,M]$, $\lambda_i (i = 1,...,k) \in [m,M]$ and $P_i > 0$ $(i = 1,...,k)$ with $\sum\limits_{i=1}^{k} P_i = 1$, $\lambda_{i_j} (j = 1, \cdots, n) \in [m, M]$, then

$$\sum_{i=1}^{k} P_i f\left(\lambda_{i_j}\right) - f\left(\sum_{i=1}^{k} P_i \lambda_{i_j}\right) \leq \alpha_j f(m) + \beta_j f(M) - f\left(\alpha_j m + \beta_j M\right).$$

Proof. Since $\lambda_{i_j} (j = 1, \cdots, n) \in [m, M]$, there are sequences $\{u_{i_j}\}, \{v_{i_j}\} \in [0, 1]$ with $u_{i_j} + v_{i_j} = 1$, such that $\lambda_{i_j} = u_{i_j} m + v_{i_j} M, j = 1, 2, \ldots, n$, hence

$$\sum_{i=1}^{k} P_i f\left(\lambda_{i_j}\right) - f\left(\sum_{i=1}^{k} P_i \lambda_{i_j}\right)$$

$$= \sum_{i=1}^{k} P_i f\left(u_{i_j} m + v_{i_j} M\right) - f\left[\sum_{i=1}^{k} P_i \left(u_{i_j} m + v_{i_j} M\right)\right]$$

$$= \sum_{i=1}^{k} P_i f\left(u_{i_j} m + \left(1 - u_{i_j}\right) M\right) - f\left[\sum_{i=1}^{k} P_i \left(u_{i_j} m + \left(1 - u_{i_j}\right) M\right)\right]$$

$$\leq \sum_{i=1}^{k} P_i \left(u_{i_j} f(m) + \left(1 - u_{i_j}\right) f(M)\right) - f\left[m \sum_{i=1}^{k} P_i u_{i_j} + M \sum_{i=1}^{k} P_i \left(1 - u_{i_j}\right)\right]$$

$$= f(m) \left(\sum_{i=1}^{k} P_i u_{i_j}\right) + f(M) \left(1 - \sum_{i=1}^{k} P_i u_{i_j}\right) - f\left[m \sum_{i=1}^{k} P_i u_{i_j} + M \left(1 - \sum_{i=1}^{k} P_i u_{i_j}\right)\right].$$

Denoting $\sum\limits_{i=1}^{k} P_i u_{i_j} = \alpha_j, 1 - \sum\limits_{i=1}^{k} P_i u_{i_j} = \beta_j$, we have that $0 \leq \alpha_j, \beta_j \leq 1$ with $\alpha_j + \beta_j = 1$.

Consequently,

$$\sum_{i=1}^{k} P_i f\left(\lambda_{i_j}\right) - f\left(\sum_{i=1}^{k} P_i \lambda_{i_j}\right) \leq \alpha_j f(m) + \beta_j f(M) - f\left(\alpha_j m + \beta_j M\right).$$

From this lemma, we have

Theorem 3.2. Let f be convex function on $I = [m, M]$, $A_i (i = 1, ..., k)$, are Hermitian matrices of order $n \times n$ with eigenvalues in $[m, M]$ and commutative matrices; $P_i > 0$ ($i = 1, ..., k$) with $\sum_{i=1}^{k} P_i = 1$, then

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right) \leq \max_{\alpha_j} \left[\alpha_j f(m) + \beta_j f(M) - f(\alpha_j m + \beta_j M)\right] I.$$

Proof. As the proof in the definition, by Lemma 3, we get

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right)$$

$$= \sum_{i=1}^{k} P_i U^* \text{diag}\left[f(\lambda_{i_1}), \ldots, f(\lambda_{i_n})\right] U - f\left[\sum_{i=1}^{k} P_i U^* \text{diag}(\lambda_{i_1}, \ldots, \lambda_{i_n}) U\right]$$

$$= U^* \text{diag}\left[\sum_{i=1}^{k} P_i f(\lambda_{i_1}), \ldots, \sum_{i=1}^{k} P_i f(\lambda_{i_n})\right] U - U^* \text{diag}\left[f\left(\sum_{i=1}^{k} P_i \lambda_{i_1}\right), \ldots, f\left(\sum_{i=1}^{k} P_i \lambda_{i_n}\right)\right] U$$

$$= U^* \text{diag}\left[\sum_{i=1}^{k} P_i f(\lambda_{i_1}) - f\left(\sum_{i=1}^{k} P_i \lambda_{i_1}\right), \ldots, \sum_{i=1}^{k} P_i f(\lambda_{i_n}) - f\left(\sum_{i=1}^{k} P_i \lambda_{i_n}\right)\right] U$$

$$\leq U^* \text{diag}\left[(\alpha_1 f(m) + \beta_1 f(M) - f(\alpha_1 m + \beta_1 M)), \ldots, (\alpha_n f(m) + \beta_n f(M) - f(\alpha_n m + \beta_n M))\right] U$$

$$\leq \max_{\alpha_j}\left[\alpha_j f(m) + \beta_j f(M) - f(\alpha_j m + \beta_j M)\right] I.$$

Apply Theorem 3.2 with $f = x^2$, we obtain the following corollary which we call it pre-Gruss inequality of Hermitan matrix.

Corollary 3.4. Let $A_i (i = 1, ..., k)$, P_i be defined as above, then

$$\sum_{i=1}^{k} P_i A_i^2 - \left(\sum_{i=1}^{k} P_i A_i\right)^2 \leq \frac{1}{4}(M - m)^2 I.$$

Proof. By Theorem 3.2, we obtain at once the following inequality

$$\sum_{i=1}^{k} P_i A_i^2 - \left(\sum_{i=1}^{k} P_i A_i\right)^2 \leq \max_{\alpha_j}\left(\alpha_j m^2 + \beta_j M^2 - (\alpha_j m + \beta_j M)^2\right) I$$

$$\leq \max_{\alpha_j}\left(\alpha_j \beta_j (M - m)^2\right) I$$

$$= \frac{1}{4}(M - m)^2 I.$$

For

$$P_1 = P_2 = \cdots = P_k = \frac{1}{k}.$$

Let us define

$$P_0 = \left\{\frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}\right\}.$$

From Theorem 3.2, we have

Corollary 3.5. Let f, $A_i(i = 1,...,k)$, P_i be defined as above, then

$$\frac{f(A_1) + f(A_2) + \cdots + f(A_k)}{k} - f\left(\frac{A_1 + A_2 + \cdots + A_k}{k}\right)$$
$$\leq \max_{p \in P_0} \left[pf(m) + (1-p)f(M) - f(pm + (1-p)M)\right] I.$$

Theorem 3.3. Let f, $A_i(i = 1,...,k)$, P_i be defined as above, then we have

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right) \leq f(m) I + f(M) I - 2\left[f\left(\frac{M+m}{2}\right)\right] I.$$

Proof. From Theorem 3.2, we have

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right)$$
$$\leq U^* \text{diag}\left[(\alpha_1 f(m) + \beta_1 f(M) - f(\alpha_1 m + \beta_1 M)), \ldots, (\alpha_n f(m) + \beta_n f(M) - f(\alpha_n m + \beta_n M))\right] U.$$

Indeed,

$$\alpha_j f(m) + \beta_j f(M) - f(\alpha_j m + \beta_j M)$$
$$= f(m) + f(M) - \left[f(\alpha_j M + \beta_j m) + f(\alpha_j m + \beta_j M)\right]$$
$$\leq f(m) + f(M) - 2\left[\frac{1}{2}f(\alpha_j M + \beta_j m) + \frac{1}{2}f(\alpha_j m + \beta_j M)\right]$$
$$= f(m) + f(M) - 2\left[f\left(\frac{M+m}{2}\right)\right].$$

Hence,

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right) \leq f(m) I + f(M) I - 2\left[f\left(\frac{M+m}{2}\right)\right] I$$

Moreover, we get further conclusions

Theorem 3.4. Let f, $A_i(i = 1,...,k)$, P_i be defined as above, and f is also a differentiable convex function, then we have

$$\sum_{i=1}^{k} P_i f(A_i) - f\left(\sum_{i=1}^{k} P_i A_i\right) \leq \frac{1}{4}(M-m)\left[f'(M) - f'(m)\right] I.$$

Proof. Since f is convex, we have

$$f(\alpha_j m + \beta_j M) \geq f(m) + \beta_j(M-m)f'(m);$$
$$f(\alpha_j m + \beta_j M) \geq f(M) + \alpha_j(m-M)f'(M).$$

Therefore,

$$\alpha_j f(m) + \beta_j f(M) - f(\alpha_j m + \beta_j M)$$
$$= \alpha_j\left[f(m) - f(\alpha_j m + \beta_j M)\right] + \beta_j\left[f(M) - f(\alpha_j m + \beta_j M)\right]$$
$$\leq \alpha_j\left[f(m) - f(M) - \beta_j(M-m)f'(m)\right] + \beta_j\left[f(M) - f(m) - \alpha_j(m-M)f'(M)\right]$$
$$\leq \alpha_j \beta_j(m-M)f'(M) + \alpha_j \beta_j(M-m)f'(m)$$
$$= \alpha_j \beta_j(M-m)\left[f'(M) - f'(m)\right]$$
$$\leq \frac{1}{4}(M-m)\left[f'(M) - f'(m)\right].$$

Hence,

$$\sum_{i=1}^{k} P_i f\left(A_i\right) - f\left(\sum_{i=1}^{k} P_i A_i\right) \leq \max_{\alpha_j}\left[\alpha_j f\left(m\right) + \beta_j f\left(M\right) - f\left(\alpha_j m + \beta_j M\right)\right] I$$

$$\leq \frac{1}{4}\left(M - m\right)\left[f'\left(M\right) - f'\left(m\right)\right] I.$$

Thus, the proof is complete.

Authors' contributions

WJ carried out Matrix operator theory studies, participated in the conception and design. XL conceived of the study, participated in its design and drafting the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests. Xue Lanlan is responsible for all the responsibility of the article appeared.

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