Weidong and Chen *Journal of Inequalities and Applications* 2011, **2011**:6
http://www.journalofinequalitiesandapplications.com/content/2011/1/6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

L_p-Dual geominimal surface area

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Abstract

Lutwak proposed the notion of L_p-geominimal surface area according to the L_p-mixed volume. In this article, associated with the L_p-dual mixed volume, we introduce the L_p-dual geominimal surface area and prove some inequalities for this notion.

2000 Mathematics Subject Classification: 52A20 52A40.

Keywords: L_p-geominimal surface area, L_p-mixed volume, L_p-dual geominimal surface area, L_p-dual mixed volume

1 Introduction and main results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in R^n. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$; denote by $V(K)$ the n-dimensional volume of body K; for the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

The notion of geominimal surface area was given by Petty [1]. For $K \in \mathcal{K}^n$, the geominimal surface area, $G(K)$, of K is defined by

$$\omega_n^{\frac{1}{n}} G(K) = \inf\{nV_1(K, Q)V(Q^*)^{\frac{1}{n}} : Q \in K^n\}.$$

Here Q^* denotes the polar of body Q and $V_1(M, N)$ denotes the mixed volume of $M, N \in \mathcal{K}^n$[2].

According to the L_p-mixed volume, Lutwak [3] introduced the notion of L_p-geominimal surface area. For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{1.1}$$

Here $V_p(M, N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$[3,4]. Obviously, if $p = 1$, $G_p(K)$ is just the geominimal surface area $G(K)$. Further, Lutwak [3] proved the following result for the L_p-geominimal surface area.

Theorem 1.A. *If* $K \in \mathcal{K}_o^n$, $p \geq 1$, *then*

$$G_p(K) \leq n\omega_n^{\frac{p}{n}} V(K)^{\frac{n-p}{n}}, \tag{1.2}$$

with equality if and only if K *is an ellipsoid.*

Lutwak [3] also defined the L_p-geominimal area ratio as follows: For $K \in K_o^n$, the L_p-geominimal area ratio of K is defined by

$$\left(\frac{G_p(K)^n}{n^n V(K)^{n-p}}\right)^{\frac{1}{p}}. \tag{1.3}$$

Lutwak [3] proved (1.3) is monotone nondecreasing in p, namely

Theorem 1.B. *If* $K \in \mathcal{K}_o^n$, $1 \le p < q$, *then*

$$\left(\frac{G_p(K)^n}{n^n V(K)^{n-p}}\right)^{\frac{1}{p}} \le \left(\frac{G_q(K)^n}{n^n V(K)^{n-q}}\right)^{\frac{1}{q}}$$

with equality if and only if K *and* T_pK *are dilates.*

Here T_pK denotes the L_p-Petty body of $K \in \mathcal{K}_o^n$[3].

Above, the definition of L_p-geominimal surface area is based on the L_p-mixed volume. In this paper, associated with the L_p-dual mixed volume, we give the notion of L_p-dual geominimal surface area as follows: For $K \in \mathcal{S}_c^n$, and $p \ge 1$, the L_p-dual geominimal surface area, $\tilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(K) = \inf\{n\tilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\}. \tag{1.4}$$

Here, $\tilde{V}_{-p}(M, N)$ denotes the L_p-dual mixed volume of $M, N \in \mathcal{S}_o^n$[3].

For the L_p-dual geominimal surface area, we proved the following dual forms of Theorems 1.A and 1.B, respectively.

Theorem 1.1. *If* $K \in \mathcal{S}_c^n$, $p \ge 1$, *then*

$$\tilde{G}_{-p}(K) \ge n\omega_n^{-\frac{p}{n}} V(K)^{\frac{n+p}{n}} \tag{1.5}$$

with equality if and only if K *is an ellipsoid centered at the origin.*

Theorem 1.2. *If* $K \in \mathcal{S}_c^n$, $1 \le p < q$, *then*

$$\left(\frac{\tilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}} \le \left(\frac{\tilde{G}_{-q}(K)^n}{n^n V(K)^{n+q}}\right)^{\frac{1}{q}} \tag{1.6}$$

with equality if and only if $K \in \mathcal{K}_o^n$.

Here

$$\left(\frac{\tilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}}$$

may be called the L_p-dual geominimal surface area ratio of $K \in \mathcal{S}_c^n$.

Further, we establish Blaschke-Santaló type inequality for the L_p-dual geominimal surface area as follows:

Theorem 1.3. *If* $K \in \mathcal{K}_c^n$, $n \ge p \ge 1$, *then*

$$\tilde{G}_{-p}(K)\tilde{G}_{-p}(K^*) \le n^2\omega_n^2 \tag{1.7}$$

with equality if and only if K *is an ellipsoid.*

Finally, we give the following Brunn-Minkowski type inequality for the L_p-dual geominimal surface area.

Theorem 1.4. *If* $K, L \in \mathcal{S}_o^n$, $p \geq 1$ *and* $\lambda, \mu \geq 0$ *(not both zero), then*

$$\tilde{G}_{-p}(\lambda \star K +_{-p} \mu \star L)^{-\frac{p}{n+p}} \geq \lambda \tilde{G}_{-p}(K)^{-\frac{p}{n+p}} + \mu \tilde{G}_{-p}(L)^{-\frac{p}{n+p}} \tag{1.8}$$

with equality if and only if K and L are dilates.

Here $\lambda \star K +_{-p} \mu \star L$ denotes the L_p-harmonic radial combination of K and L.

The proofs of Theorems 1.1-1.3 are completed in Section 3 of this paper. In Section 4, we will give proof of Theorem 1.4.

2 Preliminaries

2.1 Support function, radial function and polar of convex bodies

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot)$: $\mathbb{R}^n \to (-\infty, \infty)$, is defined by [5,6]

$$h(K, x) = \max\{x \cdot y : y \in K\}, x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) in R^n, then its radial function, $\rho_K = \rho(K,\cdot)$: $R^n \setminus \{0\} \to [0, \infty)$, is defined by [5,6]

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, u \in S^{n-1}.$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by [5,6]

$$K^* = \{x \in R^n : x \cdot y \leq 1, \ y \in K\}. \tag{2.1}$$

For $K \in \mathcal{K}_o^n$, if $\varphi \in GL(n)$, then by (2.1) we know that

$$(\phi K)^* = \phi^{-\tau} K^*. \tag{2.2}$$

Here $GL(n)$ denotes the group of general (nonsingular) linear transformations and $\varphi^{-\tau}$ denotes the reverse of transpose (transpose of reverse) of φ.

For $K \in \mathcal{K}_o^n$ and its polar body, the well-known Blaschke-Santaló inequality can be stated that [5]:

Theorem 2.A. *If* $K \in \mathcal{K}_c^n$, *then*

$$V(K)V(K^*) \leq \omega_n^2 \tag{2.3}$$

with equality if and only if K is an ellipsoid.

2.2 L_p-Mixed volume

For $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathcal{K}_o^n$ is defined by [7]

$$h(K +_p \in \cdot L, \ \cdot)^p = h(K, \ \cdot)^p + \varepsilon h(L, \ \cdot)^p,$$

where "·" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

If $K, L \in \mathcal{K}_o^n$, then for $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of K and L is defined by [4]

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The L_p-Minkowski inequality can be stated that [4]:

Theorem 2.B. *If* $K, L \in \mathcal{K}_o^n$ *and* $p \geq 1$ *then*

$$V_p(K,\ L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}} \tag{2.4}$$

with equality for $p > 1$ *if and only if K and L are dilates, for* $p = 1$ *if and only if K and L are homothetic.*

2.3 L_p-Dual mixed volume

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and λ, $\mu \geq 0$ (not both zero), the L_p harmonic-radial combination, $\lambda \star K \tilde{+}_{-p} \mu \star L \in \mathcal{S}_o$ of K and L is defined by [3]

$$\rho(\lambda \star K +_{-p} \mu \star L,\ \cdot)^{-p} = \lambda \rho(K,\ \cdot)^{-p} + \mu \rho(L,\ \cdot)^{-p}. \tag{2.5}$$

From (2.5), for $\varphi \in GL(n)$, we have that

$$\phi(\lambda \star K +_{-p} \mu \star L) = \lambda \star \phi K +_{-p} \mu \star \phi L. \tag{2.6}$$

Associated with the L_p-harmonic radial combination of star bodies, Lutwak [3] introduced the notion of L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\tilde{V}_{-p}(K,\ L)$ of the K and L is defined by [3]

$$\frac{n}{-p} \tilde{V}_{-p}(K,\ L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}. \tag{2.7}$$

The definition above and Hospital's role give the following integral representation of the L_p-dual mixed volume [3]:

$$\tilde{V}_{-p}(K,\ L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u), \tag{2.8}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.8), we get

$$\tilde{V}_{-p}(K,\ K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) dS(u). \tag{2.9}$$

The Minkowski's inequality for the L_p-dual mixed volume is that [3]

Theorem 2.C. *Let* $K, L \in \mathcal{S}_o^n$, $p \geq 1$, *then*

$$\tilde{V}_{-p}(K,\ L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}} \tag{2.10}$$

with equality if and only if K and L are dilates.

2.4 L_p-Curvature image

For $K \in \mathcal{K}_o^n$, and real $p \geq 1$, the L_p-surface area measure, $S_p(K, \cdot)$, of K is defined by [4]

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K,\ \cdot)^{1-p}. \tag{2.11}$$

Equation (2.11) is also called Radon-Nikodym derivative, it turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to surface area measure $S(K, \cdot)$.

A convex body $K \in \mathcal{K}_o^n$ is said to have an L_p-curvature function [3]$f_p(K, \cdot)$: $S^{n-1} \to \mathbb{R}$, if its L_p-surface area measure $S_p(K, \cdot)$ is absolutely continuous with respect to spherical

Lebesgue measure S, and

$$f_p(K,\ \cdot) = \frac{dS_p(K, \cdot)}{dS}.$$

Let $\mathcal{F}_o^n, \mathcal{F}_c^n$, denote set of all bodies in $\mathcal{K}_o^n$, $\mathcal{K}_c^n$, respectively, that have a positive continuous curvature function.

Lutwak [3] showed the notion of L_p-curvature image as follows: For each $K \in \mathcal{F}_o^n$ and real $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the L_p-curvature image of K, by

$$\rho(\Lambda_p K,\ \cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K,\ \cdot).$$

Note that for $p = 1$, this definition differs from the definition of classical curvature image [3]. For the studies of classical curvature image and L_p-curvature image, one may see [6,8-12].

3 L_p-Dual geominimal surface area

In this section, we research the L_p-dual geominimal surface area. First, we give a property of the L_p-dual geominimal surface area under the general linear transformation. Next, we will complete proofs of Theorems 1.1-1.3.

For the L_p-geominimal surface area, Lutwak [3] proved the following a property under the special linear transformation.

Theorem 3.A. *For* $K \in \mathcal{K}_o^n$, $p \geq 1$, *if* $\varphi \in SL(n)$, *then*

$$G_p(\phi K) = G_p(K). \tag{3.1}$$

Here $SL(n)$ denotes the group of special linear transformations.

Similar to Theorem 3.A, we get the following result of general linear transformation for the L_p-dual geominimal surface area:

Theorem 3.1. *For* $K \in \mathcal{S}_c^n$, $p \geq 1$, *if* $\varphi \in GL(n)$, *then*

$$\tilde{G}_{-p}(\phi K) = |\,det\phi\,|^{\frac{n+p}{n}} \tilde{G}_{-p}(K). \tag{3.2}$$

Lemma 3.1. *If* $K, L \in \mathcal{S}_o^n$ *and* $p \geq 1$, *then for* $\varphi \in GL(n)$,

$$\tilde{V}_{-p}(\phi K,\ \phi L) = |\,det\phi\,| \tilde{V}_{-p}(K,\ L). \tag{3.3}$$

Note that for $\varphi \in SL(n)$, proof of (3.3) may be fund in [3].

Proof. From (2.6), (2.7) and notice the fact $V(\varphi K) = |det\varphi| V(K)$, we have

$$\begin{aligned}
\frac{n}{-p}\tilde{V}_{-p}(\phi K,\ \phi L) &= \lim_{\varepsilon \to 0^+} \frac{V(\phi K +_{-p} \varepsilon \star \phi L) - V(\phi K)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} \frac{V[\phi(K +_{-p} \varepsilon \star L)] - V(\phi K)}{\varepsilon} \\
&= |\,det\phi\,| \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon} \\
&= |\,det\phi\,| \tilde{V}_{-p}(K,\ L).
\end{aligned}$$

□

Proof of Theorem 3.1. From (1.4), (3.3) and (2.2), we have

$$\begin{aligned}\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(\phi K) &= \inf\{n\tilde{V}_{-p}(\phi K,\ Q)V(Q^*)^{-\frac{p}{n}} : Q\in\mathcal{K}_c^n\}\\ &= \inf\{n|det\phi|\tilde{V}_{-p}(K,\ \phi^{-1}Q)V(Q^*)^{-\frac{p}{n}} : Q\in\mathcal{K}_c^n\}\\ &= \inf\{n|det\phi|\tilde{V}_{-p}(K,\ \phi^{-1}Q)V(\phi^{-\tau}\phi^{\tau}Q^*)^{-\frac{p}{n}} : Q\in\mathcal{K}_c^n\}\\ &= \inf\{n|det\phi||det(\phi^{-\tau})|^{-\frac{p}{n}}\tilde{V}_{-p}(K,\ \phi^{-1}Q)V((\phi^{-1}Q)^*)^{-\frac{p}{n}} : Q\in\mathcal{K}_c^n\}\\ &= |det\phi|^{\frac{n+p}{n}}\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(K).\end{aligned}$$

This immediately yields (3.2). □

Actually, using definition (1.1) and fact [13]: *If* $K, L\in\mathcal{K}_o^n$ *and* $p\geq 1$, *then for* $\varphi\in GL(n)$,

$$V_p(\phi K,\phi L) = |\,det\phi\,|V_p(K,L),$$

we may extend Theorem 3.A as follows:

Theorem 3.2. *For* $K\in\mathcal{K}_o^n$, $p\geq 1$, *if* $\varphi\in GL(n)$, *then*

$$G_p(\phi K) = |\,det\phi\,|^{\frac{n-p}{n}}G_p(K). \tag{3.4}$$

Obviously, (3.2) is dual form of (3.4). In particular, if $\varphi\in SL(n)$, then (3.4) is just (3.1).

Now we prove Theorems 1.1-1.3.

Proof of Theorem 1.1. From (2.10) and Blaschke-Santaló inequality (2.3), we have that

$$\tilde{V}_{-p}(K,\ Q)V(Q^*)^{-\frac{p}{n}} \geq V(K)^{\frac{n+p}{n}}[V(Q)V(Q^*)]^{-\frac{p}{n}} \geq \omega_n^{-\frac{2p}{n}}V(K)^{\frac{n+p}{n}}.$$

Hence, using definition (1.4), we know

$$\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(K) \geq n\omega_n^{-\frac{2p}{n}}V(K)^{\frac{n+p}{n}},$$

this yield inequality (1.5). According to the equality conditions of (2.3) and (2.10), we see that equality holds in (1.5) if and only if K and $Q\in\mathcal{K}_c^n$ are dilates and Q is an ellipsoid, i.e. K is an ellipsoid centered at the origin. □

Compare to inequalities (1.2) and (1.5), we easily get that

Corollary 3.1. *For* $K\in\mathcal{K}_o^n$, $p\geq 1$, *then for* $n > p$,

$$\tilde{G}_{-p}(K) \geq (n\omega_n)^{-\frac{2p}{n-p}}G_p(K)^{\frac{n+p}{n-p}},$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.2. Using the Hölder inequality, (2.8) and (2.9), we obtain

$$\begin{aligned}\tilde{V}_{-p}(K,\ Q) &= \frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(u)\rho_Q^{-p}(u)dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}}[\rho_K^{n+q}(u)\rho_Q^{-q}(u)]^{\frac{p}{q}}[\rho_K^{n}(u)]^{\frac{q-p}{q}}dS(u)\\ &\leq \tilde{V}_{-q}(K,\ Q)^{\frac{p}{q}}V(K)^{\frac{q-p}{q}},\end{aligned}$$

that is

$$\left(\frac{\tilde{V}_{-p}(K,Q)}{V(K)}\right)^{\frac{1}{p}} \leq \left(\frac{\tilde{V}_{-q}(K,Q)}{V(K)}\right)^{\frac{1}{q}}. \tag{3.5}$$

According to equality condition in the Hölder inequality, we know that equality holds in (3.5) if and only if K and Q are dilates.

From definition (1.4) of $\tilde{G}_{-p}(K)$, we obtain

$$\begin{aligned}\left(\frac{\tilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}} &= \inf\left\{\left(\frac{\tilde{V}_{-p}(K,Q)}{V(K)}\right)^{\frac{n}{p}} \frac{V(Q^*)^{-1}}{V(K)} : Q \in \mathcal{K}_c^n\right\} \\ &\leq \inf\left\{\left(\frac{\tilde{V}_{-q}(K,Q)}{V(K)}\right)^{\frac{n}{q}} \frac{V(Q^*)^{-1}}{V(K)} : Q \in \mathcal{K}_c^n\right\} \\ &= \left(\frac{\tilde{G}_{-q}(K)^n}{n^n V(K)^{n+q}}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.6}$$

This gives inequality (1.6).

Because of $Q \in \mathcal{K}_c^n$ in inequality (3.6), this together with equality condition of (3.5), we see that equality holds in (1.6) if and only if $K \in \mathcal{K}_c^n$. □

Proof of Theorem 1.3. From definition (1.4), it follows that for $Q \in \mathcal{K}_c^n$,

$$\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(K) \leq n\tilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}}.$$

Since $K \in \mathcal{K}_c^n$, taking K for Q, and using (2.9), we can get

$$\begin{aligned}\tilde{G}_{-p}(K) &\leq n\omega_n^{\frac{p}{n}} \tilde{V}_{-p}(K,K)\, V(K^*)^{-\frac{p}{n}} \\ &= n\omega_n^{\frac{p}{n}} V(K)\, V(K^*)^{-\frac{p}{n}}.\end{aligned} \tag{3.7}$$

Similarly,

$$\tilde{G}_{-p}(K^*) \leq n\omega_n^{\frac{p}{n}} V(K^*)\, V(K)^{-\frac{p}{n}}. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\tilde{G}_{-p}(K)\tilde{G}_{-p}(K^*) \leq n^2 \omega_n^{\frac{2p}{n}} \left[V(K)\, V(K^*)\right]^{\frac{n-p}{n}}.$$

Hence, for $n \geq p$ using (2.3), we obtain

$$\tilde{G}_{-p}(K)\tilde{G}_{-p}(K^*) \leq n^2 \omega_n^{\frac{2p}{n}} \left[\omega_n^2\right]^{\frac{n-p}{n}} = n^2\omega_n^2.$$

According to the equality condition of (2.3), we see that equality holds in (1.7) if and only if K is an ellipsoid. □

Associated with the L_p-curvature image of convex bodies, we may give a result more better than inequality (1.5) of Theorem 1.1.

Theorem 3.3. *If* $K \in \mathcal{F}_o^n$, $p \geq 1$, *then*

$$\tilde{G}_{-p}(\Lambda_p K) \geq n\omega_n^{\frac{p-n}{n}} V(\Lambda_p K)V(K)^{\frac{n-p}{n}}, \tag{3.9}$$

with equality if and only if $K \in \mathcal{F}_c^n$.

Lemma 3.2 [3]. *If* $K \in \mathcal{F}_o^n$, $p \geq 1$, *then for any* $Q \in \mathcal{S}_o^n$,

$$V_p(K, Q^*) = \frac{\omega_n \tilde{V}_{-p}(\Lambda_p K, Q)}{V(\Lambda_p K)}. \tag{3.10}$$

Proof of Theorem 3.3. From (1.4), (3.10) and (2.4), we have that

$$\begin{aligned}
\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(\Lambda_p K) &= \inf\{n\tilde{V}_{-p}(\Lambda_p K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&= \inf\{n\omega_n^{-1} V(\Lambda_p K)V_p(K, Q^*)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&\geq \inf\{n\omega_n^{-1} V(\Lambda_p K)V(K)^{\frac{n-p}{n}} V(Q^*)^{\frac{p}{n}} V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&= \inf\{n\omega_n^{-1} V(\Lambda_p K)V(K)^{\frac{n-p}{n}}\} \\
&= n\omega_n^{-1} V(\Lambda_p K)V(K)^{\frac{n-p}{n}}.
\end{aligned}$$

This yields (3.9). According to the equality condition in inequality (2.4), we see that equality holds in inequality (3.9) if and only if K and Q^* are dilates. Since $Q \in \mathcal{K}_c^n$, equality holds in inequality (3.9) if and only if $K \in \mathcal{K}_c^n$. □

Recall that Lutwak [3] proved that *if* $K \in \mathcal{F}_c^n$ *and* $p \geq 1$, *then*

$$V(\Lambda_p K) \leq \omega_n^{\frac{2p-n}{p}} V(K)^{\frac{n-p}{n}}, \tag{3.11}$$

with equality if and only if K is an ellipsoid.

From (3.9) and (3.11), we easily get that *if* $K \in \mathcal{F}_c^n$ *and* $p \geq 1$, *then*

$$\tilde{G}_{-p}(\Lambda_p K) \geq n\omega_n^{-\frac{p}{n}} V(\Lambda_p K)^{\frac{n+p}{n}}, \tag{3.12}$$

with equality if and only if K is an ellipsoid.

Inequality (3.12) just is inequality (1.5) for the L_p-curvature image.

In addition, by (1.2) and (3.9), we also have that

Corollary 3.2. *If* $K \in \mathcal{K}_c^n$, $p \geq 1$, *then*

$$\tilde{G}_{-p}(\Lambda_p K) \geq \frac{V(\Lambda_p K)}{\omega_n} G_p(K),$$

with equality if and only if K is an ellipsoid.

4 Brunn-Minkowski type inequalities

In this section, we first prove Theorem 1.4. Next, associated with the L_p-harmonic radial combination of star bodies, we give another Brunn-Minkowski type inequality for the L_p-dual geominimal surface area.

Lemma 4.1. *If* $K, L \in \mathcal{S}_o^n$, $p \geq 1$ *and* λ, $\mu \geq 0$ *(not both zero) then for any* $Q \in \mathcal{S}_o^n$,

$$\tilde{V}_{-p}(\lambda \star K \tilde{+}_{-p} \mu \star L, Q)^{-\frac{p}{n+p}} \geq \lambda \tilde{V}_{-p}(K, Q)^{-\frac{p}{n+p}} + \mu \tilde{V}_{-p}(L, Q)^{-\frac{p}{n+p}} \tag{4.1}$$

with equality if and only if K and L are dilates.

Proof. Since $-(n + p)/p < 0$, thus by (2.5), (2.8) and Minkowski's integral inequality (see [14]), we have for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
&\tilde{V}_{-p}(\lambda \star K +_{-p} \mu \star L, Q)^{-\frac{p}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(\lambda \star K +_{-p} \mu \star L, u)^{n+p}\rho(Q,u)^{-p}du\right]^{-\frac{p}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [\rho(\lambda \star K +_{-p} \mu \star L, u)^{-p}\rho(Q,u)^{\frac{p^2}{n+p}}]^{-\frac{n+p}{p}} du\right]^{-\frac{p}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [(\lambda\rho(K,u)^{-p} + \mu\rho(L,u)^{-p})\rho(Q,\ u)^{\frac{p^2}{n+p}}]^{-\frac{n+p}{p}} du\right]^{-\frac{p}{n+p}} \\
&\geq \lambda\left[\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(Q,u)^{-p}du\right]^{-\frac{p}{n+p}} \\
&\quad + \mu\left[\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n+p}\rho(Q,u)^{-p}du\right]^{-\frac{p}{n+p}} \\
&= \lambda\tilde{V}_{-p}(K,Q)^{-\frac{p}{n+p}} + \mu\tilde{V}_{-p}(L,Q)^{-\frac{p}{n+p}}.
\end{aligned}$$

According to the equality condition of Minkowski's integral inequality, we see that equality holds in (4.1) if and only if K and L are dilates. □

Proof of Theorem 1.4. From definition (1.4) and inequality (4.1), we obtain

$$\begin{aligned}
&[\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(\lambda \star K +_{-p} \mu \star L)]^{-\frac{p}{n+p}} \\
&= \inf\{[n\tilde{V}_{-p}(\lambda \star K +_{-p} \mu \star L, Q)V(Q^*)^{-\frac{p}{n}}]^{-\frac{p}{n+p}} : Q \in \mathcal{K}_c^n\} \\
&= \inf\{[n\tilde{V}_{-p}(\lambda \star K +_{-p} \mu \star L,\ Q)]^{-\frac{p}{n+p}}V(Q^*)^{\frac{p^2}{n(n+p)}} : Q \in \mathcal{K}_c^n\} \\
&\geq \inf\{[\lambda(n\tilde{V}_{-p}(K,Q))^{-\frac{p}{n+p}} + \mu(n\tilde{V}_{-p}(L,Q))^{-\frac{p}{n+p}}]V(Q^*)^{\frac{p^2}{n(n+p)}} : Q \in \mathcal{K}_c^n\} \\
&\geq \inf\{\lambda[n\tilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}}]^{-\frac{p}{n+p}} : Q \in \mathcal{K}_c^n\} \\
&\quad + \inf\{\mu[n\tilde{V}_{-p}(K,Q)V(Q^*)^{-\frac{p}{n}}]^{-\frac{p}{n+p}} : Q \in \mathcal{K}_c^n\} \\
&= \lambda[\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(K)]^{-\frac{p}{n+p}} + \mu[\omega_n^{-\frac{p}{n}}\tilde{G}_{-p}(L)]^{-\frac{p}{n+p}}.
\end{aligned}$$

This yields inequality (1.8).

By the equality condition of (4.1) we know that equality holds in (1.8) if and only if K and L are dilates. □

The notion of L_p-radial combination can be introduced as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and λ, $\mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in S_o^n$, of K and L is defined by [15]

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{4.2}$$

Under the definition (4.2) of L_p-radial combination, we also obtain the following Brunn-Minkowski type inequality for the L_p-dual geominimal surface area.

Theorem 4.1. *If $K, L \in K_c^n$, $p \geq 1$ and λ, $\mu \geq 0$ (not both zero), then*

$$\tilde{G}_{-p}(\lambda \circ K \tilde{+}_{n+p} \mu \circ L) \geq \lambda \tilde{G}_{-p}(K) + \mu \tilde{G}_{-p}(L) \tag{4.3}$$

with equality if and only if K and L are dilates.

Proof. From definitions (1.4), (4.2) and formula (2.8), we have

$$\begin{aligned}
&\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(\lambda \circ K \tilde{+}_{n+p} \mu \circ L) \\
&= \inf\{n\tilde{V}_{-p}(\lambda \circ K \tilde{+}_{n+p} \mu \circ L, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&= \inf\{n[\lambda \tilde{V}_{-p}(K, Q) + \mu \tilde{V}_{-p}(L, Q)]V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&= \inf\{n\lambda \tilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} + n\mu \tilde{V}_{-p}(L, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&\geq \inf\{n\lambda \tilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&\quad + \inf\{n\mu \tilde{V}_{-p}(L, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_c^n\} \\
&= \omega_n^{-\frac{p}{n}} \lambda \tilde{G}_{-p}(K) + \omega_n^{-\frac{p}{n}} \mu \tilde{G}_{-p}(L).
\end{aligned}$$

Thus

$$\tilde{G}_{-p}(\lambda \circ K \tilde{+}_{n+p} \mu \circ L) \geq \lambda \tilde{G}_{-p}(K) + \mu \tilde{G}_{-p}(L).$$

The equality holds if and only if $\lambda \circ K \tilde{+}_{n+p} \mu \circ L$ are dilates with K and L, respectively. This mean that equality holds in (4.3) if and only if K and L are dilates. □

Acknowledgements
We wish to thank the referees for this paper. Research is supported in part by the Natural Science Foundation of China (Grant No. 10671117) and Science Foundation of China Three Gorges University.

Authors' contributions
In the article, WW complete the proof of Theorems 1.1-1.3, 3.1-3.3, QC give the proof of Theorems 1.4 and 4.1. WW carry out the writing of whole manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 1 December 2010 Accepted: 17 June 2011 Published: 17 June 2011

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doi:10.1186/1029-242X-2011-6
Cite this article as: Weidong and Chen: **L_p-Dual geominimal surface area.** *Journal of Inequalities and Applications* 2011 **2011**:6.