Rachůnková and Tomeček *Boundary Value Problems* 2014, 2014:118
http://www.boundaryvalueproblems.com/content/2014/1/118

Boundary Value Problems
a SpringerOpen Journal

RESEARCH ARTICLE Open Access

Existence principle for higher-order nonlinear differential equations with state-dependent impulses via fixed point theorem

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Dedicated to Professor Ivan Kiguradze for his merits in mathematical sciences

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Abstract

The paper provides an existence principle for a general boundary value problem of the form $\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h(t, u(t), \dots, u^{(n-1)}(t))$, a.e. $t \in [a,b] \subset \mathbb{R}$, $\ell_k(u, u', \dots, u^{(n-1)}) = c_k$, $k = 1, \dots, n$, with the state-dependent impulses $u^{(j)}(t+) - u^{(j)}(t-) = J_{ij}(u(t-), u'(t-), \dots, u^{(n-1)}(t-))$, where the impulse points t are determined as solutions of the equations $t = \gamma_i(u(t-), u'(t-), \dots, u^{(n-2)}(t-))$, $i = 1, \dots, p$, $j = 0, \dots, n-1$. Here, $n, p \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{R}$, the functions a_j/a_n, $j = 0, \dots, n-1$, are Lebesgue integrable on $[a,b]$ and h/a_n satisfies the Carathéodory conditions on $[a,b] \times \mathbb{R}^n$. The impulse functions J_{ij}, $i = 1, \dots, p$, $j = 0, \dots, n-1$, and the barrier functions γ_i, $i = 1, \dots, p$, are continuous on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. The functionals ℓ_k, $k = 1, \dots, n$, are linear and bounded on the space of left-continuous regulated (*i.e.* having finite one-sided limits at each point) on $[a,b]$ vector functions. Provided the data functions h and J_{ij} are bounded, transversality conditions which guarantee that each possible solution of the problem in a given region crosses each barrier γ_i at the unique impulse point τ_i are presented, and consequently the existence of a solution to the problem is proved.
MSC: Primary 34B37; secondary 34B10; 34B15

Keywords: nonlinear higher-order ODE; state-dependent impulses; general linear boundary conditions; transversality conditions; fixed point

1 Introduction

In this paper we are interested in the nonlinear ordinary differential equation of the nth-order ($n \geq 2$) with state-dependent impulses and general linear boundary conditions on the interval $[a,b] \subset \mathbb{R}$. Studies of real-life problems with state-dependent impulses can be found *e.g.* in [1–6]. Here we consider the equation

$$\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = h\bigl(t, u(t), \dots, u^{(n-1)}(t)\bigr), \quad \text{a.e. } t \in [a,b], \tag{1}$$

subject to the impulse conditions

$$\left.\begin{array}{l} u^{(j)}(t+) - u^{(j)}(t-) = J_{ij}(u(t-), u'(t-), \dots, u^{(n-1)}(t-)), \\ \text{where } t = \gamma_i(u(t-), u'(t-), \dots, u^{(n-2)}(t-)) \\ \text{for } i = 1, \dots, p, j = 0, \dots, n-1, \end{array}\right\} \tag{2}$$

and the linear boundary conditions

$$\ell_k\big(u, u', \dots, u^{(n-1)}\big) = c_k, \quad k = 1, \dots, n. \tag{3}$$

In what follows we use this notation. Let $k, m, n \in \mathbb{N}$. By $\mathbb{R}^{m\times n}$ we denote the set of all matrices of the type $m \times n$ with real valued coefficients. Let A^T denote the transpose of $A \in \mathbb{R}^{m\times n}$. Let $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ be the set of all n-dimensional column vectors $c = (c_1, \dots, c_n)^T$, where $c_i \in \mathbb{R}$, $i = 1, \dots, n$, and $\mathbb{R} = \mathbb{R}^{1\times 1}$. By $\mathbb{C}(\mathbb{R}^n; \mathbb{R}^m)$ we denote the set of all mappings $x : \mathbb{R}^n \to \mathbb{R}^m$ with continuous components. By $\mathbb{L}^\infty([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{L}^1([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{G}_\mathrm{L}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{AC}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{BV}([a,b]; \mathbb{R}^{m\times n})$, $\mathbb{C}^k([a,b]; \mathbb{R}^{m\times n})$, we denote the sets of all mappings $x : [a,b] \to \mathbb{R}^{m\times n}$ whose components are, respectively, essentially bounded functions, Lebesgue integrable functions, left-continuous regulated functions, absolutely continuous functions, functions with bounded variation and functions with continuous derivatives of the kth order on the interval $[a,b]$. By $\mathrm{Car}([a,b] \times \mathbb{R}^n; \mathbb{R})$ we denote the set of all functions $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the Carathéodory conditions on the set $[a,b] \times \mathbb{R}^n$. Finally, by χ_M we denote the characteristic function of the set $M \subset \mathbb{R}$.

Note that a mapping $u : [a,b] \to \mathbb{R}^n$ is left-continuous regulated on $[a,b]$ if for each $t \in (a,b]$ and each $s \in [a,b)$ there exist finite limits

$$u(t) = u(t-) = \lim_{\tau\to t-} u(\tau), \qquad u(s+) = \lim_{\tau\to s+} u(\tau).$$

$\mathbb{G}_\mathrm{L}([a,b]; \mathbb{R}^n)$ is a linear space, and equipped with the sup-norm $\|\cdot\|_\infty$ it is a Banach space (see [7, Theorem 3.6]). In particular, we set

$$\|u\|_\infty = \max_{i\in\{1,\dots,n\}} \Big(\sup_{t\in[a,b]} \big|u_i(t)\big| \Big) \quad \text{for } u = (u_1, \dots, u_n)^T \in \mathbb{G}_\mathrm{L}\big([a,b]; \mathbb{R}^n\big).$$

A function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the Carathéodory conditions on $[a,b] \times \mathbb{R}^n$ if

- $f(\cdot, x) : [a,b] \to \mathbb{R}$ is measurable for all $x \in \mathbb{R}^n$,
- $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
- for each compact set $K \subset \mathbb{R}^n$ there exists a function $m_K \in \mathbb{L}^1([a,b]; \mathbb{R})$ such that $|f(t,x)| \le m_K(t)$ for a.e. $t \in [a,b]$ and each $x \in K$.

In this paper we provide sufficient conditions for the solvability of problem (1)-(3). This problem is a generalization of problems studied in the papers [8–10] which are devoted to the second-order differential equation. Other types of initial or boundary value problems for the first- or second-order differential equations with state-dependent impulses can be found in [11–19]. To get the existence results for problem (1)-(3), we exploit the paper [20] with fixed-time impulsive problems.

Here we assume that

$$\left.\begin{array}{l}
n \ge 2, \frac{a_j}{a_n} \in \mathbb{L}^1([a,b]; \mathbb{R}), j = 0, \dots, n-1, \frac{h(t,x)}{a_n(t)} \in \mathrm{Car}([a,b] \times \mathbb{R}^n; \mathbb{R}), \\
c_j \in \mathbb{R}, J_{ij} \in \mathbb{C}(\mathbb{R}^n; \mathbb{R}), \gamma_i \in \mathbb{C}(\mathbb{R}^{n-1}; \mathbb{R}), i = 1, \dots, p, j = 0, \dots, n-1, \\
\ell_k : \mathbb{G}_\mathrm{L}([a,b]; \mathbb{R}^n) \to \mathbb{R} \text{ is a linear bounded functional, } \textit{i.e.} \\
\ell_k(z) = K_k z(a) + \int_a^b V_k(t)\, \mathrm{d}[z(t)], z \in \mathbb{G}_\mathrm{L}([a,b]; \mathbb{R}^{n\times 1}), \\
\text{where } K_k \in \mathbb{R}^{1\times n}, V_k \in \mathbb{BV}([a,b]; \mathbb{R}^{1\times n}), k = 1, \dots, n, n, p \in \mathbb{N}.
\end{array}\right\} \tag{4}$$

Remark 1 The integral in formula (4) is the Kurzweil-Stieltjes integral, whose definition and properties can be found in [21]. The fact that each linear bounded functional on $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^{n\times 1})$ can be written uniquely in the form described in (4) is proved in [22]. See also [20].

Now let us define a solution of problem (1)-(3).

Definition 2 A function $u \in \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ is said to be a solution of problem (1)-(3) if u satisfies (1) for a.e. $t \in [a,b]$ and fulfils conditions (2) and (3).

2 Problem with impulses at fixed times

In the paper [20] we have found an operator representation to the special type of problem (1)-(3) having impulses at fixed times. This is the case that the barrier functions γ_i in (2) are constant functions, *i.e.* there exist $t_1,\dots,t_p \in \mathbb{R}$ satisfying $a < t_1 < \cdots < t_p < b$ such that

$$\gamma_i(x_0,x_1,\dots,x_{n-2}) = t_i \quad \text{for } i = 1,\dots,p, x_0,x_1,\dots,x_{n-2} \in \mathbb{R}. \tag{5}$$

In this case, each solution of the problem crosses ith barrier at same time instant $\tau_i = t_i$ for $i = 1,\dots,p$.

Note that boundary value problems for higher-order differential equations with impulses at fixed times have been studied for example in [23–31] and for delay higher-order impulsive equations in [32, 33].

Let us summarize the results of the paper [20] according to our needs. Assume that the linear homogeneous problem

$$\left.\begin{aligned} &\sum_{j=0}^{n} a_j(t)u^{(j)}(t) = 0, \quad \text{a.e. } t \in [a,b], \\ &\ell_k\big(u,u',\dots,u^{(n-1)}\big) = 0, \quad k = 1,\dots,n, \end{aligned}\right\} \tag{6}$$

has only the trivial solution. Let $\{\tilde{u}_1,\dots,\tilde{u}_n\}$ be a fundamental system of solutions of the differential equation from (6), W be their Wronski matrix and w its first row, *i.e.*

$$W(t) = \begin{pmatrix} \tilde{u}_1(t) & \cdots & \tilde{u}_n(t) \\ \tilde{u}_1'(t) & \cdots & \tilde{u}_n'(t) \\ \tilde{u}_1^{(n-1)}(t) & \cdots & \tilde{u}_n^{(n-1)}(t) \end{pmatrix}, \qquad w(t) = \big(\tilde{u}_1(t),\dots,\tilde{u}_n(t)\big), \quad t \in [a,b]. \tag{7}$$

Denote

$$\ell(W) = \Big(\ell_i\big(\tilde{u}_j,\tilde{u}_j',\dots,\tilde{u}_j^{(n-1)}\big)\Big)_{i,j=1}^{n}. \tag{8}$$

From [20, Lemma 8] (see also Chapter 3 in [34]) it follows that the unique solvability of (6) is equivalent to the condition

$$\det \ell(W) \neq 0. \tag{9}$$

Further assume (9), consider V_j, $j = 1, \dots, n$, from (4), and denote

$$V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ \cdots \\ V_n(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0(t)}{a_n(t)} & -\frac{a_1(t)}{a_n(t)} & -\frac{a_2(t)}{a_n(t)} & \cdots & -\frac{a_{n-1}(t)}{a_n(t)} \end{pmatrix},$$

$t \in [a,b]$ and

$$H(\tau) = -\big[\ell(W)\big]^{-1}\left(\int_\tau^b V(s)A(s)W(s)\,\mathrm{d}s \cdot W^{-1}(\tau) + V(\tau)\right), \quad \tau \in [a,b]. \tag{10}$$

If we denote by H_{ij} and ω_{ij} elements of the matrices H and W^{-1}, respectively, that is,

$$H(\tau) = \big(H_{ij}(\tau)\big)_{i,j=1}^n, \qquad W^{-1}(\tau) = \big(\omega_{ij}(\tau)\big)_{i,j=1}^n, \tag{11}$$

we can define functions g_j, $j = 1, \dots, n$, as

$$g_j(t,\tau) = \sum_{k=1}^n \tilde{u}_k(t)\big(H_{kj}(\tau) + \chi_{(\tau,b]}(t)\omega_{kj}(\tau)\big), \quad t, \tau \in [a,b]. \tag{12}$$

For each fixed $\tau \in [a,b]$ the functions $\frac{\partial^k g_j(t,\tau)}{\partial \tau^k}$, $k = 0, 1, \dots, n-1$, will be understood as right-continuous extensions at $t = a$ and left-continuous extensions at $t = \tau$ and $t = b$. In this way the Green's function of problem (6) is built (*cf.* Remark 6).

Remark 3 In order to state one of the main results of [20] we introduce the set of all functions u continuous on the intervals $[a,t_1], (t_1,t_2], \dots, (t_p,b]$, with t_i from (5), having their derivatives $u', \dots, u^{(n-1)}$ continuously extendable onto these intervals. This set is denoted by $\mathbb{PC}^{n-1}([a,b])$. For $u \in \mathbb{PC}^{n-1}([a,b])$ we define

$$u^{(k)}(a) = u^{(k)}(a+), \qquad u^{(k)}(t_i) = u^{(k)}(t_i-) \quad \text{for } k = 1, \dots, n-1, i = 1, \dots, p.$$

Equipped with the standard addition, scalar multiplication, and with the norm

$$\|u\| = \sum_{k=0}^{n-1} \big\|u^{(k)}\big\|_\infty, \quad u \in \mathbb{PC}^{n-1}\big([a,b]\big),$$

$\mathbb{PC}^{n-1}([a,b])$ forms a Banach space.

Now we are ready to state the operator representation theorem for the problem with impulses at fixed times $a < t_1 < \cdots < t_p < b$ which has the form

$$\sum_{j=0}^n a_j(t)u^{(j)}(t) = h\big(t, u(t), \dots, u^{(n-1)}(t)\big), \quad \text{a.e. } t \in [a,b], \tag{13}$$

$$u^{(j)}(t_i+) - u^{(j)}(t_i) = J_{ij}\big(u(t_i), u'(t_i), \dots, u^{(n-1)}(t_i)\big), \quad i = 1, \dots, p, j = 0, \dots, n-1, \tag{14}$$

$$\ell_k\bigl(u,u',\ldots,u^{(n-1)}\bigr)=c_k,\quad k=1,\ldots,n. \tag{15}$$

Theorem 4 [20, Theorem 17] *Let* (4), (9) *hold, and let* W, w, $\ell(W)$ *and* g_j, $j=1,\ldots,n$ *be defined in* (7), (8), *and* (12). *Then* $u\in\mathbb{PC}^{n-1}([a,b])$ *is a fixed point of an operator* $\mathcal{H}:\mathbb{PC}^{n-1}([a,b])\to\mathbb{PC}^{n-1}([a,b])$ *defined by*

$$\left.\begin{aligned}(\mathcal{H}u)(t)={}&\int_a^b\frac{g_n(t,s)}{a_n(s)}h\bigl(s,u(s),\ldots,u^{(n-1)}(s)\bigr)\,\mathrm{d}s\\&+\sum_{j=1}^n\sum_{i=1}^p g_j(t,t_i)J_{i,j-1}\bigl(u(t_i),\ldots,u^{(n-1)}(t_i)\bigr)\\&+w(t)\bigl[\ell(W)\bigr]^{-1}(c_1,\ldots,c_n)^T,\end{aligned}\right\} \tag{16}$$

$t\in[a,b]$, *if and only if* u *is a solution of problem* (13)-(15). *Moreover, the operator* $\mathcal{H}$ *is completely continuous.*

Remark 5 Let us note that the row vector

$$w(t)\bigl[\ell(W)\bigr]^{-1}$$

does not depend on the choice of a fundamental system of solutions $\tilde{u}_1,\ldots,\tilde{u}_n$, but only on the data of problem (6).

Remark 6 Let us put

$$J_{ij}=0,\quad i=1,\ldots,p, j=0,\ldots,n-1,\qquad c_k=0,\quad k=1,\ldots,n$$

and

$$h(t,x)=h_0(t)\in\mathbb{L}^1\bigl([a,b];\mathbb{R}\bigr)\quad\text{for } x\in\mathbb{R}^n.$$

Then the operator $\mathcal{H}$ in Theorem 4 can be written as

$$(\mathcal{H}_0u)(t)=\int_a^b\frac{g_n(t,s)}{a_n(s)}h_0(s)\,\mathrm{d}s.$$

Theorem 4 implies that u is a fixed point of $\mathcal{H}_0$ if and only if u is a solution of the problem

$$\sum_{j=0}^n a_j(t)u^{(j)}(t)=h_0(t),\qquad \ell_j\bigl(u,u',\ldots,u^{(n-1)}\bigr)=0,\quad j=1,\ldots,n. \tag{17}$$

Therefore a (unique) solution of problem (17) has the form

$$u(t)=\int_a^b\frac{g_n(t,s)}{a_n(s)}h_0(s)\,\mathrm{d}s,$$

and consequently $\frac{g_n(t,s)}{a_n(s)}$ is the Green's function of (6).

Remark 7 Under the assumption (9) we are allowed using (11) to define the functions

$$\left.\begin{aligned} g_j^{[1]}(t,\tau) &= \sum_{k=1}^n \tilde{u}_k(t) H_{kj}(\tau), \\ g_j^{[2]}(t,\tau) &= \sum_{k=1}^n \tilde{u}_k(t)\bigl(H_{kj}(\tau) + \omega_{kj}(\tau)\bigr) \end{aligned}\right\} \tag{18}$$

for $t, \tau \in [a,b]$, $j = 1, \dots, n$. Obviously, due to (12),

$$g_j(t,\tau) = \begin{cases} g_j^{[1]}(t,\tau) & \text{for } a \le t \le \tau \le b, \\ g_j^{[2]}(t,\tau) & \text{for } a \le \tau < t \le b, \end{cases} \tag{19}$$

for $j = 1, \dots, n$. Let us stress that $g_j^{[\nu]}$, as well as g_j, do not depend on the choice of fundamental system $\tilde{u}_1, \dots, \tilde{u}_n$, but only on the data of problem (6). The functions $g_j^{[\nu]}$ possess crucial properties for our approach. From their definition it follows that for each $\tau \in [a,b]$

$$\frac{\partial^k g_j^{[\nu]}}{\partial t^k}(\cdot,\tau) \in \mathbb{AC}\bigl([a,b];\mathbb{R}\bigr) \tag{20}$$

for $\nu = 1,2$, $j = 1, \dots, n$, $k = 0, \dots, n-1$. Moreover, for each $\nu = 1,2$, $j = 1, \dots, n$, $k = 0, \dots, n-1$, there exists a constant $C_{\nu jk} > 0$ such that

$$\left|\frac{\partial^k g_j^{[\nu]}}{\partial t^k}(t,\tau)\right| \le C_{\nu jk} \quad \text{and} \quad \left|\frac{\partial^k g_j}{\partial t^k}(t,\tau)\right| \le \max_{\nu=1,2} C_{\nu jk} \quad t,\tau \in [a,b]. \tag{21}$$

This follows from the definition of $g_j^{[\nu]}$ ($\nu = 1,2$), from the fact $w \in \mathbb{C}^{n-1}([a,b];\mathbb{R}^{1\times n})$ and from the boundedness of the matrices W^{-1} and H (*cf.* (7), (10) and (11)).

3 Transversality conditions

The most results for differential equations with state-dependent impulses concern initial value problems. Theorems about the existence, uniqueness or extension of solutions of initial value problems, and about intersections of such solutions with barriers γ_i can be found for example in [35, Chapter 5].

A different approach has to be used when boundary value problems with state-dependent impulses are discussed and boundary conditions are imposed on a solution anywhere in the interval $[a,b]$ including unknown points of impulses. This is the case of problem (1)-(3).

Our approach is based on the existence of a fixed point of an operator $\mathcal{F}$ in some set $\bar{\Omega} = \bar{\mathcal{B}}^{p+1}$ (*cf.* Lemma 12), where $\bar{\mathcal{B}} \subset \mathbb{C}^{n-1}([a,b];\mathbb{R})$ is a ball defined in (28). In order to get a fixed point, we need to prove for functions of $\bar{\mathcal{B}}$ assertions about their transversality through barriers. Such assertions are contained in Lemmas 9 and 10 and it is important that they are valid for all functions in $\bar{\mathcal{B}}$ and not only for solutions of problem (1), (2).

Remark 8 Having the lemmas about the transversality, we will prove in Section 4 the existence of a solution u of problem (1)-(3), which has the following property:

$$\left.\begin{array}{l}\text{for each } i \in \{1,\dots,p\} \text{ there exists a unique } \tau_i \in (a,b) \text{ such that} \\ \tau_i = \gamma_i(u(\tau_i-), u'(\tau_i-),\dots,u^{(n-2)}(\tau_i-)), a < \tau_1 < \dots < \tau_p < b, \\ \text{and the restrictions } u|_{[a,\tau_1]}, u|(\tau_1,\tau_2],\dots,u|_{(\tau_p,b]} \text{ have absolutely} \\ \text{continuous derivatives of the } (n-1)\text{th order.}\end{array}\right\} \tag{22}$$

Consider real numbers K_j, $j = 0,1,\dots,n-1$, and denote

$$\mathcal{A}_n = \{(x_0,x_1,\dots,x_{n-1}) \in \mathbb{R}^n : |x_0| \le K_0,\dots,|x_{n-1}| \le K_{n-1}\}. \tag{23}$$

Now, we are ready to formulate the following transversality conditions:

$$a < \min_{\mathcal{A}_{n-1}} \gamma_1 \le \max_{\mathcal{A}_{n-1}} \gamma_{i-1} < \min_{\mathcal{A}_{n-1}} \gamma_i \le \max_{\mathcal{A}_{n-1}} \gamma_p < b, \quad i = 2,\dots,p, \tag{24}$$

$$\left.\begin{array}{l}\text{for each } i = 1,\dots,p, k = 0,\dots,n-2 \text{ there exists } L_{ik} \in [0,\infty) \text{ such that} \\ \text{if } (x_0,x_1,\dots,x_{n-2}), (y_0,y_1,\dots,y_{n-2}) \text{ belong to } \mathcal{A}_{n-1}, \text{ then} \\ |\gamma_i(x_0,x_1,\dots,x_{n-2}) - \gamma_i(y_0,y_1,\dots,y_{n-2})| \le \sum_{j=0}^{n-2} L_{ij}|x_j - y_j|, \\ i = 1,\dots,p,\end{array}\right\} \tag{25}$$

$$\sum_{j=0}^{n-2} L_{ij}K_{j+1} < 1 \quad \text{for } i = 1,\dots,p, \tag{26}$$

$$\left.\begin{array}{l}\gamma_i(x_0 + J_{i0}(x_0,\dots,x_{n-1}),\dots,x_{n-2} + J_{i,n-2}(x_0,\dots,x_{n-1})) \\ \quad \le \gamma_i(x_0,\dots,x_{n-2}), \quad (x_0,\dots,x_{n-1}) \in \mathcal{A}_n, i = 1,\dots,p.\end{array}\right\} \tag{27}$$

Let us define the set

$$\mathcal{B} = \{u \in \mathbb{C}^{n-1}([a,b];\mathbb{R}) : \|u^{(j)}\|_\infty < K_j \text{ for } j = 0,\dots,n-1\}. \tag{28}$$

Our current goal is to find a continuous functional $\mathcal{P}_i$ for $i = 1,\dots,p$, which maps each function u from $\overline{\mathcal{B}}$ to some time instant τ_i of (2).

Lemma 9 *Let K_j, $j = 0,\dots,n-1$, L_{ik}, $i = 1,\dots,p$, $k = 0,\dots,n-2$, be real numbers satisfying* (26), *and let $\mathcal{A}_n$ and $\mathcal{B}$ be given by* (23) *and* (28), *respectively. Finally, assume that γ_i, $i = 1,\dots,p$, satisfy* (24), (25), *and choose $u \in \overline{\mathcal{B}}$. Then the function*

$$\sigma(t) = \gamma_i(u(t),u'(t),\dots,u^{(n-2)}(t)) - t, \quad t \in [a,b], \tag{29}$$

is continuous and decreasing on $[a,b]$ and it has a unique root in the interval (a,b), i.e. there exists a unique solution of the equation

$$t = \gamma_i(u(t),\dots,u^{(n-2)}(t)). \tag{30}$$

Proof Let $u \in \overline{\mathcal{B}}$, $i \in \{1,\dots,p\}$. By (24),

$$\sigma(a) = \gamma_i(u(a),u'(a),\dots,u^{(n-2)}(a)) - a > 0,$$

$$\sigma(b) = \gamma_i\big(u(b), u'(b), \dots, u^{(n-2)}(b)\big) - b < 0$$

is valid. This together with the fact that σ is continuous shows that σ has at least one root in (a,b). Now, we will prove that σ is decreasing, by a contradiction. Let $s_1, s_2 \in (a,b)$, $s_1 < s_2$ be such that

$$\sigma(s_1) = \sigma(s_2),$$

i.e.

$$\gamma_i\big(u(s_1), \dots, u^{(n-2)}(s_1)\big) - \gamma_i\big(u(s_2), \dots, u^{(n-2)}(s_2)\big) = s_1 - s_2.$$

From (25), (26), (28), and the Mean Value Theorem we obtain

$$\begin{aligned} 0 < |s_1 - s_2| &= \big|\gamma_i\big(u(s_1), \dots, u^{(n-2)}(s_1)\big) - \gamma_i\big(u(s_2), \dots, u^{(n-2)}(s_2)\big)\big| \\ &\le \sum_{j=0}^{n-2} L_{ij}\big|u^{(j)}(s_1) - u^{(j)}(s_2)\big| \le \sum_{j=0}^{n-2} L_{ij}K_{j+1}|s_1 - s_2| < |s_1 - s_2|, \end{aligned}$$

which is a contradiction.

According to Lemma 9, we can define a functional $\mathcal{P}_i : \overline{\mathcal{B}} \to (a,b)$ by

$$\mathcal{P}_i u = \tau_i, \quad u \in \overline{\mathcal{B}}, \tag{31}$$

where τ_i is a solution of (30), *i.e.* a unique root of the function σ from Lemma 9, for $i = 1, \dots, p$. □

Lemma 10 *Let the assumptions of Lemma* 9 *be satisfied. The functionals* $\mathcal{P}_i$, $i = 1, \dots, p$, *are continuous.*

Proof Let $u_m, u \in \overline{\mathcal{B}}$, for $m \in \mathbb{N}$ such that

$$u_m \to u \quad \text{in } \mathbb{C}^{n-1}\big([a,b];\mathbb{R}\big) \text{ as } m \to \infty. \tag{32}$$

Let us choose $i \in \{1, \dots, p\}$ and prove that $\mathcal{P}_i u_m \to \mathcal{P}_i u$ as $m \to \infty$. We denote

$$\tau = \mathcal{P}_i u, \qquad \tau_m = \mathcal{P}_i u_m, \quad m \in \mathbb{N}.$$

From Lemma 9 it follows that $\tau, \tau_m \in (a,b)$ are the unique roots of the functions

$$\sigma(t) = \gamma_i\big(u(t), \dots, u^{(n-2)}(t)\big) - t, \qquad \sigma_m(t) = \gamma_i\big(u_m(t), \dots, u_m^{(n-2)}(t)\big) - t, \quad t \in [a,b],$$

and these functions are strictly decreasing. Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$ be such that $\tau - \epsilon, \tau + \epsilon \in (a,b)$. Then $\sigma(\tau - \epsilon) > 0$ and $\sigma(\tau + \epsilon) < 0$. According to (32) we see that $\sigma_m \to \sigma$ uniformly on $[a,b]$, in particular $\sigma_m(\tau - \epsilon) \to \sigma(\tau - \epsilon)$ and $\sigma_m(\tau + \epsilon) \to \sigma(\tau + \epsilon)$ as $m \to \infty$. These facts imply that

$$\sigma_m(\tau - \epsilon) > 0 \quad \text{and} \quad \sigma_m(\tau + \epsilon) < 0 \quad \text{for a.e. } m \in \mathbb{N}.$$

From the continuity of σ_m and the Intermediate Value Theorem it follows that

$$\mathcal{P}_i u_m = \tau_m \in (\tau - \epsilon, \tau + \epsilon) = (\mathcal{P}_i u - \epsilon, \mathcal{P}_i u + \epsilon) \quad \text{for a.e. } m \in \mathbb{N},$$

which completes the proof. □

Our next step is to define an appropriate operator representation of the BVP with state-dependent impulses. The first idea would be a direct exploitation of the operator $\mathcal{H}$ from Theorem 4, putting $\mathcal{P}_i u$ in place of t_i. This is not possible for many reasons. First, each $\mathcal{P}_i$ acts on the space of functions having continuous derivatives - but we need functions having p discontinuities. Even if we would overcome this difficulty we arrive at a problem of choosing an appropriate Banach space on which $\mathcal{H}$ would be acting. According to Remark 8, we search a solution u of problem (1)-(3), which has its jumps (together with $u, u', \dots, u^{(n-1)}$) at the points $\tau_i = \mathcal{P}_i u \in (a,b)$, $i = 1, \dots, p$ (see (31)). In general, these points are different for different solutions. Consequently, such solutions have to be searched in the Banach space $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$. But then there is a difficulty with the continuity of such operator. In fact the operator $\mathcal{H}$ from (16) having $\mathcal{P}_i u$ in place of t_i is not continuous in the space $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$ (*cf.* Remark 6.2 and Example 6.3 in [36]).

Therefore, we choose the way here, which we have developed in our joint papers [8–10]. The main idea of our approach lies in representing the solution u of problem (1)-(3) by an ordered $(p+1)$-tuple $(u_1, \dots, u_{p+1}) \in [\mathbb{C}^{n-1}([a,b];\mathbb{R})]^{p+1}$ as follows:

$$u(t) = \begin{cases} u_1(t), & t \in [a, \mathcal{P}_1 u_1], \\ u_2(t), & t \in (\mathcal{P}_1 u_1, \mathcal{P}_2 u_2], \\ \dots & \dots \\ u_{p+1}(t), & t \in (\mathcal{P}_p u_p, b]. \end{cases} \tag{33}$$

Consequently, we work with the space

$$X = \left[\mathbb{C}^{n-1}\big([a,b];\mathbb{R}\big)\right]^{p+1}$$

equipped with the norm

$$\big\|(u_1, \dots, u_{p+1})\big\| = \sum_{i=1}^{p+1} \sum_{j=0}^{n-1} \big\|u_i^{(j)}\big\|_\infty \quad \text{for } (u_1, \dots, u_{p+1}) \in X.$$

It is well known that X is a Banach space.

4 Main results

Let us turn our attention to problem (1)-(3) with state-dependent impulses under the assumptions (4) and (9). In our approach we will make use of the tools introduced in the previous sections.

In addition we assume

$$\left.\begin{array}{l} \text{there exists } m \in \mathbb{L}^1([a,b];\mathbb{R}), A_{ij} \in \mathbb{R} \text{ such that} \\ \left|\frac{h(t,x)}{a_n(t)}\right| \le m(t) \text{ for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}^n, \\ |J_{ij}(x)| \le A_{ij} \text{ for each } i = 1, \dots, p, j = 0, \dots, n-1. \end{array}\right\} \tag{34}$$

Consider $c_1,\dots,c_n$ from (3), w from (7) and $\ell(W)$ from (8), and denote

$$M = \int_a^b m(t)\,\mathrm{d}t, \qquad c_0 = (c_1,\dots,c_n)^T, \qquad D_r = \max_{t\in[a,b]} w^{(r)}(t)\bigl[\ell(W)\bigr]^{-1}c_0, \tag{35}$$

and

$$K_r = M\max_{\nu=1,2}\{C_{\nu nr}\} + \sum_{j=1}^{n}\sum_{k=1}^{p}\max_{\nu=1,2}\{C_{\nu jr}\}A_{k,j-1} + D_r, \tag{36}$$

for $r = 0,\dots,n-1$, where $C_{\nu jr}$ are constants from (21).

Remark 11 Let us note that the constants D_r from (35) do not depend on the choice of the fundamental system of solutions $\tilde{u}_1,\dots,\tilde{u}_n$, but only on the coefficients a_i of the differential equation (1) and on the operators ℓ_j from (3) (and, of course, on the constants c_j).

Now, we are ready to construct a convenient operator for a representation of problem (1)-(3). Let us choose its domain as the closure of the set

$$\Omega = \mathcal{B}^{p+1} \subset X,$$

where $\mathcal{B}$ is defined in (28) with K_j from (36).

Now, we have to modify the operator $\mathcal{H}$ from Theorem 4 using $g_j^{[1]}$ and $g_j^{[2]}$ instead of the Green's functions g_j, that is, we define an operator $\mathcal{F} : \overline{\Omega} \to X$ by $\mathcal{F}(u_1,\dots,u_{p+1}) = (x_1,\dots,x_{p+1})$ with

$$\left.\begin{aligned} x_i(t) = {}& \sum_{k=1}^{p+1}\int_{\tau_{k-1}}^{\tau_k} g_n(t,s)\frac{h(s,u_k(s),\dots,u_k^{(n-1)}(s))}{a_n(s)}\,\mathrm{d}s \\ &+ \sum_{j=1}^{n}\biggl(\sum_{i\le k\le p} g_j^{[1]}(t,\tau_k)J_{k,j-1}\bigl(u_k(\tau_k),\dots,u_k^{(n-1)}(\tau_k)\bigr) \\ &+ \sum_{1\le k<i} g_j^{[2]}(t,\tau_k)J_{k,j-1}\bigl(u_k(\tau_k),\dots,u_k^{(n-1)}(\tau_k)\bigr)\biggr) + w(t)\bigl[\ell(W)\bigr]^{-1}c_0 \end{aligned}\right\} \tag{37}$$

for $i = 1,\dots,p+1$, $t \in [a,b]$, where

$$\tau_k = \mathcal{P}_k u_k \quad \text{for } k = 1,\dots,p, \tau_0 = a, \tau_{p+1} = b,$$

and W, w, g_j, $g_j^{[1]}$, $g_j^{[2]}$, $j = 1,\dots,n$, and c_0 are from (7), (12), (18), and (35), respectively.

Let us compare (16) for the operator $\mathcal{H}$ with (37) for the operator $\mathcal{F}$. The first term in (16) expresses a solution of homogeneous boundary value problem without impulses. This term is decomposed in (37) on subintervals which depend on the choice of $(p+1)$-tuple $(u_1,\dots,u_{p+1})$. The second term in (16) caused (according to the discontinuity of functions g_j) needed impulses of solutions of the fixed-time impulsive problem (13)-(15). We significantly modify this term in (37) in such a way that, instead of discontinuous functions g_j which have jumps at the points $\tau_k = P_k u_k$, we use smooth functions $g_j^{[1]}$, $g_j^{[2]}$ defined in (18). Due to this modification the operator $\mathcal{F}$ maps one tuple of smooth functions

$u_1, \dots, u_{p+1}$ onto another tuple of smooth functions $x_1, \dots, x_{p+1}$, and we will be able to prove the compactness of $\mathcal{F}$ on $\overline{\Omega}$.

In the next lemma we arrive at a justification of our definition.

Lemma 12 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. If* $(u_1, \dots, u_{p+1})$ *is a fixed point of the operator* $\mathcal{F}$*, then the function* u *defined by* (33) *is a solution of problem* (1)-(3) *satisfying* (22).

Proof Let $\mathcal{B}$ be defined by (28) and $\Omega = \mathcal{B}^{p+1}$. Further, let $(u_1, \dots, u_{p+1}) \in \overline{\Omega}$ be such that $\mathcal{F}(u_1, \dots, u_{p+1}) = (u_1, \dots, u_{p+1})$. For each $i \in \{1, \dots, p+1\}$, we have $u_i \in \overline{\mathcal{B}}$, and hence by Lemma 9 and (31), there exists a unique solution $\tau_i = P_i u_i$ of the equation $t = \gamma_i(u_i(t), \dots, u_i^{(n-2)}(t))$. Due to (24), the inequalities $a < \tau_1 < \cdots < \tau_p < b$ are valid and u can be defined by (33). We will prove that u is a fixed point of the operator $\mathcal{H}$ from Theorem 4, taking the space $\mathbb{PC}^{n-1}([a,b])$ from Remark 3 with

$$t_i = \tau_i, \quad i = 1, \dots, p.$$

Denote

$$\begin{aligned} &\tau_0 = a, \qquad \tau_{p+1} = b, \qquad \mathcal{I}_1 = [\tau_0, \tau_1], \qquad \mathcal{I}_2 = (\tau_1, \tau_2], \\ &\mathcal{I}_3 = (\tau_2, \tau_3], \qquad \dots, \qquad \mathcal{I}_{p+1} = (\tau_p, \tau_{p+1}], \end{aligned}$$

and choose $i \in \{1, \dots, p+1\}$, $t \in \mathcal{I}_i$. Then, according to (33), we have

$$\begin{aligned} u(t) = u_i(t) &= \sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\big(s, u_k(s), \dots, u_k^{(n-1)}(s)\big)\,\mathrm{d}s \\ &\quad + \sum_{j=1}^{n} \Bigg(\sum_{i \le k \le p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\big(u_k(\tau_k), \dots, u_k^{(n-1)}(\tau_k)\big) \\ &\quad + \sum_{1 \le k < i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\big(u_k(\tau_k), \dots, u_k^{(n-1)}(\tau_k)\big) \Bigg) + w(t)\big[\ell(W)\big]^{-1} c_0 \\ &= \sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\big(s, u(s), \dots, u^{(n-1)}(s)\big)\,\mathrm{d}s \\ &\quad + \sum_{j=1}^{n} \Bigg(\sum_{i \le k \le p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\big(u(\tau_k), \dots, u^{(n-1)}(\tau_k-)\big) \\ &\quad + \sum_{1 \le k < i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\big(u(\tau_k), \dots, u^{(n-1)}(\tau_k-)\big) \Bigg) + w(t)\big[\ell(W)\big]^{-1} c_0. \end{aligned}$$

Of course we have

$$\sum_{k=1}^{p+1} \int_{\mathcal{I}_k} \frac{g_n(t,s)}{a_n(s)} h\big(s, u(s), \dots, u^{(n-1)}(s)\big)\,\mathrm{d}s = \int_a^b \frac{g_n(t,s)}{a_n(s)} h\big(s, u(s), \dots, u^{(n-1)}(s)\big)\,\mathrm{d}s.$$

Let $k \in \mathbb{N}$ be such that $i \leq k \leq p$. Then $t \leq \tau_i \leq \tau_k$ and therefore (19) gives

$$g_j^{[1]}(t,\tau_k) = g_j(t,\tau_k) \quad \text{for } j = 1,\dots,n.$$

Let $k \in \mathbb{N}$ be such that $1 \leq k < i$ (such k exists only if $i > 1$). Then $t > \tau_{i-1} \geq \tau_k$ and therefore we get by (19)

$$g_j^{[2]}(t,\tau_k) = g_j(t,\tau_k) \quad \text{for } j = 1,\dots,n.$$

These facts imply that

$$\begin{aligned}
&\sum_{i\leq k\leq p} g_j^{[1]}(t,\tau_k) J_{k,j-1}\big(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\big) \\
&\qquad + \sum_{1\leq k<i} g_j^{[2]}(t,\tau_k) J_{k,j-1}\big(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\big) \\
&\quad = \sum_{i\leq k\leq p} g_j(t,\tau_k) J_{k,j-1}\big(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\big) \\
&\qquad + \sum_{1\leq k<i} g_j(t,\tau_k) J_{k,j-1}\big(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\big) \\
&\quad = \sum_{k=1}^{p} g_j(t,\tau_k) J_{k,j-1}\big(u(\tau_k),\dots,u^{(n-1)}(\tau_k-)\big),
\end{aligned}$$

for $j = 1,\dots,n$. Consequently, by virtue of (16) and Theorem 4, u is a solution of problem (13)-(15). Clearly u fulfils equation (1) a.e. on $[a,b]$ and satisfies the boundary conditions (3). In addition, since u fulfils the impulse conditions (14) with $t_i = \tau_i$, where $\tau_i = \gamma_i(u_i(\tau_i),\dots,u_i^{(n-2)}(\tau_i)) = \gamma_i(u(\tau_i),\dots,u^{(n-2)}(\tau_i-))$, $i = 1,\dots,p$, we see that u also fulfils the state-dependent impulse conditions (2). According to Remark 8, it remains to prove that $\tau_1,\dots,\tau_p$ are the only instants at which the function u crosses the barriers $\gamma_1,\dots,\gamma_p$, respectively. To this aim, due to (24) and (33), it suffices to prove that

$$t \neq \gamma_i\big(u_{i+1}(t),u'_{i+1}(t),\dots,u_{i+1}^{(n-2)}(t)\big) \quad \text{for } t \in (\tau_i,b],\ i = 1,\dots,p. \tag{38}$$

Choose an arbitrary $i \in \{1,\dots,p\}$ and consider σ from (29). Since u fulfils (2), we have

$$\sigma(\tau_i-) = 0.$$

Let us denote

$$\psi(t) = \gamma_i\big(u_{i+1}(t),u'_{i+1}(t),\dots,u_{i+1}^{(n-2)}(t)\big) - t, \quad t \in [a,b].$$

From Lemma 9 it follows that ψ is decreasing. So, by virtue of (38), it suffices to prove that

$$\psi(\tau_i) \leq 0. \tag{39}$$

Using (33), (2), and (27), we have

$$\begin{aligned}\psi(\tau_i) &= \gamma_i\big(u_{i+1}(\tau_i),\dots,u_{i+1}^{(n-2)}(\tau_i)\big) - \tau_i = \gamma_i\big(u(\tau_i+),\dots,u^{(n-2)}(\tau_i+)\big) - \tau_i \\ &= \gamma_i\big(u(\tau_i-) + J_{i0}\big(u(\tau_i-),\dots,u^{(n-1)}(\tau_i-)\big),\dots,u^{(n-2)}(\tau_i-) \\ &\quad + J_{i,n-2}\big(u(\tau_i-),\dots,u^{(n-1)}(\tau_i-)\big)\big) - \tau_i \\ &\le \gamma_i\big(u(\tau_i-),\dots,u^{(n-2)}(\tau_i-)\big) - \tau_i = 0,\end{aligned}$$

which yields (39). This completes the proof. □

Lemma 13 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. Then the operator* $\mathcal{F}$ *from* (37) *has a fixed point in* $\overline{\Omega}$.

Proof The last term $\omega(t)[\ell(W)]^{-1}c_0$ in (37) is the same as in (16) for the compact operator $\mathcal{H}$. Therefore it suffices to prove the compactness of the operator $\mathcal{F}$ on $\overline{\Omega}$ for $c_0 = 0$. To do it we can use the same arguments as in the proof of Lemma 6 in [9], where the second-order state-dependent impulsive problem is investigated. In particular, the compactness of $\mathcal{F}$ on $\overline{\Omega}$ is a consequence of the following properties of functions and functionals contained in (37):

- the first term in (37) can be written in the form

$$\begin{aligned}&\sum_{k=1}^{p+1}\int_{\tau_{k-1}}^{\tau_k} g_n(t,s)\frac{h(s,u_k(s),\dots,u_k^{(n-1)}(s))}{a_n(s)}\,\mathrm{d}s \\ &\quad = \int_a^b g_n(t,s)\sum_{k=1}^{p+1}\frac{h(s,u_k(s),\dots,u_k^{(n-1)}(s))}{a_n(s)}\chi_{(\tau_{k-1},\tau_k)}(s)\,\mathrm{d}s,\end{aligned}$$

  where $\tau_k = \mathcal{P}_k u_k$ for $k = 1,\dots,p$, $\tau_0 = a$, $\tau_{p+1} = b$,
- $\mathcal{P}_k$ are continuous on $\overline{\mathcal{B}}$ (due to Lemma 10),
- $\frac{h(t,x)}{a_n(t)} \in \mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R})$,
- $g_j^{[1]}, g_j^{[2]}$ satisfy (20), g_n satisfies (19),
- J_{kj} are continuous on $\mathbb{R}^n$.

For the application of the Schauder Fixed Point Theorem it remains to prove that

$$\mathcal{F}(\overline{\Omega}) \subset \overline{\Omega}. \tag{40}$$

Let $(x_1,\dots,x_{p+1}) = \mathcal{F}(u_1,\dots,u_{p+1})$ for some $(u_1,\dots,u_{p+1}) \in \overline{\Omega}$. Then, by (21), (34), (35), and (37), we have

$$\big|x_i^{(r)}(t)\big| \le M\max_{\nu=1,2}\{C_{\nu nr}\} + \sum_{j=1}^{n}\sum_{k=1}^{p}\max_{\nu=1,2}\{C_{\nu jr}\}A_{k,j-1} + D_r$$

for $i = 1,\dots,p+1$, $r = 0,\dots,n-1$, $t \in [a,b]$. From (36) we get

$$\big\|x_i^{(r)}\big\|_\infty \le K_r, \quad i = 1,\dots,p+1, r = 0,\dots,n-1,$$

and so $\mathcal{F}(u_1,\dots,u_{p+1}) \in \overline{\Omega}$. We have proved (40), and consequently there exists at least one fixed point of $\mathcal{F}$ in $\overline{\Omega}$. □

Theorem 14 *Let assumptions* (4), (9), (23)-(27), (34)-(36) *be satisfied. Then there exists at least one solution to problem* (1)-(3) *satisfying* (22).

Proof The assertion follows directly from Lemma 12 and Lemma 13. □

Remark 15 The existence result from Theorem 14 can be extended to unbounded functions h and J_{ij} by means of the method of *a priori* estimates. This can be found for the special case $n = 2$ in [10].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and read and approved the final draft.

Acknowledgements
The authors were supported by the grant IGA_PrF_2014028. The authors sincerely thank the anonymous referees for their valuable comments and suggestions.

Received: 20 November 2013 Accepted: 1 May 2014 Published: 14 May 2014

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10.1186/1687-2770-2014-118
Cite this article as: Rachůnková and Tomeček: **Existence principle for higher-order nonlinear differential equations with state-dependent impulses via fixed point theorem.** *Boundary Value Problems* 2014, 2014:118