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Some inequalities involving k-gamma and k-beta functions with applications

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Abstract

In this paper, we present some inequalities involving k-gamma and k-beta functions via some classical inequalities like the Chebychev inequality for synchronous (asynchronous) mappings, and the Grüss and the Ostrowski inequality. Also, we give a new proof of the log-convexity of the k-gamma and k-beta functions by using the Hölder inequality.

Keywords: k-gamma; k-beta; inequalities; log-convexity

1 Introduction

In this section, we present some fundamental relations for k-gamma and k-beta functions introduced by the researchers [1–7]. The second and third section is devoted to the applications of some integral inequalities like the Chebychev, Grüss, and Ostrowski inequalities. In the last section, we prove the log-convexity of the k-gamma and k-beta functions.

Recently, Diaz and Pariguan [1] introduced the generalized k-gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, x \in \mathbb{C} \setminus kZ^- \tag{1}$$

and also gave the properties of said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. The Γ_k is based on the repeated appearance of the expression of the following form:

$$\alpha(\alpha + k)(\alpha + 2k)(\alpha + 3k) \cdots \bigl(\alpha + (n-1)k\bigr). \tag{2}$$

The function of the variable α given by the statement (2), denoted by $(\alpha)_{n,k}$, is called the Pochhammer k-symbol. We obtain the usual Pochhammer symbol $(\alpha)_n$ by taking $k = 1$. The definition given in (1) is the generalization of $\Gamma(x)$ and the integral form of Γ_k is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt, \quad \mathrm{Re}(x) > 0. \tag{3}$$

From (3), we can easily show that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{4}$$

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The same authors defined the k-beta function as

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, \tag{5}$$

and the integral form of $\beta_k(x,y)$ is

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt. \tag{6}$$

From the definition of $\beta_k(x,y)$ given in (5) and (6), we can easily prove that

$$\beta_k(x,y) = \frac{1}{k}\beta\left(\frac{x}{k}, \frac{y}{k}\right). \tag{7}$$

Also, the researchers [2–6] have worked on the generalized k-gamma and k-beta functions and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{8}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \tag{9}$$

$$\Gamma_k(k) = 1, \quad k > 0, \tag{10}$$

$$\Gamma_k(x) = a^{\frac{x}{k}}\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}a}\,dt, \quad a \in \mathbb{R}, \tag{11}$$

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha), \quad k > 0, \alpha \in \mathbb{R}, \tag{12}$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!, \quad k > 0, n \in \mathbb{N}, \tag{13}$$

$$\Gamma_k\left((2n+1)\frac{k}{2}\right) = k^{\frac{2n-1}{2}}\frac{(2n)!\sqrt{\pi}}{2^n n!}, \quad k > 0, n \in \mathbb{N}. \tag{14}$$

Using (5) and (7), we see that, for $x, y > 0$ and $k > 0$, the following properties of the k-beta function are satisfied (see [2, 3] and [7]):

$$\beta_k(x+k,y) = \frac{x}{x+y}\beta_k(x,y), \tag{15}$$

$$\beta_k(x,y+k) = \frac{y}{x+y}\beta_k(x,y), \tag{16}$$

$$\beta_k(xk,yk) = \frac{1}{k}\beta(x,y), \tag{17}$$

$$\beta_k(mk,mk) = \frac{[(m-1)!]^2}{k(2m-1)!}, \quad m \in \mathbb{N}, \tag{18}$$

$$\beta_k(x,k) = \frac{1}{x}, \qquad \beta_k(k,y) = \frac{1}{y}. \tag{19}$$

Note that when $k \to 1$, $\beta_k(x,y) \to \beta(x,y)$.

2 Main results: inequalities via the Chebychev integral inequality

In this section, we prove some inequalities which involve k-gamma and k-beta functions by using some natural inequalities [8]. The following result is known in the literature as the Chebychev integral inequality for synchronous (asynchronous) functions. Here, we use this result to prove some k-analog inequalities.

Lemma 2.1 *Let $f, g, h : I \subseteq \mathbb{R} \to \mathbb{R}$ be such that $h(x) \geq 0$ for all $x \in I$ and h, hfg, hf, and hg are integrable on I. If f, g are synchronous (asynchronous) on I, i.e.,*

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq (\leq)\, 0 \quad \text{for all } x, y \in I, \tag{20}$$

then we have the inequality (*see* [9, 10])

$$\int_I h(x)\,dx \int_I h(x)f(x)g(x)\,dx \geq (\leq) \int_I h(x)f(x)\,dx \int_I h(x)g(x)\,dx. \tag{21}$$

This lemma can be proved by using the Korkine identity [11],

$$\begin{aligned} &\int_I h(x)\,dx \int_I h(x)f(x)g(x)\,dx - \int_I h(x)f(x)\,dx \int_I h(x)g(x)\,dx \\ &\quad = \frac{1}{2}\int_I\int_I h(x)h(y)\big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\,dx\,dy. \end{aligned} \tag{22}$$

Theorem 2.2 *If m, n, p, and q are positive real numbers satisfying the condition*

$$(p - m)(q - n) \leq (\geq)\, 0, \tag{23}$$

then, for the k-beta function, we have the inequality

$$\beta_k(m, n)\beta_k(p, q) \geq (\leq)\, \beta_k(p, n)\beta_k(m, q), \quad k > 0. \tag{24}$$

Proof For $k > 0$, consider the mappings $f, g, h : [0, 1] \to [0, \infty)$ given by

$$f(x) = x^{\frac{p-m}{k}}, \qquad g(x) = (1 - x)^{\frac{q-n}{k}} \quad \text{and} \quad h(x) = x^{\frac{m}{k}-1}(1 - x)^{\frac{n}{k}-1}.$$

Now, differentiation of f and g gives

$$f'(x) = \frac{(p - m)}{k}x^{\frac{p-m}{k}-1}, \qquad g'(x) = \frac{(n - q)}{k}(1 - x)^{\frac{q-n}{k}-1}, \quad x \in (0, 1).$$

As $k > 0$, so using (22) and (23), we see that the mappings f and g are synchronous (asynchronous) having the same (opposite) monotonicity on $[0, 1]$ and h is non-negative on $[0, 1]$. Thus, using the Chebychev integral inequality for the functions f, g, and h defined above, we have

$$\begin{aligned} &\frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1 - x)^{\frac{n}{k}-1}\,dx \cdot \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1 - x)^{\frac{n}{k}-1}x^{\frac{p-m}{k}}(1 - x)^{\frac{q-n}{k}}\,dx \\ &\quad \geq (\leq)\, \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1 - x)^{\frac{n}{k}-1}x^{\frac{p-m}{k}}\,dx \cdot \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1 - x)^{\frac{n}{k}-1}(1 - x)^{\frac{q-n}{k}}\,dx. \end{aligned}$$

This implies

$$\begin{aligned}&\frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}\,dx\cdot\frac{1}{k}\int_0^1 x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\,dx\\&\quad\geq(\leq)\,\frac{1}{k}\int_0^1 x^{\frac{p}{k}-1}(1-x)^{\frac{n}{k}-1}\,dx\cdot\frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{q}{k}-1}\,dx.\end{aligned}$$

Applying (6), we get the required inequality (24). □

Corollary 2.3 *For positive real numbers m, n, p, and q, we have*

$$\Gamma_k(p+n)\Gamma_k(q+m)\geq(\leq)\,\Gamma_k(p+q)\Gamma_k(m+n),\quad k>0. \tag{25}$$

Proof Using (5) and inequality (24), we have

$$\begin{aligned}&\frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}\frac{\Gamma_k(p)\Gamma_k(q)}{\Gamma_k(p+q)}\geq(\leq)\,\frac{\Gamma_k(p)\Gamma_k(n)}{\Gamma_k(p+n)}\frac{\Gamma_k(m)\Gamma_k(q)}{\Gamma_k(m+q)}\\&\quad\Rightarrow\quad\Gamma_k(p+n)\Gamma_k(q+m)\geq(\leq)\,\Gamma_k(p+q)\Gamma_k(m+n).\end{aligned}$$

□

Corollary 2.4 *For $m,p>0$, the following inequality holds for the k-beta function*:

$$\beta_k(m,p)\geq\left[\beta_k(p,p)\beta_k(m,m)\right]^{\frac{1}{2}},\quad k>0. \tag{26}$$

Proof Setting $q=p$ and $n=m$ in Theorem 2.2, we get

$$(p-m)(q-n)=(p-m)^2\geq 0.$$

Thus, Corollary 2.4 follows. We have

$$\beta_k(p,p)\beta_k(m,m)\leq\beta_k(p,m)\beta_k(m,p)=\left[\beta_k(m,p)\right]^2.$$

□

Remarks 2.5 By (5) and (26), we can deduce the following inequality for the k-gamma function:

$$\Gamma_k(p+m)\leq\left[\Gamma_k(2p)\Gamma_k(2m)\right]^{\frac{1}{2}}. \tag{27}$$

Setting $2p=s$ and $2m=t$ in (27), we get

$$\Gamma_k\left(\frac{s+t}{2}\right)\leq\sqrt{\Gamma_k(s)\Gamma_k(t)}.$$

From the above result, we conclude that for two positive numbers s and t, the geometric mean of $\Gamma_k(s)$ and $\Gamma_k(t)$ is greater than or equal to Γ_k (arithmetic mean of s and t).

Now, the Chebychev inequality is used for an infinite interval. For this purpose, see the following theorem employing the inequality [8].

Theorem 2.6 *Let m, p, and r be positive real numbers such that $p > r > -m$. If $r(p-m-r) \geq (\leq)\, 0$, then*

$$\Gamma_k(m)\Gamma_k(p) \geq (\leq)\, \Gamma_k(p-r)\Gamma_k(m+r), \quad k > 0. \tag{28}$$

Proof For $k > 0$, define the mappings $f, g, h : [0,\infty) \to [0,\infty)$ given by

$$f(t) = t^{p-r-m}, \qquad g(t) = t^r \quad \text{and} \quad h(t) = t^{m-1}e^{-\frac{t^k}{k}}.$$

If $r(p-m-r) \geq (\leq)\, 0$, holds and $k > 0$, then we can assert that the mappings f and g are synchronous (asynchronous) $]0,\infty[$. Thus, using the Chebychev inequality for the interval $I = (0,\infty)$ along with the functions f, g, and h defined above, we can write

$$\begin{aligned} &\int_0^\infty t^{m-1}e^{-\frac{t^k}{k}}\,dt \int_0^\infty t^{p-r-m}t^r t^{m-1}e^{-\frac{t^k}{k}}\,dt \\ &\quad \geq (\leq) \int_0^\infty t^{p-r-m}t^{m-1}e^{-\frac{t^k}{k}}\,dt \int_0^\infty t^r t^{m-1}e^{-\frac{t^k}{k}}\,dt. \end{aligned}$$

This implies

$$\int_0^\infty t^{m-1}e^{-\frac{t^k}{k}}\,dt \int_0^\infty t^{p-1}e^{-\frac{t^k}{k}}\,dt \geq (\leq) \int_0^\infty t^{p-r-1}e^{-\frac{t^k}{k}}\,dt \int_0^\infty t^{r+m-1}e^{-\frac{t^k}{k}}\,dt.$$

By (3), we get the required Theorem 2.6. □

Corollary 2.7 *If p, m, and r are positive real numbers satisfying the conditions of Theorem* 2.6, *then we deduce*

$$\beta_k(p,m) \geq (\leq)\, \beta_k(p-r, m+r), \quad k > 0.$$

Proof Using the property $\frac{\Gamma_k(p)\Gamma_k(m)}{\Gamma_k(p+m)} = \beta_k(p,m)$ and from the inequality (28), we can derive Corollary 2.7. □

Corollary 2.8 *If $k, p > 0$ and $q \in \mathbb{R}$ with $|q| < p$, then*

$$\Gamma_k(p) \leq \big[\Gamma_k(p-q)\Gamma_k(p+q)\big]^{\frac{1}{2}}. \tag{29}$$

Proof Setting $m = p$ and $r = q$ in Theorem 2.6, we get $r(p-m-r) = -q^2 \leq 0$ and inequality (28) provides the desired Corollary 2.8. □

Definition 2.9 Two positive real numbers a and b are said to be similarly (oppositely) unitary if (see [8])

$$(a-1)(b-1) \geq (\leq)\, 0. \tag{30}$$

Theorem 2.10 *If $a, b > 0$ are similarly (oppositely) unitary, then*

$$\Gamma_k(a+b+k-1) \geq (\leq)\, \frac{ab\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(k+1)}. \tag{31}$$

Proof For $k > 0$, consider the mappings $f, g, h : [0, \infty) \to [0, \infty)$ defined by

$$f(t) = t^{a-1}, \qquad g(t) = t^{b-1} \quad \text{and} \quad h(t) = t^k e^{-\frac{t^k}{k}}.$$

If the condition $(a-1)(b-1) \geq (\leq)\, 0$ holds and $k > 0$, then clearly the mappings f and g are synchronous (asynchronous) on $[0, \infty)$. Thus, by the Chebychev integral inequality along with the functions f, g, and h defined above, we have

$$\int_0^\infty t^k e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{a-1} t^{b-1} t^k e^{-\frac{t^k}{k}}\, dt \geq (\leq) \int_0^\infty t^{a-1} t^k e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{b-1} t^k e^{-\frac{t^k}{k}}\, dt$$

and

$$\int_0^\infty t^k e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{a+b+k-2} e^{-\frac{t^k}{k}}\, dt \geq (\leq) \int_0^\infty t^{a+k-1} e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{b+k-1} e^{-\frac{t^k}{k}}\, dt.$$

By the definition of the k-gamma function and then using (8), we have

$$\begin{aligned} &\Gamma_k(k+1)\Gamma_k(a+b+k-1) \geq (\leq)\, \Gamma_k(a+k)\Gamma_k(b+k), \\ &\Gamma_k(a+b+k-1) \geq (\leq)\, \frac{ab\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(k+1)}. \end{aligned}$$ □

Corollary 2.11 *If the condition $(a-1)(b-1) \geq (\leq)\, 0$ holds and $k > 0$, then we have*

$$\beta_k(a,b) \geq (\leq)\, \frac{1}{ab} \frac{\Gamma_k(k+1)\Gamma_k(a+b+k-1)}{\Gamma_k(a+b)}. \tag{32}$$

Proof Obvious result from (5) and Theorem 2.10. □

Remarks 2.12 The results proved here are k-analog of theorems as given in [10]. Using $k = 1$, we have the results about classical gamma and beta functions.

Theorem 2.13 *If a, b, and k are positive real numbers such that a and b are similarly (oppositely) unitary, then*

$$\Gamma_k\big((a+b)k\big) \geq (\leq)\, \frac{kab\Gamma_k(ak)\Gamma_k(bk)}{(a+b)}. \tag{33}$$

Proof For $k > 0$, consider the mappings $f, g, h : [0, \infty) \to [0, \infty)$ defined by

$$f(t) = t^{ak}, \qquad g(t) = t^{bk} \quad \text{and} \quad h(t) = t^{k-1} e^{-\frac{t^k}{k}}.$$

If the conditions of Theorem 2.10 hold and $k > 0$, then clearly the mappings f and g are synchronous (asynchronous) on $[0, \infty)$. Thus, by the Chebychev integral inequality along with the choice of the functions f, g, and h defined above, we have

$$\int_0^\infty t^{k-1} e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{ak} t^{bk} t^{k-1} e^{-\frac{t^k}{k}}\, dt \geq (\leq) \int_0^\infty t^{ak} t^{k-1} e^{-\frac{t^k}{k}}\, dt \int_0^\infty t^{bk} t^{k-1} e^{-\frac{t^k}{k}}\, dt$$

and

$$\int_0^\infty t^{k-1}e^{-\frac{t^k}{k}}\,dt\int_0^\infty t^{ak+bk+k-1}e^{-\frac{t^k}{k}}\,dt \ge (\le) \int_0^\infty t^{ak+k-1}e^{-\frac{t^k}{k}}\,dt\int_0^\infty t^{bk+k-1}e^{-\frac{t^k}{k}}\,dt.$$

By the definition of the k-gamma function and then using (8) and (10), we have

$$\begin{aligned}&\Gamma_k(k)\Gamma_k(ak+bk+k) \ge (\le)\, \Gamma_k(ak+k)\Gamma_k(bk+k)\\ &\quad\Rightarrow\quad \Gamma_k(a+b)k \ge (\le)\, \frac{kab\Gamma_k(ak)\Gamma_k(bk)}{(a+b)}.\end{aligned}$$

□

Corollary 2.14 *For all $n \in \mathbb{N}$ and $a, k > 0$, prove that*

$$n\Gamma_k(nak) \ge (ak)^{n-1}\big[\Gamma_k(ak)\big]^n.$$

Proof Replacing b by $a, 2a, 3a, \ldots, (n-1)a$ in the inequality (33), we get

$$\begin{aligned}&\Gamma_k(2ak) \ge \frac{a}{2}k\Gamma_k(ak)\Gamma_k(ak),\\ &\Gamma_k(3ak) \ge \frac{2a}{3}k\Gamma_k(2ak)\Gamma_k(ak),\\ &\vdots\\ &\Gamma_k(nak) \ge \frac{(n-1)a}{n}k\Gamma_k\big((n-1)ak\big)\Gamma_k(ak).\end{aligned}$$

By multiplying the above inequalities, we have the desired Corollary 2.14. □

3 Main results via Grüss and Ostrowski inequalities

In 1935, Grüss established an integral inequality which provides an estimation for the integral of a product in terms of the product of integrals [8]. Here, we use this inequality to prove some inequalities involving k-gamma and k-beta functions. We also use the weighted version of the said inequality which allows us to obtain the inequalities directly for k-gamma function. The following lemma is used to prove some k-analog inequalities.

Lemma 3.1 *Let f and g be two functions defined and integrable on $[a, b]$. If m, M, n, and N are given real constants such that $m \le f(x) \le M$ and $n \le g(x) \le N$ for all $x \in [a, b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ &\quad\le \frac{1}{4}(M-m)(N-n),\end{aligned}\tag{34}$$

and the constant $\frac{1}{4}$ is the best possible (*see* [10]).

Theorem 3.2 *Let p, q, r, s, and k be positive real numbers, then*

$$\begin{aligned}&\big|\beta_k(p+r+k, q+s+k) - k\beta_k(p+k, q+k)\beta_k(r+k, s+k)\big|\\ &\quad\le \frac{1}{4k}\frac{p^{\frac{p}{k}}q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}\frac{r^{\frac{r}{k}}s^{\frac{s}{k}}}{(r+s)^{\frac{r+s}{k}}}.\end{aligned}\tag{35}$$

Proof Consider the functions defined by

$$f_{p,q}(x) = x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}, \qquad f_{r,s}(x) = x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}, \quad x \in [0,1], k > 0.$$

For the application of the Grüss inequality, we have to find the minima and maxima of $f_{a,b}(x)$ $(a, b, k > 0)$. Thus

$$\frac{d}{dx} f_{a,b}(x) = \frac{1}{k} x^{\frac{a}{k}-1}(1-x)^{\frac{b}{k}-1}\bigl[a-(a+b)x\bigr].$$

Here, we see that the solution of $f'_{a,b}(x) = 0$ in the interval $(0,1)$ is $x_0 = \frac{a}{a+b}$. Also, $f'_{a,b}(x) > 0$ on $(0, x_0)$ and $f'_{a,b}(x) < 0$ on $(x_0, 1)$. We conclude that x_0 is the maximum point in the interval $(0,1)$ and consequently

$$m_{a,b} = \inf_{x\in[0,1]} f_{a,b}(x) = 0 = m \quad \text{(say)} \quad \text{and}$$

$$M_{a,b} = \sup_{x\in[0,1]} f_{a,b}(x) = f_{a,b}\left(\frac{a}{a+b}\right) = \frac{a^{\frac{a}{k}} b^{\frac{b}{k}}}{(a+b)^{\frac{a+b}{k}}} = M \quad \text{(say)}.$$

Hence, by the Grüss inequality, we get

$$\begin{aligned}
&\left|\int_0^1 f_{p,q}(x) f_{r,s}(x)\,dx - \int_0^1 f_{p,q}(x)\,dx \int_0^1 f_{r,s}(x)\,dx\right| \le \frac{1}{4}(M_{p,q}-m_{p,q})(M_{r,s}-m_{r,s}) \\
&\quad\Rightarrow \quad \left|\int_0^1 x^{\frac{p}{k}}(1-x)^{\frac{q}{k}} x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}\,dx - \int_0^1 x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}\,dx \int_0^1 x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}\,dx\right| \\
&\qquad\qquad \le \frac{1}{4k}\frac{p^{\frac{p}{k}} q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}\frac{r^{\frac{r}{k}} s^{\frac{s}{k}}}{(r+s)^{\frac{r+s}{k}}}.
\end{aligned}$$

Rearranging the terms on left-hand side and using (6) with simple algebraic computation, we reach the required proof. □

Theorem 3.3 *Let p, q, and k be positive real numbers, then prove that*

$$\left|\beta_k(p+k, q+k) - \frac{k}{(p+k)(q+k)}\right| \le \frac{1}{4k} \tag{36}$$

and an equivalent statement is given by

$$\max\left\{0, \frac{3k^2 - pq - pk - qk}{4k(p+k)(q+k)}\right\} \le \beta_k(p+k, q+k) \le \frac{5k^2 + pq + pk + qk}{4k(p+k)(q+k)}. \tag{37}$$

Proof Consider the functions defined by

$$f(x) = x^{\frac{p}{k}}, \qquad g(x) = (1-x)^{\frac{q}{k}}, \quad x \in [0,1], k > 0.$$

For minima and maxima of $f(x)$ and $g(x)$, we have

$$\inf_{x\in[0,1]} f(x) = \inf_{x\in[0,1]} g(x) = 0; \qquad \sup_{x\in[0,1]} f(x) = \sup_{x\in[0,1]} g(x) = 1.$$

Also,

$$\int_0^1 f(x)\,dx = \frac{k}{p+k}; \qquad \int_0^1 g(x)\,dx = \frac{k}{q+k}.$$

By using the Grüss inequality, we get

$$\left|\int_0^1 x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}\,dx - \int_0^1 x^{\frac{p}{k}}\,dx \int_0^1 (1-x)^{\frac{q}{k}}\,dx\right| \le \frac{1}{4}(1-0)(1-0).$$

Using the definition of the k-beta function, we have

$$\begin{aligned} &\left|k\beta_k(p+k,q+k) - \frac{k}{p+k}\frac{k}{q+k}\right| \le \frac{1}{4} \\ &\quad\Rightarrow \quad \left|\beta_k(p+k,q+k) - \frac{k}{(p+k)(q+k)}\right| \le \frac{1}{4k}. \end{aligned}$$

Algebraic computation provides the equivalent inequality (37). □

Corollary 3.4 *If we use* (5), *the inequality* (36) *yields*

$$\begin{aligned} &\left|\frac{\Gamma_k(p+k)\Gamma_k(q+k)}{\Gamma_k(p+q+2k)} - \frac{k}{(p+k)(q+k)}\right| \le \frac{1}{4k}, \\ &\left|(p+k)\Gamma_k(p+k)(q+k)\Gamma_k(q+k) - k\Gamma_k(p+q+2k)\right| \\ &\quad\le \frac{1}{4k}(p+k)(q+k)\Gamma_k(p+q+2k). \end{aligned}$$

By using (8), *we get the following inequality*:

$$\left|\Gamma_k(p+2k)\Gamma_k(q+2k) - k\Gamma_k(p+q+2k)\right| \le \frac{1}{4k}(p+k)(q+k)\Gamma_k(p+q+2k).$$

Now, we discuss the weighted version of the Grüss inequality which is used to generalize the previous Theorems 3.2 and 3.3. The weighted version is given in the following lemma and generalized results in the form of propositions.

Lemma 3.5 *Let f and g be two functions satisfying the conditions of Theorem* 3.2 *and* $h:[a,b]\to[0,\infty)$ *is such that* $\int_a^b h(x)\,dx > 0$, *then*

$$\begin{aligned} &\left|\frac{1}{\int_a^b h(x)\,dx}\int_a^b f(x)g(x)h(x)\,dx \right. \\ &\qquad \left. - \frac{1}{\int_a^b h(x)\,dx}\int_a^b f(x)h(x)\,dx \frac{1}{\int_a^b h(x)\,dx}\int_a^b g(x)h(x)\,dx\right| \\ &\quad\le \frac{1}{4}(M-m)(N-n) \end{aligned} \tag{38}$$

and the constant $\frac{1}{4}$ is best possible.

Proof Similar to the classical one (see [8]). □

Proposition 3.6 *Let m, n, p, q, and k be positive real numbers and $r, s > -k$; then we have*

$$\begin{aligned}&\bigl|\beta_k(r+k,s+k)\beta_k(m+p+r+k,n+q+s+k)\\&\quad-\beta_k(m+r+k,n+s+k)\beta_k(p+r+k,q+s+k)\bigr|\\&\quad\le\frac{1}{4k}\frac{m^{\frac{m}{k}}n^{\frac{n}{k}}}{(m+n)^{\frac{m+n}{k}}}\frac{p^{\frac{p}{k}}q^{\frac{q}{k}}}{(p+q)^{\frac{p+q}{k}}}\beta_k^2(r+k,s+k).\end{aligned}\tag{39}$$

Proof Straightforward by considering the choice of the following functions along with Lemma 3.5 which generalizes Theorem 3.2:

$$\begin{aligned}&f_{m,n}(x)=x^{\frac{m}{k}}(1-x)^{\frac{n}{k}}=f(x),\qquad f_{p,q}(x)=x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}=g(x),\\&f_{r,s}(x)=x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}=h(x),\quad x\in[0,1],k>0.\end{aligned}$$ □

Proposition 3.7 *Let p, q, and k be positive real numbers and $r, s > -k$, then*

$$\begin{aligned}&\bigl|\beta_k(r+k,s+k)\beta_k(p+r+k,q+s+k)-\beta_k(p+r+k,s+k)\beta_k(r+k,q+s+k)\bigr|\\&\quad\le\frac{1}{4k}\beta_k^2(r+k,s+k).\end{aligned}\tag{40}$$

Proof Using Lemma 3.5 by considering the choice of functions defined by

$$x^{\frac{p}{k}}=f(x),\qquad(1-x)^{\frac{q}{k}}=g(x),\qquad f_{r,s}(x)=x^{\frac{r}{k}}(1-x)^{\frac{s}{k}}=h(x),\quad x\in[0,1],k>0,$$

we can prove Proposition 3.7, which is the generalization of Theorem 3.3. □

The weighted version of the Grüss inequality allows us to obtain the inequalities directly for k-gamma function (see the following theorem).

Theorem 3.8 *Let a, b, and c are positive real numbers, then prove that*

$$\begin{aligned}&\left|\frac{1}{3^{\frac{a+b+c+1}{k}}}\Gamma_k(a+b+c+1)\Gamma_k(c+1)-\frac{1}{2^{\frac{a+b+2c+2}{k}}}\Gamma_k(a+c+1)\Gamma_k(b+c+1)\right|\\&\quad\le\frac{1}{4}\left(\frac{a}{e}\right)^{\frac{a}{k}}\cdot\left(\frac{b}{e}\right)^{\frac{b}{k}}\Gamma_k^2(c+1),\quad k>0.\end{aligned}\tag{41}$$

Proof Let the mapping $f_a(t)=t^ae^{-\frac{t^k}{k}}$ be defined on $(0,\infty)$, then

$$f_a'(t)=t^{a-1}e^{-\frac{t^k}{k}}\bigl[a-t^k\bigr];$$

$f_a'(t)=0$ gives the unique solution $t_0=a^{\frac{1}{k}}$, which implies that f_a is an increasing function on $(0,t_0)$ and decreasing on (t_0,∞). Thus f_a has a maximum value at $t_0=a^{\frac{1}{k}}$ *i.e.*, $f_a(a^{\frac{1}{k}})=a^{\frac{a}{k}}/e^{\frac{a}{k}}$. Using Lemma 3.5, we get

$$\begin{aligned}&\left|\frac{1}{\int_0^xf_c(t)\,dt}\int_0^xf_a(t)f_b(t)f_c(t)\,dt-\frac{1}{\int_0^xf_c(t)\,dt}\int_0^xf_a(t)f_c(t)\,dt\frac{1}{\int_0^xf_c(t)\,dt}\int_0^xf_b(t)f_c(t)\,dt\right|\\&\quad\le\frac{1}{4}\Bigl(\sup_{t\in[0,x]}f_a(t)-\inf_{t\in[0,x]}f_a(t)\Bigr)\Bigl(\sup_{t\in[0,x]}f_b(t)-\inf_{t\in[0,x]}f_b(t)\Bigr)\end{aligned}$$

which (for all $x > 0$) is equivalent to

$$\left|\int_0^x t^{a+b+c} e^{-3\frac{t^k}{k}}\,dt \int_0^x t^c e^{-\frac{t^k}{k}}\,dt - \int_0^x t^{a+c} e^{-2\frac{t^k}{k}}\,dt \int_0^x t^{b+c} e^{-2\frac{t^k}{k}}\,dt\right|$$
$$\leq \frac{1}{4}\left(\frac{a^{\frac{a}{k}}}{e^{\frac{a}{k}}} - 0\right)\left(\frac{b^{\frac{b}{k}}}{e^{\frac{b}{k}}} - 0\right)\left(\int_0^x t^c e^{-\frac{t^k}{k}}\,dt\right)^2.$$

Since all the integrals involved here are convergent on $[0, \infty)$, we have

$$\left|\int_0^\infty t^{a+b+c} e^{-3\frac{t^k}{k}}\,dt \int_0^\infty t^c e^{-\frac{t^k}{k}}\,dt - \int_0^\infty t^{a+c} e^{-2\frac{t^k}{k}}\,dt \int_0^\infty t^{b+c} e^{-2\frac{t^k}{k}}\,dt\right|$$
$$\leq \frac{1}{4} \cdot \frac{a^{\frac{a}{k}} b^{\frac{b}{k}}}{e^{\frac{a+b}{k}}}\left(\int_0^\infty t^c e^{-\frac{t^k}{k}}\,dt\right)^2. \tag{42}$$

Now, a simple change of variable $3t^k = s^k$, above integrals can be changed into k-gamma functions as

$$\int_0^\infty t^{a+b+c} e^{-3\frac{t^k}{k}}\,dt = \frac{1}{3^{\frac{1}{k}}} \int_0^\infty \frac{s^{a+b+c}}{3^{\frac{a+b+c}{k}}} e^{-\frac{s^k}{k}}\,ds = \frac{1}{3^{\frac{a+b+c+1}{k}}} \Gamma_k(a+b+c+1).$$

Similarly, we have

$$\int_0^\infty t^{a+c} e^{-2\frac{t^k}{k}}\,dt = \frac{1}{2^{\frac{a+c+1}{k}}} \Gamma_k(a+c+1) \quad \text{and} \quad \int_0^\infty t^{b+c} e^{-2\frac{t^k}{k}}\,dt = \frac{1}{2^{\frac{b+c+1}{k}}} \Gamma_k(b+c+1).$$

From (42) along with the above results, we get Theorem 3.8. □

Now, we mention here another inequality which is known in literature as the Ostrowski inequality. The following lemmas contain the said integral inequality [12] which is used to prove the inequalities involving k-beta function.

Lemma 3.9 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b), whose derivative is bounded on (a, b) and let $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, for all $x \in [a, b]$, we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{(a+b)}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{43}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one and the following lemma is the generalization of the inequality (43) which has been proved in [13].

Lemma 3.10 *Let $u : [a, b] \to \mathbb{R}$ be a L-Lipschitzian mapping on $[a, b]$, i.e.,*

$$\left|u(x) - u(y)\right| \leq L|x - y| \quad \textit{for all } x, y \in [a, b]. \tag{44}$$

Then, for all $x \in [a, b]$, we have

$$\left|\int_a^b u(t)\,dt - u(x)(b-a)\right| \leq L(b-a)^2\left[\frac{1}{4} + \frac{(x - \frac{(a+b)}{2})^2}{(b-a)^2}\right]. \tag{45}$$

The constant $\frac{1}{4}$ is best possible.

Remarks 3.11 If we assume that $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and the derivative f' is bounded on (a,b) we can put instead of L the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang in [11]. Now we are able to prove our result for the k-beta function.

Theorem 3.12 *If $k > 0$ and $p, q > 2 + k$ and $x \in [0,1]$, then we have the following inequality*:

$$\begin{aligned} &\left|\beta_k(p,q) - \frac{1}{k}x^{\frac{p-k}{k}}(1-x)^{\frac{q-k}{k}}\right| \\ &\quad \le \max\{p-1,q-1\}\frac{1}{k^2}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right]. \end{aligned} \tag{46}$$

Proof Consider the mapping $l_{p,q} : (0,1) \to \mathbb{R}$, defined by $l_{p,q} = x^{\frac{p}{k}}(1-x)^{\frac{q}{k}}$, $k > 0$. For $p, q > 1 + k$, differentiation gives

$$l'_{p,q}(x) = \frac{1}{k}x^{\frac{p-k}{k}}(1-x)^{\frac{q-k}{k}}\bigl[p-(p+q)x\bigr], \quad x \in (0,1)$$

and

$$\begin{aligned} l'_{p-1,q-1}(x) &= \frac{1}{k}x^{\frac{p-k-1}{k}}(1-x)^{\frac{q-k-1}{k}}\bigl[p-1-(p+q-2)x\bigr] \\ &= \frac{1}{k}l_{p-k-1,q-k-1}\bigl[p-1-(p+q-2)x\bigr]. \end{aligned}$$

The solution of $l'_{p-1,q-1}(x) = 0$ in the interval $(0,1)$ is $x_0 = \frac{p-1}{p+q-2}$. Also, $f'_{p-1,q-1}(x) > 0$ on $(0,x_0)$ and $f'_{p-1,q-1}(x) < 0$ on $(x_0,1)$. This shows that x_0 is the maximum point, so

$$\operatorname*{Sup}_{x\in(0,1)} l_{p-1,q-1}(x) = l_{p-1,q-1}(x_0) = \frac{(p-1)^{\frac{p}{k}-1}(q-1)^{\frac{q}{k}-1}}{(p+q-2)^{\frac{p+q}{k}-2}}, \quad p,q > 1+k.$$

Consequently, for all $x \in [0,1]$, we have

$$\begin{aligned} \bigl|l'_{p-1,q-1}(x)\bigr| &\le \frac{1}{k}\bigl|l_{p-k-1,q-k-1}(x)\bigr| \operatorname*{Sup}_{x\in(0,1)}\bigl|(p-1)-(p+q-2)x\bigr| \\ &\le \max\{p-1,q-1\}\frac{1}{k}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}. \end{aligned}$$

Thus, for $p, q > 2 + k$, we get

$$\bigl\|l'_{p-1,q-1}(x)\bigr\|_\infty \le \max\{p-1,q-1\}\frac{1}{k}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}. \tag{47}$$

Taking the function $f(x) = l_{p-1,q-1}(x)$, $x \in [0,1]$ and using the inequalities (47) and (45) along with Remarks 3.11, we have

$$\begin{aligned} &\left|\int_0^1 x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\,dx - x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}(1-0)\right| \\ &\quad \le \max\{p-1,q-1\}\frac{1}{k}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right], \end{aligned}$$

$$\begin{aligned}&\left|\beta_k(p,q)-\frac{1}{k}x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\right|\\&\quad\le\max\{p-1,q-1\}\frac{1}{k^2}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}\left[\frac{1}{4}+\left(x-\frac{1}{2}\right)^2\right].\end{aligned}$$

□

Corollary 3.13 *The best inequality that can be obtained for* $p,q>2+k$ *and* $x\in[0,1]$ *is*

$$\left|\beta_k(p,q)-\frac{1}{k}\frac{1}{2^{\frac{p+q}{k}-2}}\right|\le\frac{1}{4k^2}\max\{p-1,q-1\}\frac{(p-2)^{(p-k-1)/k}(q-2)^{(q-k-1)/k}}{(p+q-4)^{(p+q-2k-2)/k}}.$$

Proof Setting $x=\frac{1}{2}$, we get the desired result. □

Here, we use the Ostrowski inequality for the mappings of bounded variation [14] involving k-beta function. For this purpose, we need the following lemmas.

Lemma 3.14 *Let* $u:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$. *Then, for all* $x\in[a,b]$, *we have the inequality* (*see* [10])

$$\left|\int_a^b u(t)\,dt-u(x)(b-a)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u),\tag{48}$$

where $\bigvee_a^b(u)$ *indicates the total variation of* u *and the constant* $\frac{1}{2}$ *is the best possible.*

Lemma 3.15 *If* $u:[a,b]\to\mathbb{R}$ *is continuous and differentiable on* $[a,b]$, u' *is continuous on* (a,b) *and* $\|u'\|_1=\int_a^b|u'(t)|\,dt<\infty$, *then, for all* $x\in[a,b]$, *we have the inequality* (*see* [10])

$$\left|\int_a^b u(t)\,dt-u(x)(b-a)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\|u'\|_1.\tag{49}$$

Theorem 3.16 *If* $k>0$ *and* $p,q>1+k$, $x\in[0,1]$, *then we have the following inequality*:

$$\left|\beta_k(p,q)-\frac{1}{k}x^{\frac{p-k}{k}}(1-x)^{\frac{q-k}{k}}\right|\le\max\{p-1,q-1\}\frac{1}{k}\beta_k(p-k,q-k)\left[\frac{1}{2}+\left|x-\frac{1}{2}\right|\right].\tag{50}$$

Proof Consider the mapping as in Theorem 3.12. Now, we have

$$\begin{aligned}l'_{p-1,q-1}(t)&=\frac{1}{k}x^{\frac{p-k-1}{k}}(1-x)^{\frac{q-k-1}{k}}\left[p-1-(p+q-2)t\right]\\&=\frac{1}{k}l_{p-k-1,q-k-1}\left[p-1-(p+q-2)x\right].\end{aligned}$$

Also, $|p-1-(p+q-2)t|\le\max\{p-1,q-1\}$ for all $t\in[0,1]$, so

$$\begin{aligned}\|l'_{p-1,q-1}\|_1&=\frac{1}{k}\int_0^1 l_{p-k-1,q-k-1}(t)\left|p-1-(p+q-2)t\right|dt\\&\le\frac{1}{k}\max\{p-1,q-1\}\|l_{p-k-1,q-k-1}\|_1\\\Rightarrow\quad&\|l'_{p-1,q-1}\|_1\le\max\{p-1,q-1\}\beta_k(p-k,q-k),\quad p,q>1+k,k>0.\end{aligned}$$

Now, applying the previous lemmas for $u(t) = l_{p-1,q-1}$, $x \in [0,1]$, we conclude

$$\begin{aligned}\left|\int_0^1 l_{p-1,q-1}(t)\,dt - x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}\right| &\le \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right] \bigvee_0^1 (l_{p-1,q-1}) \\ &\le \max\{p-1,q-1\}\beta_k(p-k,q-k)\left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right].\end{aligned}$$

By definition of the k-beta function, we have the required Theorem 3.16. □

Corollary 3.17 *Let $p,q > 1+k$, then the best inequality that can be obtained from inequality* (50) *is*

$$\left|\beta_k(p,q) - \frac{1}{k}\frac{1}{2^{\frac{p+q}{k}-2}}\right| \le \frac{1}{2k}\max\{p-1,q-1\}\beta_k(p-k,q-k). \tag{51}$$

Proof Setting $x = \frac{1}{2}$ in the inequality (50), we get the desired result. □

4 Log-convexity of the k-gamma and k-beta functions

Many authors, see [15–18] and the references therein, have worked on convexity, log-convexity, and exponential convexity of different functions including the Euler gamma function. Lately, Diaz and Pariguan [1] worked on the convexity of k-gamma function and proved that the said function is logarithmically convex. They used the limit form of k-gamma function for this purpose. Here, we give a new technique to prove the log-convexity of the k-gamma function. Also, we prove the log-convexity of the k-beta function which is the k-analog result [10].

Definition 4.1 In [15], a function $f : (a,b) \to \mathbb{R}$ is said to be convex if for any $x,y \in (a,b)$ and $\lambda \in (0,1)$

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda)f(y). \tag{52}$$

The above definition shows that when we move from x to y, the line joining the points $(x,f(x))$ and $(y,f(y))$ lies always above the graph of f.

Definition 4.2 If $f > 0$ and $\log f$ is convex, then f is called a log-convex function *i.e.*, $\forall x,y \in I$ (an interval) and $\lambda \in (0,1)$, we have

$$\begin{aligned}&\log f\big(\lambda x + (1-\lambda)y\big) \le \lambda \log f(x) + (1-\lambda)\log f(y) = \log\big(f^{\lambda}(x)f^{(1-\lambda)}(y)\big) \\ &\quad\Rightarrow\quad f\big(\lambda x + (1-\lambda)y\big) \le \big(f^{\lambda}(x)f^{(1-\lambda)}(y)\big).\end{aligned} \tag{53}$$

Lemma 4.3 (Hölder inequality) *If p and q are positive real numbers satisfying the condition $\frac{1}{p} + \frac{1}{q} = 1$, then for integrable functions $f,g : [a,b] \to \mathbb{R}$, we have*

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g|^q\,dx\right)^{\frac{1}{q}}. \tag{54}$$

Proof The proof of the above inequality is available in [19]. □

Theorem 4.4 *For $k > 0$, prove that $\Gamma_k : (0,\infty) \to \mathbb{R}$ is log-convex or $\log\Gamma_k$ is convex* (*proved in* [1], *but we give a new proof here*).

Proof Let p and q be positive real numbers satisfying the condition $\frac{1}{p} + \frac{1}{q} = 1$. By the definition of $\Gamma_k(x)$, we have

$$\begin{aligned}
\Gamma_k\left(\frac{x}{p} + \frac{y}{q}\right) &= \int_0^\infty t^{(\frac{x}{p}+\frac{y}{q}-1)} e^{-\frac{t^k}{k}}\,dt \\
&= \int_0^\infty t^{(\frac{x}{p}+\frac{y}{q}-(\frac{1}{p}+\frac{1}{q}))} e^{-\frac{t^k}{k}(\frac{1}{p}+\frac{1}{q})}\,dt \\
&= \int_0^\infty t^{(\frac{x}{p}-\frac{1}{p})} e^{-\frac{t^k}{kp}} t^{(\frac{y}{q}-\frac{1}{q})} e^{-\frac{t^k}{kq}}\,dt \\
&= \int_0^\infty \left(t^{x-1} e^{-\frac{t^k}{k}}\right)^{\frac{1}{p}} \left(t^{y-1} e^{-\frac{t^k}{k}}\right)^{\frac{1}{q}}\,dt.
\end{aligned}$$

By Lemma 4.3, we conclude that

$$\Gamma_k\left(\frac{x}{p} + \frac{y}{q}\right) \le \left(\int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\,dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{y-1} e^{-\frac{t^k}{k}}\,dt\right)^{\frac{1}{q}}.$$

This implies

$$\Gamma_k\left(\frac{x}{p} + \frac{y}{q}\right) \le \left(\Gamma_k(x)\right)^{\frac{1}{p}} \left(\Gamma_k(y)\right)^{\frac{1}{q}}.$$

Let $\lambda = \frac{1}{p}$, $(1-\lambda) = \frac{1}{q}$, then $\lambda \in (0,1)$ and

$$\begin{aligned}
&\Gamma_k\big(\lambda x + (1-\lambda)y\big) \le \big(\Gamma_k(x)\big)^{\lambda} \big(\Gamma_k(y)\big)^{(1-\lambda)} \\
&\quad \Rightarrow \quad \log\big(\Gamma_k\big(\lambda x + (1-\lambda)y\big)\big) \le \lambda \log\Gamma_k(x) + (1-\lambda)\log\Gamma_k(y)
\end{aligned}$$

for $x, y \in (0,\infty)$, $\log\Gamma_k$ is convex *i.e.*, Γ_k is log-convex. □

Remarks 4.5 By Theorem 4.4, the function Γ_k is log-convex. Also, every log-convex function is convex [15], so the k-gamma function is convex.

Theorem 4.6 *For $k > 0$, prove that the function β_k is logarithmically convex on $(0,\infty) \times (0,\infty)$ as a function of two variables.*

Proof Let $(p,q), (m,n) \in (0,\infty)^2$ and $a, b \ge 0$ with $a + b = 1$, we have

$$\beta_k\big[a(p,q) + b(m,n)\big] = \beta_k(ap + bm, aq + bn). \tag{55}$$

Using (6) on right-hand side of the inequality (55), we get

$$\begin{aligned}
\beta_k\big[a(p,q) + b(m,n)\big] &= \frac{1}{k}\int_0^1 t^{\frac{ap+bm}{k}-1}(1-t)^{\frac{aq+bn}{k}-1}\,dt \\
&= \frac{1}{k}\int_0^1 t^{\frac{ap+bm}{k}-(a+b)}(1-t)^{\frac{aq+bn}{k}-(a+b)}\,dt
\end{aligned}$$

$$= \frac{1}{k}\int_0^1 t^{a(\frac{p}{k}-1)}(1-t)^{a(\frac{q}{k}-1)}t^{b(\frac{m}{k}-1)}(1-t)^{b(\frac{n}{k}-1)}\,dt$$
$$= \frac{1}{k}\int_0^1 \left[t^{\frac{p}{k}-1}(1-t)^{\frac{q}{k}-1}\right]^a \times \left[t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}\right]^b dt.$$

Thus, we get

$$\beta_k\bigl[a(p,q)+b(m,n)\bigr] \le \frac{1}{k}\bigl[k\beta_k(p,q)\bigr]^a\bigl[k\beta_k(m,n)\bigr]^b = k^{a+b-1}\bigl[\beta_k(p,q)\bigr]^a\bigl[\beta_k(m,n)\bigr]^b.$$

Here, $\lambda = a$, $(1-\lambda) = b$, then $\lambda \in (0,1)$, which shows the logarithmic convexity of β_k on $(0,\infty)^2$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by AR and SM. The both authors contributed equally to the writing of this paper. The authors NS and FS read and approved the final manuscript.

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