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Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces

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Abstract

The purpose of this paper is to study the strong convergence theorems of Moudafi's viscosity approximation methods for a nonexpansive mapping T in CAT(0) spaces without the property $\mathcal{P}$. For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; *i.e.*,

$$x_t = tf(x_t) \oplus (1-t)Tx_t$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $x_0 \in C$ is arbitrarily chosen and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions. We prove that the iterative schemes $\{x_t\}$ and $\{x_n\}$ converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is the unique solution of the variational inequality (VIP)

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T).$$

By using the concept of quasilinearization, we remark that the proof is different from that of Shi and Chen in J. Appl. Math. 2012:421050, 2012. In fact, strong convergence theorems for two given iterative schemes are established in CAT(0) spaces without the property $\mathcal{P}$.

Keywords: viscosity approximation method; nonexpansive mapping; variational inequality; CAT(0) space; common fixed point

1 Introduction

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}_2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of an appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{CN}$$

This is the (CN)-inequality of Bruhat and Tits [1]. In fact (*cf.* [2], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)-inequality.

It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$-trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

It is proved in [2] that a normed linear space satisfies the (CN)-inequality if and only if it satisfies the parallelogram identity, *i.e.*, is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [5] introduced the concept of quasilinearization as follows:

Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a map $\langle\cdot,\cdot\rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \frac{1}{2}\bigl(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\bigr) \quad (a, b, c, d \in X). \tag{1}$$

It is easily seen that $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = \langle\overrightarrow{cd}, \overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle = -\langle\overrightarrow{ba}, \overrightarrow{cd}\rangle$, and $\langle\overrightarrow{ax}, \overrightarrow{cd}\rangle + \langle\overrightarrow{xb}, \overrightarrow{cd}\rangle = \langle\overrightarrow{ab}, \overrightarrow{cd}\rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle\overrightarrow{ab}, \overrightarrow{cd}\rangle \leq d(a, b)d(c, d) \tag{2}$$

for all $a, b, c, d \in X$. It is known [5, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 2010, Kakavandi and Amini [6] introduced the concept of a dual space for CAT(0) spaces as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t, a, b)(x) = t\langle\overrightarrow{ab}, \overrightarrow{ax}\rangle, \tag{3}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with the Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$ for all $t \in \mathbb{R}$ and $a,b \in X$, where

$$L(f) = \sup\left\{\frac{f(x) - f(y)}{d(x,y)} : x,y \in X, x \neq y\right\}$$

is the Lipschitz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\big((t,a,b),(s,c,d)\big) = L\big(\Theta(t,a,b) - \Theta(s,c,d)\big).$$

Lemma 1.1 [6, Lemma 2.1] *$D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle = s\langle\overrightarrow{cd},\overrightarrow{xy}\rangle$ for all $x,y \in X$.*

For a complete CAT(0) space (X,d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X,\mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, D defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $t\overrightarrow{ab} := (t,a,b)$ is

$$[t\overrightarrow{ab}] = \big\{s\overrightarrow{cd} : t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle = s\langle\overrightarrow{cd},\overrightarrow{xy}\rangle\ \forall x,y \in X\big\}.$$

The set $X^* := \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with a metric D, which is called the dual metric space of (X,d).

Recently, Dehghan and Rooin [7] introduced the duality mapping in CAT(0) spaces and studied its relation with subdifferential by using the concept of quasilinearization. Then they presented a characterization of a metric projection in CAT(0) spaces as follows.

Theorem 1.2 [7, Theorem 2.4] *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle\overrightarrow{yu},\overrightarrow{ux}\rangle \geq 0 \quad \text{for all } y \in C.$$

Let C be a nonempty subset of a complete CAT(0) space X. Then the mapping T of C into itself is called *nonexpansive* iff $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in C$. A point $x \in C$ is called a fixed point of T if $x = Tx$. We denote by $F(T)$ the set of all fixed points of T. Kirk [8] showed that the fixed point set of a nonexpansive mapping T is closed and convex. A mapping f of C into itself is called *contraction* with coefficient $\alpha \in (0,1)$ iff $d(fx,fy) \leq \alpha d(x,y)$ for all $x,y \in C$. Banach's contraction principle [9] guarantees that f has a unique fixed point when C is a nonempty closed convex subset of a complete metric space. The existence of fixed points and convergence theorems for several mappings in CAT(0) spaces have been investigated by many authors (see also [10–16]).

In 2010, Saejung [15] studied the convergence theorems of the following Halpern's iterations for a nonexpansive mapping T: Let u be fixed and $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tu \oplus (1-t)Tx$; *i.e.*,

$$x_t = tu \oplus (1-t)Tx_t, \tag{4}$$

where $t \in [0,1]$ and $x_0, u \in C$ are arbitrarily chosen and

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{5}$$

where $\{\alpha_n\} \subset (0,1)$. It is proved in [15] that $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ which is nearest to u ($\tilde{x} = P_{F(T)}u$), and $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ which is nearest to u under certain appropriate conditions on $\{\alpha_n\}$, where $P_C x$ is a metric projection from X onto C.

In 2012, Shi and Chen [16] studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T: For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; *i.e.*,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{6}$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{7}$$

where $\{\alpha_n\} \subset (0,1)$. They proved that $\{x_t\}$ defined by (6) converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of a CAT(0) space satisfying the property $\mathcal{P}$, *i.e.*, if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they also obtained that $\{x_n\}$ defined by (7) converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

All of the above bring us the following conjectures.

Question 1.3 Could we obtain the strong convergence of both $\{x_t\}$ and $\{x_n\}$ defined by (6) and (7) respectively, in the framework of a CAT(0) space without the property $\mathcal{P}$?

The purpose of this paper is to study the strong convergence theorems of the iterative schemes (6) and (7) in CAT(0) spaces without the property $\mathcal{P}$. We prove that the iterative schemes (6) and (7) converge strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is the unique solution of the variational inequality (VIP)

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{8}$$

By using the concept of quasilinearization, we remark that the proof given below is different from that of Shi and Chen [16]. In fact, strong convergence theorems for two given iterative schemes are established in CAT(0) spaces without the property $\mathcal{P}$.

2 Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y) \quad \text{and} \quad d(z,y) = (1-t)d(x,y).$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$.

The following lemmas play an important role in our paper.

Lemma 2.1 [2, Proposition 2.2] *Let X be a CAT(0) space, $p,q,r,s \in X$ and $\lambda \in [0,1]$. Then*

$$d\bigl(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s\bigr) \le \lambda d(p,r) + (1-\lambda)d(q,s).$$

Lemma 2.2 [11, Lemma 2.4] *Let X be a CAT(0) space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

$$d\bigl(\lambda x \oplus (1-\lambda)y, z\bigr) \le \lambda d(x,z) + (1-\lambda)d(y,z).$$

Lemma 2.3 [11, Lemma 2.5] *Let X be a CAT(0) space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

$$d^2\bigl(\lambda x \oplus (1-\lambda)y, z\bigr) \le \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

The concept of Δ-convergence introduced by Lim [17] in 1976 was shown by Kirk and Panyanak [18] in CAT(0) spaces to be very similar to weak convergence in the Banach space setting. Next, we give the concept of Δ-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

It is known from Proposition 7 of [14] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. The uniqueness of an asymptotic center implies that the CAT(0) space X satisfies Opial's property, *i.e.*, for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that '$I - T$ is demiclosed at zero' if the conditions $\{x_n\} \subseteq C$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.4 [18] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.5 [13] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.6 [13] *If C is a closed convex subset of X and $T : C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

Having the notion of quasilinearization, Kakavandi and Amini [6] introduced the following notion of convergence.

A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w-converges to $x \in X$ if $\lim_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$, *i.e.*, $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [6, Proposition 2.5], but it is showed in ([19, Example 4.7]) that the converse is not valid. However, the following lemma shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Lemma 2.7 [19, Theorem 2.6] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0$ for all $y \in X$.*

Lemma 2.8 [20, Lemma 2.1] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \ge 0,$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that
1. $\sum_{n=0}^{\infty} \alpha_n = \infty$;
2. $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

The following two lemmas can be obtained from elementary computation. For convenience of the readers, we include the details.

Lemma 2.9 *Let X be a complete CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds*:

$$d^2(x, u) \le d^2(y, u) + 2\langle\overrightarrow{xy}, \overrightarrow{xu}\rangle.$$

Proof

$$\begin{aligned}
d^2(y, u) - d^2(x, u) - 2\langle\overrightarrow{yx}, \overrightarrow{xu}\rangle &= d^2(y, u) - d^2(x, u) - 2\langle\overrightarrow{yu}, \overrightarrow{xu}\rangle - 2\langle\overrightarrow{ux}, \overrightarrow{xu}\rangle \\
&= d^2(y, u) - d^2(x, u) - 2\langle\overrightarrow{yu}, \overrightarrow{xu}\rangle + 2d^2(x, u) \\
&= d^2(y, u) + d^2(x, u) - 2\langle\overrightarrow{yu}, \overrightarrow{xu}\rangle \\
&\ge d^2(y, u) + d^2(x, u) - 2d(y, u)d(x, u) \\
&= \big(d(y, u) - d(x, u)\big)^2 \ge 0.
\end{aligned}$$

Hence

$$d^2(x,u) \le d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle. \qquad \square$$

Lemma 2.10 *Let X be a CAT(0) space. For any $u, v \in X$ and $t \in [0,1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Proof (i) It follows from the (CN)-inequality that

$$\begin{aligned}
2\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle &= d^2(u_t, y) + d^2(x, u_t) - d^2(u_t, u_t) - d^2(x,y) \\
&\le td^2(u,y) + (1-t)d^2(v,y) - t(1-t)d^2(u,v) + d^2(x,u_t) \\
&\quad - d^2(u_t,u_t) - d^2(x,y) \\
&= td^2(u,y) + td^2(x,u_t) - td^2(u,u_t) - td^2(x,y) \\
&\quad + (1-t)d^2(v,y) + (1-t)d^2(x,u_t) - (1-t)d^2(v,u_t) - (1-t)d^2(x,y) \\
&\quad + td^2(u,u_t) + (1-t)d^2(v,u_t) - t(1-t)d^2(u,v) \\
&= t\bigl[d^2(u,y) + d^2(x,u_t) - d^2(u,u_t) - d^2(x,y)\bigr] \\
&\quad + (1-t)\bigl[d^2(v,y) + d^2(x,u_t) - d^2(v,u_t) - d^2(x,y)\bigr] \\
&\quad + t(1-t)^2 d^2(u,v) + (1-t)t^2 d^2(u,v) - t(1-t)d^2(u,v) \\
&= t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle.
\end{aligned}$$

(ii) The proof is similar to (i). $\square$

3 Variational inequalities in CAT(0) spaces

In this section, we present strong convergence theorems of Moudafi's viscosity methods in CAT(0) spaces. Our first result is the continuous version of Theorem 2.2 of Shi and Chen [16]. By using the concept of quasilinearization, we note that the proof given below is different from that of Shi and Chen.

For any $t \in (0,1]$ and a contraction f with coefficient $\alpha \in (0,1)$, define the mapping $S_t : C \to C$ by

$$G_t = tf(x) \oplus (1-t)Tx, \quad \forall x \in C. \tag{9}$$

It is not hard to see that G_t is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned}
&d\bigl(G_t(x), G_t(y)\bigr) \\
&\quad = d\bigl(tf(x) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty\bigr) \\
&\quad \le d\bigl(tf(x) \oplus (1-t)Tx, tf(y) \oplus (1-t)Tx\bigr) \\
&\qquad + d\bigl(tf(y) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty\bigr) \\
&\quad \le td\bigl(f(x), f(y)\bigr) + (1-t)d(Tx, Ty)
\end{aligned}$$

$$\begin{aligned} &\leq t\alpha d(x,y) + (1-t)d(x,y) \\ &= \bigl(1-t(1-\alpha)\bigr)d(x,y). \end{aligned}$$

This implies that G_t is a contraction mapping. Then there exists a unique $u \in C$ such that

$$u = G_t(u) = tf(u) \oplus (1-t)Tu.$$

Let $x_t \in C$ be the unique fixed point of G_t. Thus

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{10}$$

Theorem 3.1 *Let C be a closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For each $t \in (0,1]$, let $\{x_t\}$ be given by*

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{11}$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\bigl\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \bigr\rangle \geq 0, \quad x \in F(T). \tag{12}$$

Proof We first show that $\{x_t\}$ is bounded. For any $p \in F(T)$, we have that

$$\begin{aligned} d(x_t,p) &= d\bigl(tf(x_t) \oplus (1-t)Tx_t, p\bigr) \\ &\leq td\bigl(f(x_t),p\bigr) + (1-t)d(Tx_t,p) \\ &\leq td\bigl(f(x_t),p\bigr) + (1-t)d(x_t,p). \end{aligned}$$

Then

$$\begin{aligned} d(x_t,p) &\leq d\bigl(f(x_t),p\bigr) \leq d\bigl(f(x_t),f(p)\bigr) + d\bigl(f(p),p\bigr) \\ &\leq \alpha d(x_t,p) + d\bigl(f(p),p\bigr). \end{aligned}$$

This implies that

$$d(x_t,p) \leq \frac{1}{1-\alpha} d\bigl(f(p),p\bigr).$$

Hence $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$. We get that

$$\begin{aligned} \lim_{t\to 0} d(x_t, Tx_t) &= \lim_{t\to 0} d\bigl(tf(x_t) \oplus (1-t)Tx_t, Tx_t\bigr) \\ &\leq \lim_{t\to 0} \bigl[td\bigl(f(x_t), Tx_t\bigr) + (1-t)d(Tx_t, Tx_t)\bigr] \\ &\leq \lim_{t\to 0} td\bigl(f(x_t), Tx_t\bigr) = 0. \end{aligned}$$

Assume that $\{t_n\} \subset (0,1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T).$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, 2.6, we may assume that $\{x_n\}$ Δ-converges to a point $\tilde{x}$ and $\tilde{x} \in F(T)$. It follows from Lemma 2.10 (i) that

$$\begin{aligned} d^2(x_n, \tilde{x}) &= \langle \overrightarrow{x_n\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\ &\leq \alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle + (1-\alpha_n)\langle \overrightarrow{Tx_n\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\ &\leq \alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle + (1-\alpha_n) d(Tx_n, \tilde{x}) d(x_n, \tilde{x}) \\ &\leq \alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle + (1-\alpha_n) d^2(x_n, \tilde{x}). \end{aligned}$$

It follows that

$$\begin{aligned} d^2(x_n, \tilde{x}) &\leq \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\ &= \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_n\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\ &\leq d\big(f(x_n), f(\tilde{x})\big) d(x_n, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\ &\leq \alpha d^2(x_n, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle, \end{aligned}$$

and thus

$$d^2(x_n, \tilde{x}) \leq \frac{1}{1-\alpha} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle. \tag{13}$$

Since $\{x_n\}$ Δ-converges to $\tilde{x}$, by Lemma 2.7, we have

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0. \tag{14}$$

It follows from (13) that $\{x_n\}$ converges strongly to $\tilde{x}$.

Next, we show that $\tilde{x}$ solves the variational inequality (12). Applying Lemma 2.3, for any $q \in F(T)$,

$$\begin{aligned} d^2(x_t, q) &= d^2\big(tf(x_t) \oplus (1-t)Tx_t, q\big) \\ &\leq td^2\big(f(x_t), q\big) + (1-t)d^2(Tx_t, q) - t(1-t)d^2\big(f(x_t), Tx_t\big) \\ &\leq td^2\big(f(x_t), q\big) + (1-t)d^2(x_t, q) - t(1-t)d^2\big(f(x_t), Tx_t\big). \end{aligned}$$

It implies that

$$d^2(x_t, q) \leq d^2\big(f(x_t), q\big) - (1-t)d^2\big(f(x_t), Tx_t\big).$$

Taking the limit through $t = t_n \to 0$, we can get that

$$d^2(\tilde{x}, q) \leq d^2\big(f(\tilde{x}), q\big) - d^2\big(f(\tilde{x}), \tilde{x}\big).$$

Hence

$$0 \leq \frac{1}{2}\big[d^2(\tilde{x},\bar{x}) + d^2\big(f(\tilde{x}),q\big) - d^2(\tilde{x},q) - d^2\big(f(\tilde{x}),\tilde{x}\big)\big] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}}\rangle, \quad \forall q \in F(T).$$

That is, $\tilde{x}$ solves the inequality (12).

Finally, we show that the entire net $\{x_t\}$ converges to $\tilde{x}$, assume $x_{s_n} \to \hat{x}$, where $s_n \to 0$. By the same argument, we get that $\hat{x} \in F(T)$ and solves the variational inequality (12), *i.e.*,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \leq 0, \tag{15}$$

and

$$\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}}\rangle \leq 0. \tag{16}$$

Adding up (15) and (16), we get that

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\
&= \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\
&= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\
&\geq \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - d\big(f(\hat{x}),f(\tilde{x})\big)d(\hat{x},\tilde{x}) \\
&\geq d^2(\tilde{x},\hat{x}) - \alpha d(\hat{x},\tilde{x})d(\hat{x},\tilde{x}) \\
&= d^2(\tilde{x},\hat{x}) - \alpha d^2(\hat{x},\tilde{x}) \\
&\geq (1-\alpha)d^2(\tilde{x},\hat{x}).
\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x},\hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the net x_t converges strongly to $\tilde{x}$ which is the unique solution to the variational inequality (12). This completes the proof. □

Remark 3.2 We give the different proof of [16, Theorem 2.2]. In fact, the property $\mathcal{P}$ imposed on a CAT(0) space X is removed.

If $f \equiv u$, then the following result can be obtained directly from Theorem 3.1.

Corollary 3.3 [15, Lemma 2.2] *Let C be a closed convex subset of a complete CAT(0) space X, and let $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0,1]$, let u be fixed and $\{x_t\}$ be given by*

$$x_t = tu \oplus (1-t)Tx_t. \tag{17}$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ which is nearest to u which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}}\rangle \geq 0, \quad x \in F(T). \tag{18}$$

Theorem 3.4 *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) T x_n, \quad \forall n \geq 0, \tag{19}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ which is equivalent to the variational inequality (12).

Proof We first show that the sequence $\{x_n\}$ is bounded. For any $p \in F(T)$, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d\big(\alpha_n f(x_n) \oplus (1 - \alpha_n) T x_n, p\big) \\ &\leq \alpha_n d\big(f(x_n), p\big) + (1 - \alpha_n) d(T x_n, p) \\ &\leq \alpha_n \big(d\big(f(x_n), f(p)\big) + d\big(f(p), p\big)\big) + (1 - \alpha_n) d(T x_n, p) \\ &\leq \max\left\{ d(x_n, p), \frac{1}{1 - \alpha} d\big(f(p), p\big) \right\}. \end{aligned}$$

By induction, we have

$$d(x_n, p) \leq \max\left\{ d(x_0, p), \frac{1}{1 - \alpha} d\big(f(p), p\big) \right\},$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{T x_n\}$ and $\{f(x_n)\}$. Next, we claim that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. To this end, we observe that

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(\alpha_n f(x_n) \oplus (1 - \alpha_n) T x_n, \alpha_{n-1} f(x_{n-1}) \oplus (1 - \alpha_{n-1}) T x_{n-1}\big) \\ &\leq d\big(\alpha_n f(x_n) \oplus (1 - \alpha_n) T x_n, \alpha_n f(x_n) \oplus (1 - \alpha_n) T x_{n-1}\big) \\ &\quad + d\big(\alpha_n f(x_n) \oplus (1 - \alpha_n) T x_{n-1}, \alpha_n f(x_{n-1}) \oplus (1 - \alpha_n) T x_{n-1}\big) \\ &\quad + d\big(\alpha_n f(x_{n-1}) \oplus (1 - \alpha_n) T x_{n-1}, \alpha_{n-1} f(x_{n-1}) \oplus (1 - \alpha_{n-1}) T x_{n-1}\big) \\ &\leq (1 - \alpha_n) d(T x_n, T x_{n-1}) + \alpha_n d\big(f(x_n), f(x_{n-1})\big) \\ &\quad + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), T x_{n-1}\big) \\ &\leq (1 - \alpha_n) d(x_n, x_{n-1}) + \alpha_n d\big(f(x_n), f(x_{n-1})\big) + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), T x_{n-1}\big) \\ &\leq (1 - \alpha_n) d(x_n, x_{n-1}) + \alpha_n \alpha d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), T x_{n-1}\big) \\ &= \big(1 - \alpha_n(1 - \alpha)\big) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), T x_{n-1}\big). \end{aligned}$$

By the conditions (ii) and (iii) and Lemma 2.8, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{20}$$

It follows from (20) and condition (i) that

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x_n, x_{n+1}) + d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, Tx_n\big) \\ &\le d(x_n, x_{n+1}) + \alpha_n d\big(f(x_n), Tx_n\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{21}$$

Let $\{x_t\}$ be a net in C such that

$$x_t = tf(x_t) \oplus (1-t)Tx_t.$$

By Theorem 3.1, we have that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x} \in F(T)$, which solves the variational inequality (12). Now, we claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \le 0.$$

It follows from Lemma 2.10 (i) that

$$\begin{aligned} d^2(x_t, x_n) &= \langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\ &\le t\langle \overrightarrow{f(x_t)x_n}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{Tx_t x_n}, \overrightarrow{x_t x_n} \rangle \\ &= t\langle \overrightarrow{f(x_t)f(\tilde{x})}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{\tilde{x}x_t}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\ &\quad + (1-t)\langle \overrightarrow{Tx_t Tx_n}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{Tx_n x_n}, \overrightarrow{x_t x_n} \rangle \\ &\le t\alpha d(x_t, \tilde{x})d(x_t, x_n) + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + td(\tilde{x}, x_t)d(x_t, x_n) + td^2(x_t, x_n) \\ &\quad + (1-t)d^2(x_t, x_n) + (1-t)d(Tx_n, x_n)d(x_t, x_n) \\ &\le t\alpha d(x_t, \tilde{x})M + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + td(\tilde{x}, x_t)M + td^2(x_t, x_n) \\ &\quad + (1-t)d^2(x_t, x_n) + (1-t)d(Tx_n, x_n)M \\ &\le d^2(x_t, x_n) + t\alpha d(x_t, \tilde{x})M + td(\tilde{x}, x_t)M + d(Tx_n, x_n)M + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle, \end{aligned}$$

where $M \ge \sup_{m,n\ge 1}\{d(x_t, x_n)\}$. This implies that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \le (1+\alpha)d(x_t, \tilde{x})M + \frac{d(Tx_n, x_n)}{t}M. \tag{22}$$

Taking the limit as $n \to \infty$ first and then $t \to 0$, the inequality (22) yields

$$\limsup_{t\to 0}\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \le 0.$$

Since $x_t \to \tilde{x}$ as $t \to 0$ and by the continuity of a metric distance d, we have, for any fixed $n \ge 0$,

$$\begin{aligned} &\lim_{t\to 0} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \\ &\quad = \lim_{t\to 0} \frac{1}{2}\big[d^2\big(f(\tilde{x}), x_t\big) + d^2(\tilde{x}, x_n) - d^2\big(f(\tilde{x}), x_n\big) - d^2(\tilde{x}, x_t)\big] \end{aligned}$$

$$= \frac{1}{2}\big[d^2\big(f(\tilde{x}),\tilde{x}\big) + d^2(\tilde{x},x_n) - d^2\big(f(\tilde{x}),x_n\big) - d^2(\tilde{x},\tilde{x})\big]$$
$$= \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle.$$

It implies that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle < \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle + \varepsilon, \quad \forall t \in (0,\delta). \tag{23}$$

Thus, by the upper limit as $n \to \infty$ first and then $t \to 0$, the inequality in (23), we get that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \le \varepsilon.$$

Since ε is arbitrary, it follows that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \le 0.$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $y_n = \alpha_n\tilde{x} \oplus (1-\alpha_n)Tx_n$. It follows from Lemma 2.9 and Lemma 2.10 (i), (ii) that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) &\le d^2(y_n,\tilde{x}) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\le \big(\alpha_n d(\tilde{x},\tilde{x}) + (1-\alpha_n)d(Tx_n,\tilde{x})\big)^2 \\
&\quad + 2\big[\alpha_n \langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)\langle \overrightarrow{Tx_n y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)Tx_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\quad + (1-\alpha_n)\alpha_n \langle \overrightarrow{Tx_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)(1-\alpha_n)\langle \overrightarrow{Tx_n Tx_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\big] \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)Tx_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\quad + (1-\alpha_n)\alpha_n \langle \overrightarrow{Tx_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + (1-\alpha_n)^2 d(Tx_n, Tx_n)d(x_{n+1}\tilde{x})\big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle\big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n\alpha\big(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\big) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) &\le \frac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
&\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n^2 M,
\end{aligned}$$

where $M \ge \sup_{n\ge 0}\{d^2(x_n,\tilde{x})\}$. It then follows that

$$d^2(x_{n+1},\tilde{x}) \le \big(1-\alpha_n'\big)d^2(x_n,\tilde{x}) + \alpha_n'\beta_n',$$

where

$$\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad \text{and} \quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}M + \frac{1}{(1-\alpha)}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.$$

Applying Lemma 2.8, we can conclude that $x_n \to \tilde{x}$. This completes the proof. □

Remark 3.5 We give the different proof of [16, Theorem 2.3]. In fact, the property $\mathcal{P}$ imposed on a CAT(0) space X is removed.

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 3.4.

Corollary 3.6 [15, Theorem 2.3] *Let C be a closed convex subset of a complete CAT*(0) *space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u, x_0 \in C$ be arbitrarily chosen and $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{24}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ which is nearest to u which is equivalent to the following variational inequality (18).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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