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GNM ordered variational inequality system with ordered Lipschitz continuous mappings in an ordered Banach space

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Full list of author information is available at the end of the article**Abstract**

The main purpose of this paper is to introduce and study a new class of generalized nonlinear mixed ordered variational inequalities systems with ordered Lipschitz continuous mappings in ordered Banach spaces. Then, applying the matrix analysis and the vector-valued mapping fixed point analysis method, an existence theorem of solutions for this kind of the system is established. Furthermore, based on the existence theorem and the new ordered B -restricted-accretive mappings, a general algorithm for solving the systems is introduced and applied to the approximation solvability of the systems on hand. The obtained results seem to be general in nature.

MSC: 49J40; 47H06**Keywords:** general nonlinear mixed ordered variational inequality system; ordered Lipschitz continuous mappings; B -restricted-accretive mappings; iterative algorithm; convergence; ordered Banach space

1 Introduction

Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . Let $F_{ij}, g, f : X \times X \rightarrow X$ be single-valued nonlinear ordered compression mappings, and let F_{ij} be a Lipschitz continuous mapping ($i, j = 1, 2$), and for any $x, y \in X$,

$$\text{range}(f) \cap \text{dom}(F_{11}(\cdot, y)) \cap \text{range}(g) \cap \text{dom}(F_{22}(x, \cdot)) \neq \emptyset,$$

we consider the following problem:

For $u, v \in X$, find $x, y \in X$ such that

$$\begin{cases} u \leq F_{11}(f(x), y) + F_{12}(y, x), \\ v \leq F_{21}(x, y) \oplus F_{22}(x, g(y)), \end{cases} \quad (1.1)$$

which is called a generalized nonlinear mixed ordered variational inequality system (GNM ordered variational inequality system) with ordered Lipschitz continuous mappings in an ordered Banach space. Obviously, system (1.1) belongs to a new class of generalized nonlinear mixed ordered variational inequality systems with the \oplus calculation.

For a suitable choice of the mappings u, v, f, g, F_{ij} ($i, j = 1, 2$) and the space X , a number of known classes of ordered variational inequalities, which have been studied by the authors as special cases of system (1.1) in the Banach space (see [1, 2]).

Remark 1.1 Some special cases of system (1.1):

- (1) Let $F_{12}(\cdot, \cdot) = F_{21}(\cdot, \cdot) = F_{22}(\cdot, \cdot) = 0$ be zero operators, $u = v = \theta$ and $F_{11}(f(x), y) = A(x)$ for any $y \in X$, then system (1.1) becomes the following problem: Find $x \in X$ such that

$$\theta \leq A(f(x)), \tag{1.2}$$

which is called a generalized nonlinear ordered variational inequality (a generalized nonlinear ordered equation, as changed \geq to $=$) in an ordered Banach space (see [1]).

- (2) Let $F_{11}(\cdot, \cdot) = F_{12}(\cdot, \cdot) = 0$ be zero operators, $F_{21}(f(x), y) = A(x)$, $u = v = \theta$ and $F_{22}(x, g(y)) = F(x, g(x))$ for any $y = x$, then system (1.1) becomes the following problem: Find $x \in X$ such that

$$\theta \leq A(x) \oplus F(x; g(x)), \tag{1.3}$$

which is called a new class of general nonlinear ordered variational inequality (a general nonlinear ordered equation, as changed \geq to $=$) in an ordered Banach space (see [2]).

- (3) Let $F_{21}(\cdot, \cdot) = F_{22}(\cdot, \cdot) = 0$ be zero operators and $u = v = \theta$, then system (1.1) becomes the following problem: Find $x, y \in X$ such that

$$\theta \leq F_{11}(f(x), y) + F_{12}(y, x), \tag{1.4}$$

which is studied by many authors in a Banach space (see [3] *et al.*).

In recent years, though we have succeeded in the area of studies of variational inequality (inclusion) systems, yet, the studies of ordered variational inequality (inclusion) systems are beginning in very recent research works on an ordered Banach space (see [1, 2, 4–9]). From 1999 till present, some new and interesting problems for systems of variational inequalities (inclusions) have been introduced and studied in this field (see [1–32]).

Very recently, the approximation solution for general nonlinear ordered variational inequalities and ordered equations [1, 2] and a nonlinear ordered inclusion problem [8, 9] have been studied by Li in an ordered Banach space. For details, we refer the reader to [1–32] and the references therein.

2 Preliminaries

We need to recall the following concepts and results for solving system (1.1).

Definition 2.1 [21] Let X be a real ordered Banach space with a norm $\|\cdot\|$, a normal cone \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} , for $x, y \in X$, if $x \leq y$ (or $y \leq x$) holds, then x and y are said to be a comparison between each other (denoted by $x \propto y$ for $x \leq y$ and $y \leq x$).

Lemma 2.2 [1] *Let X be a real ordered Banach space with a norm $\| \cdot \|$, a normal cone \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} , for arbitrary $x, y \in X$, $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ express the least upper bound of the set $\{x, y\}$ and the greatest lower bound of the set $\{x, y\}$ on the partial ordered relation \leq , respectively. Suppose that $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist, some binary operators can be defined as follows:*

- (1) $x \vee y = \text{lub}\{x, y\}$;
- (2) $x \wedge y = \text{glb}\{x, y\}$;
- (3) $x \oplus y = (x - y) \vee (y - x)$.

\vee , \wedge , and \oplus are called OR, AND, and XOR operations, respectively. For arbitrary $x, y, w \in X$, the following relations hold:

- (1) $x \oplus y = y \oplus x$;
- (2) $x \oplus x = \theta$;
- (3) $\theta \leq x \oplus \theta$;
- (4) let λ be real, then $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
- (5) if x, y , and w can be compared with each other, then

$$(x \oplus y) \leq x \oplus w + w \oplus y;$$

- (6) let $(x + y) \vee (u + v)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then

$$(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u);$$

- (7) if x, y, z, w can be compared with each other, then

$$(x \wedge y) \oplus (z \wedge w) \leq ((x \oplus z) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus z));$$

- (8) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, if $x \propto \theta$.

Lemma 2.3 [5] *If $x \propto y$, then $\text{lub}\{x, y\}$, and $\text{glb}\{x, y\}$ exist, $x - y \propto y - x$, and $\theta \leq (x - y) \vee (y - x)$.*

Lemma 2.4 [5] *If for any natural number n , $x \propto y_n$, and $y_n \rightarrow y^*$ ($n \rightarrow \infty$), then $x \propto y^*$.*

Lemma 2.5 *Let X be a real ordered Banach space with a norm $\| \cdot \|$, a zero θ , a normal cone \mathbf{P} , a normal constant N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . $A : X \rightarrow X$ is comparative, then for $x, y \in X$, if $x \propto y$ [1], then*

- (1) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
- (2) $\|x \oplus y\| = \|x - y\| \leq N\|x \oplus y\|$,
- (3) $\lim_{x \rightarrow x_0} \|A(x) - A(x_0)\| = 0$ if and only if $\lim_{x \rightarrow x_0} A(x) \oplus A(x_0) = \theta$.

Proof Result (1) is obvious; (2) follows from (1), Definition 2.2 in [1], Lemma 2.7 in [2]; (3) follows from (1) and (2). This completes the proof. \square

Definition 2.6 Let X be a real ordered Banach space, and let $F : X \times X \rightarrow X$ be a mapping. The operator $F : X \times X \rightarrow X$ is said to be an ordered Lipschitz continuous with constants (μ, ν) if $x \propto y, u \propto v$, then $F(u, x) \propto F(v, y)$, and there exist constants $\mu, \nu > 0$ such that

$$F(u, x) \oplus F(v, y) \leq \mu(u \oplus v) + \nu(x \oplus y).$$

Definition 2.7 [2] Let X be a real ordered Banach space, let $B : X \rightarrow X$ be a mapping, and let I be an identity mapping on X . A mapping $A : X \rightarrow X$ is said to be a B -restricted-accretive mapping if A, B and $A \wedge B : x \in X \rightarrow A(x) \wedge B(x) \in X$ all are comparisons, and they are comparisons with each other, and there exist two constants $0 < \alpha_1, \alpha_2 \leq 1$ such that for arbitrary $x, y \in X$,

$$\begin{aligned} & (A(x) \wedge B(x) + I(x)) \oplus (A(y) \wedge B(y) + I(y)) \\ & \leq \alpha_1((A(x) \wedge B(x)) \oplus (A(y) \wedge B(y))) + \alpha_2(x \oplus y) \end{aligned}$$

holds, where I is an identity mapping on X .

Definition 2.8 Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . If $X \times X$ is a product Banach space with the normal $\|\cdot\|$ and an ordered relation \leq , and the following conditions are satisfied:

- (1) $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for any $(x, y) \in X \times X$;
- (2) $(x_1, y_1) \propto (x_2, y_2)$ if and only if $x_1 \propto x_2, y_1 \propto y_2$, and $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2, y_1 \leq y_2$ in X ;
- (3)

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2),$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2),$$

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 \oplus x_2, y_1 \oplus y_2).$$

Then $X \times X$ is called an ordered product Banach space.

Definition 2.9 Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . Let $X \times X$ be an ordered product Banach space. For a vector-valued mapping $\vec{G} = (G_1, G_2)$ (or $(G_1, G_2)^T$) : $X \times X \rightarrow X \times X$ in $X \times X$, if there exists a point $(x^*, y^*) \in X \times X$ such that

$$\vec{G}(x^*, y^*) = (G_1, G_2)(x^*, y^*) = (x^*, y^*),$$

then (x^*, y^*) is called a fixed point of vector-valued mapping \vec{G} in ordered product Banach space.

The following results are obvious.

Lemma 2.10 Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . Let $X \times X$ be an ordered product Banach space. For sequences $\{x_n\}$ and $\{y_n\}$ in X , in $X \times X$,

$$(x_n, y_n) \rightarrow (x^*, y^*) \text{ if and only if } x_n \rightarrow x^* \text{ and } y_n \rightarrow y^* \text{ as } n \rightarrow \infty. \quad (2.1)$$

Lemma 2.11 Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} . Let

$X \times X$ be an ordered product Banach space. Let $\vec{G} = (G_1, G_2)$ (or $(G_1, G_2)^T$): $X \times X \rightarrow X \times X$ be a vector-valued mapping in $X \times X$ if for any $(x_i, y_i) \in X \times X (i = 1, 2)$, $(x_1, y_1) \propto (x_2, y_2)$, and there exist a constance $1 > \delta > 0$ such that

$$\|(G_1, G_2)(x_1, y_1) \oplus (G_1, G_2)(x_2, y_2)\| \leq \delta \|(x_1, y_1) \oplus (x_2, y_2)\|, \tag{2.2}$$

then (G_1, G_2) has a fixed point in $X \times X$.

Proof This directly follows from Lemma 2.2, Lemma 2.5(2) and the contraction mapping principle. \square

3 Approximation solution for GNM system (1.1)

In this section, we will change from the solution of system (1.1) to finding a fixed point for a vector-valued mapping, and by using the vector-valued mapping fixed point analysis method, show the convergence of the approximation sequences of the solution for system (1.1) in an ordered product Banach space.

Lemma 3.1 *Let X be a real ordered Banach space with a norm $\| \cdot \|$, a zero θ , a normal cone \mathbf{P} , a normal constance N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} , and let $X \times X$ be an ordered product Banach space. If $g, f, B_i : X \rightarrow X$ are ordered compressions, $F_{ij} : X \times X \rightarrow X$ is order (μ_{ij}, ν_{ij}) -Lipschitz continuous, and let g, f, B_1, B_2 and F_{ij} be comparison mappings with each other (where $i, j = 1, 2$). Then system (1.1) has a solution (x^*, y^*) if and only if there exist two ordered compressions B_1 and B_2 such that the vector-valued mapping $\vec{G} = (G_1(x, y), G_2(x, y)) : X \times X \rightarrow X \times X$,*

$$\begin{aligned} G_1(x, y) &= (F_{11}(f(x), y) + F_{12}(y, x) - u) \wedge B_1(x) + I(x) \quad \text{and} \\ G_2(x, y) &= (F_{21}(x, y) \oplus F_{22}(x, g(y)) - v) \wedge B_2(y) + I(y), \end{aligned} \tag{3.1}$$

has the fixed point (x^*, y^*) in an ordered Banach space $X \times X$.

Proof Let (x^*, y^*) be a fixed point of the vector-valued mapping (3.1), then, obviously, (x^*, y^*) is a solution of system (1.1).

On the other hand, choosing

$$B_1(x) = \begin{cases} \theta, & \text{if } \theta \leq F_{11}(f(x), y) + F_{12}(y, x) - u, \\ \zeta_1 x, & \text{otherwise} \end{cases}$$

and

$$B_2(y) = \begin{cases} \theta, & \text{if } \theta \leq F_{21}(x, y) \oplus F_{22}(x, g(y)) - v, \\ \zeta_2 y, & \text{otherwise,} \end{cases}$$

where $1 > \zeta_1, \zeta_2 > 0$, if (x^*, y^*) is a solution of system (1.1), then by using [1, 2],

$$\begin{aligned} (F_{11}(f(x^*), y^*) + F_{12}(y^*, x^*) - u) \wedge B_1(x^*) + I(x^*) &= x^*, \\ (F_{21}(x^*, y^*) \oplus F_{22}(x^*, g(y^*)) - v) \wedge B_2(y^*) + I(y^*) &= y^* \end{aligned}$$

hold. Therefore, (x^*, y^*) is a fixed point of the vector-valued mapping (3.1), where the mappings B_1 and B_2 are ordered compressions [2]. This completes the proof. \square

Theorem 3.2 *Let X be a real ordered Banach space with a norm $\|\cdot\|$, a zero θ , a normal cone \mathbf{P} , a normal constant N of \mathbf{P} and a partial ordered relation \leq defined by the cone \mathbf{P} , and let $X \times X$ be an ordered product Banach space. Let g and f be ordered compressions with respect to γ_g and γ_f , respectively; let $B_i : X \rightarrow X$ be an ordered compression mapping with ζ_i , let $F_{ij} : X \times X \rightarrow X$ be an ordered (μ_{ij}, ν_{ij}) -Lipschitz continuous ($i, j = 1, 2$), and let g, f, B_1, B_2 and F_{ij} ($i, j = 1, 2$) be comparison mappings with each other. If $F_{11} + F_{12} - u$ is a B_1 -restricted-accretive mapping with (α_1, α_2) , and $F_{21} \oplus F_{22} - v$ is a B_2 -restricted-accretive mapping with (β_1, β_2) , and*

$$N \max \{ \alpha_1(\mu_{11}\gamma_f + \nu_{12} + \zeta_1) + \alpha_2, \alpha_1(\nu_{11} + \mu_{12}), \beta_1(\mu_{21} \oplus \mu_{22}), (\beta_1\nu_{21} \oplus \nu_{22}\gamma_g) + \zeta_2 + \beta_2 \} < 1 \tag{3.2}$$

holds, then for the general nonlinear mixed ordered variational inequality system (1.1), there exists a solution (x^, y^*) .*

Proof Let X be a real ordered Banach space, and let $X \times X$ be an ordered product Banach space. Setting

$$\begin{aligned} G_1(x, y) &= (F_{11}(f(x), y) + F_{12}(y, x) - u) \wedge B_1(x) + I(x), \\ G_2(x, y) &= (F_{21}(x, y) \oplus F_{22}(x, g(y)) - v) \wedge B_2(y) + I(y). \end{aligned} \tag{3.3}$$

Since $F_{11} + F_{12} - u$ is a B_1 -restricted-accretive mapping with $(\alpha_{11}, \alpha_{12})$, $F_{21} \oplus F_{22} - v$ is a B_2 -restricted-accretive mapping with (β_{11}, β_{12}) , and F_{11} and F_{12} are ordered (μ_{11}, ν_{11}) -Lipschitz continuous and (μ_{12}, ν_{12}) -Lipschitz continuous, respectively, then for any given $x_i, y_j \in X$ and $x_i \times y_j$ ($i, j = 1, 2$), by Lemma 2.2(6), (7), Definition 2.7 and [1, 2], we can obtain the following inequalities:

$$\begin{aligned} \theta &\leq G_1(x_1, y_1) \oplus G_1(x_2, y_2) \\ &= ((F_{11}(f(x_1), y_1) + F_{12}(y_1, x_1) - u) \wedge B_1(x_1) + I(x_1)) \\ &\quad \oplus ((F_{11}(f(x_2), y_2) + F_{12}(y_2, x_2) - u) \wedge B_1(x_2) + I(x_2))) \\ &\leq \alpha_1(((F_{11}(f(x_1), y_1) + F_{12}(y_1, x_1) - u) \wedge B_1(x_1)) \\ &\quad \oplus ((F_{11}(f(x_2), y_2) + F_{12}(y_2, x_2) - u) \wedge B_1(x_2)) + \alpha_2(x_1 \oplus x_2)) \\ &\leq \alpha_{11}(((F_{11}(f(x_1), y_1) + F_{12}(y_1, x_1) - u) \\ &\quad \oplus (F_{11}(f(x_2), y_2) + F_{12}(y_2, x_2) - u) \vee (B_1(x_1) \oplus B_1(x_2)))) + \alpha_2(x_1 \oplus x_2)) \\ &\leq \alpha_1(((F_{11}(f(x_1), y_1) \oplus F_{11}(f(x_2), y_2)) \\ &\quad + ((F_{12}(y_1, x_1) - u) \oplus (F_{12}(y_2, x_2) - u)) \vee \zeta_1(x_1 \oplus x_2))) + \alpha_2(x_1 \oplus x_2)) \\ &\leq \alpha_1(((F_{11}(f(x_1), y_1) \oplus F_{11}(f(x_2), y_2)) \\ &\quad + (F_{12}(y_1, x_1) \oplus F_{12}(y_2, x_2))) \vee \zeta_1(x_1 \oplus x_2)) + \alpha_2(x_1 \oplus x_2)) \end{aligned}$$

$$\begin{aligned}
 &\leq (\alpha_1(\mu_{11}\gamma_f(x_1 \oplus x_2) + \nu_{11}(y_1 \oplus y_2)) \\
 &\quad + \alpha_1(\mu_{12}(y_1 \oplus y_2) + \nu_{12}(x_1 \oplus x_2))) \vee \zeta_1(x_1 \oplus x_2) + \alpha_2(x_1 \oplus x_2) \\
 &\leq (\alpha_1((\mu_{11}\gamma_f + \nu_{12}) + \zeta_1) + \alpha_2)(x_1 \oplus x_2) + \alpha_1(\nu_{11} + \mu_{12})(y_1 \oplus y_2)
 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
 \theta &\leq G_2(x_1, y_1) \oplus G_2(x_2, y_2) \\
 &= ((F_{21}(x_1, y_1) \oplus F_{22}(x_1, g(y_1)) - \nu) \wedge B_2(y_1) + I(y_1)) \\
 &\quad \oplus ((F_{21}(x_2, y_2) \oplus F_{22}(x_2, g(y_2)) - \nu) \wedge B_2(y_2) + I(y_2)) \\
 &\leq \beta_1(((F_{21}(x_1, y_1) \oplus F_{21}(x_2, y_2)) - \nu) \wedge B_2(y_1)) \\
 &\quad \oplus ((F_{22}(x_1, g(y_1)) \oplus F_{22}(x_2, g(y_2))) - \nu) \wedge B_2(y_2)) + \beta_2(y_1 \oplus y_2) \\
 &\leq \beta_1(((F_{21}(x_1, y_1) \oplus F_{21}(x_2, y_2))) \\
 &\quad \oplus (F_{22}(x_1, g(y_1)) \oplus F_{22}(x_2, g(y_2)))) \vee (B_2(y_1) \oplus B_2(y_2))) + \beta_2(y_1 \oplus y_2) \\
 &\leq \beta_1(((F_{21}(x_1, y_1) \oplus F_{21}(x_2, y_2))) \\
 &\quad \oplus (F_{22}(x_1, g(y_1)) \oplus F_{22}(x_2, g(y_2)))) \vee \zeta_2(y_1 \oplus y_2)) + \beta_2(y_1 \oplus y_2) \\
 &\leq \beta_1(((\mu_{21}(x_1 \oplus x_2) + \nu_{21}(y_1 \oplus y_2)) \\
 &\quad \oplus (\mu_{22}(x_1 \oplus x_2) + \nu_{22}\gamma_g(y_1 \oplus y_2))) \vee \zeta_2(y_1 \oplus y_2)) + \beta_2(y_1 \oplus y_2) \\
 &\leq \beta_1(((\mu_{21}(x_1 \oplus x_2) \oplus \mu_{22}(x_1 \oplus x_2)) \\
 &\quad + (\nu_{21}(y_1 \oplus y_2) \oplus \nu_{22}\gamma_g(y_1 \oplus y_2))) \vee \zeta_2(y_1 \oplus y_2)) + \beta_2(y_1 \oplus y_2) \\
 &\leq \beta_1(\mu_{21} \oplus \mu_{22})(x_1 \oplus x_2) + ((\beta_1\nu_{21} \oplus \nu_{22}\gamma_g) + \zeta_2 + \beta_2)(y_1 \oplus y_2);
 \end{aligned} \tag{3.5}$$

and

$$(G_1, G_2)(x_1, y_1) \oplus (G_1, G_2)(x_2, y_2) \leq \Psi(x_1, y_1) \oplus (x_2, y_2), \tag{3.6}$$

where

$$\Psi = \begin{pmatrix} \alpha_1(\mu_{11}\gamma_f + \nu_{12} + \zeta_1) + \alpha_2 & \alpha_1(\nu_{11} + \mu_{12}) \\ \beta_1(\mu_{21} \oplus \mu_{22}) & (\beta_1\nu_{21} \oplus \nu_{22}\gamma_g) + \zeta_2 + \beta_2 \end{pmatrix}. \tag{3.7}$$

Further, by [27] and Definition 2.2 in [1], we have

$$\|(G_1, G_2)(x_1, y_1) \oplus (G_1, G_2)(x_2, y_2)\| \leq N\|\Psi\| \|(x_1, y_1) \oplus (x_2, y_2)\|, \tag{3.8}$$

where

$$\begin{aligned}
 \|\Psi\| &= \max\{\alpha_1(\mu_{11}\gamma_f + \nu_{12} + \zeta_1) + \alpha_2, \alpha_1(\nu_{11} + \mu_{12}), \\
 &\quad \beta_1(\mu_{21} \oplus \mu_{22}), (\beta_1\nu_{21} \oplus \nu_{22}\gamma_g) + \zeta_2 + \beta_2\},
 \end{aligned}$$

and N is a normal constant of \mathbf{P} .

It follows from (3.8) and the assumption condition (3.2) that $0 < N\|\Psi\| < 1$, and hence the vector-valued mapping

$$(G_1, G_2)^T = \left((F_{11}(f, \cdot) + F_{12}(\cdot, \cdot) - u) \wedge B_1 + I(\cdot), (F_{21}(\cdot, \cdot) \oplus F_{22}(\cdot, g)) - v \vee B_2 + I(\cdot) \right)^T$$

has a fixed point (x^*, y^*) for Lemma 2.11, in an ordered Banach space $X \times X$, which is a solution for system (1.1) by Lemma 3.1. This completes the proof. \square

Theorem 3.3 *Let the assumption conditions in Theorem 3.2 and (3.2) hold, that is,*

$$N \max \left\{ \alpha_1(\mu_{11}\gamma_f + \nu_{12}) + \zeta_1 + \alpha_2, \alpha_1(\nu_{11} + \mu_{12}), \beta_1(\mu_{21} \oplus \mu_{22}), (\beta_1\nu_{21} \oplus \nu_{22}\gamma_g) + \zeta_2 + \beta_2 \right\} < 1. \tag{3.9}$$

Then the iterative sequence $\{(x_n, y_n)\}$ generated by the following algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \rho)x_n + \rho \left((F_{11}(f(x_n), y_n) + F_{12}(y_n, x_n)) \wedge B_1(x_n) + I(x_n) \right), \\ y_{n+1} &= (1 - \sigma)y_n + \sigma \left((F_{21}(x_n, y_n) \oplus F_{22}(x_n, g(y_n))) \wedge B_2(y_n) + I(y_n) \right) \end{aligned} \tag{3.10}$$

for any $x_0, y_0 \in X$, $x_0 \propto y_0$, $(x_0, y_0) \propto (x_1, y_1)$ and $1 > \rho, \sigma > 0$, converges strongly to (x^, y^*) , which is a solution of system (1.1).*

Proof Let the assumption conditions in Theorem 3.2 hold. For any given $x_0, y_0 \in X$ and $x_0 \propto y_0$, $(x_0, y_0) \propto (x_1, y_1)$, setting

$$\begin{aligned} G_1(x, y) &= (F_{11}(f(x), y) + F_{12}(y, x) - u) \wedge B_1(x) + I(x), \\ G_2(x, y) &= (F_{21}(x, y) \oplus F_{22}(x, g(y)) - v) \wedge B_2(y) + I(y), \end{aligned} \tag{3.11}$$

then for any $1 > \rho, \sigma > 0$, by algorithm (3.10), (3.4) and (3.5), we have

$$\begin{aligned} \theta &\leq x_{n+1} \oplus x_n \\ &= [(1 - \rho)x_n + \rho G_1(x_n, y_n)] \oplus [(1 - \rho)x_{n-1} + \rho G_1(x_{n-1}, y_{n-1})] \\ &\leq (1 - \rho)(x_{n-1} \oplus x_n) + \rho(G_1(x_n, y_n) \oplus G_1(x_{n-1}, y_{n-1})) \\ &\leq (1 - \rho)(x_{n-1} \oplus x_n) + \rho(G_1(x_n, y_n) \oplus G_1(x_{n-1}, y_{n-1})) \\ &\leq [(1 - \rho) + \rho(\alpha_1(\mu_{11}\gamma_f + \nu_{12} + \zeta_1) + \alpha_2)](x_{n-1} \oplus x_n) \\ &\quad + \rho\alpha_1(\nu_{11} + \mu_{12})(y_{n-1} \oplus y_n) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \theta &\leq y_{n+1} \oplus y_n \\ &= [(1 - \sigma)y_n + \sigma G_2(x_n, y_n)] \oplus [(1 - \sigma)y_{n-1} + \sigma G_2(x_{n-1}, y_{n-1})] \\ &\leq (1 - \sigma)(y_{n-1} \oplus y_n) + \sigma(G_2(x_n, y_n) \oplus G_2(x_{n-1}, y_{n-1})) \\ &\leq (1 - \sigma)(y_{n-1} \oplus y_n) \end{aligned}$$

$$\begin{aligned}
 & + \sigma (\beta_1(\mu_{21} \oplus \mu_{22})(x_{n-1} \oplus x_n) + ((\beta_1 v_{21} \oplus v_{22} \gamma_g) + \zeta_2 + \beta_2)(y_{n-1} \oplus y_n)) \\
 \leq & [(1 - \sigma) + \sigma ((\beta_1 v_{21} \oplus v_{22} \gamma_g) + \zeta_2 + \beta_2)](y_{n-1} \oplus y_n) \\
 & + \sigma \beta_1(\mu_{21} \oplus \mu_{22})(x_{n-1} \oplus x_n) \\
 \leq & \sigma \beta_1(\mu_{21} \oplus \mu_{22})(x_{n-1} \oplus x_n) \\
 & + [(1 - \sigma) + \sigma ((\beta_1 v_{21} \oplus v_{22} \gamma_g) + \zeta_2 + \beta_2)](y_{n-1} \oplus y_n), \tag{3.13}
 \end{aligned}$$

combining (3.12) and (3.13), and by Definition 2.8, we have

$$(x_{n+1}, y_{n+1}) \oplus (x_n, y_n) \leq \Sigma(x_n, y_n) \oplus (x_{n-1}, y_{n-1}),$$

where

$$\begin{aligned}
 \Sigma = & \begin{pmatrix} 1 - \rho & 0 \\ 0 & 1 - \sigma \end{pmatrix} \\
 & + \begin{pmatrix} \rho(\alpha_1(\mu_{11} \gamma_f + v_{12} + \zeta_1) + \alpha_2) & \rho \alpha_1(v_{11} + \mu_{12}) \\ \sigma \beta_1(\mu_{21} \oplus \mu_{22}) & \sigma((\beta_1 v_{21} \oplus v_{22} \gamma_g) + \zeta_2 + \beta_2) \end{pmatrix}.
 \end{aligned}$$

Since (3.2) holds, that is,

$$\begin{aligned}
 N \max \{ & \alpha_1(\mu_{11} \gamma_f + v_{12} + \zeta_1) + \alpha_2, \alpha_1(v_{11} + \mu_{12}), \\
 & \beta_1(\mu_{21} \oplus \mu_{22}), (\beta_1 v_{21} \oplus v_{22} \gamma_g) + \zeta_2 + \beta_2 \} < 1,
 \end{aligned}$$

the inequality $N \|\Sigma\| < 1$ is true. It follows that $(x_n, y_n)^T \rightarrow (x^*, y^*)^T$ strongly from Lemma 2.11.

Since g, f, B_1, B_2 and F_{ij} ($i, j = 1, 2$) are ordered compressions, and they are comparisons of each other, so that

$$\begin{aligned}
 x^* & = (1 - \rho)x^* + \rho((F_{11}(f(x^*), y^*) + F_{12}(y^*, x^*) - u) \wedge B_1(x^*) + I(x^*)), \\
 y^* & = (1 - \sigma)y^* + \sigma(((F_{21}(x^*, y^*) \oplus F_{22}(x^*, g(y^*))) - v) \wedge B_2(y^*) + I(y^*))
 \end{aligned}$$

holds. Therefore, (x^*, y^*) is a fixed point of the vector-valued mapping

$$((F_{11}(f(x), y) + F_{12}(y, x) - u) \wedge B_1(x) + I(x), (F_{21}(x, y) \oplus F_{22}(x, g(y)) - v) \wedge B_2(y) + I(y)).$$

By using Lemma 3.1, (x^*, y^*) is a solution of system (1.1). This completes the proof. \square

Remark 3.4 For a suitable choice of the mappings g, f, B_1, B_2 and F_{ij} ($i, j = 1, 2$), we can obtain several known results [1] and [2] as special cases of Theorem 3.2, 3.3.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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