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On a more accurate half-discrete Hardy-Hilbert-type inequality related to the kernel of exponential function

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Abstract

By applying the weight functions, the technique of real analysis and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of exponential function with the best possible constant factor expressed by the gamma function is given. The more accurate equivalent forms, the operator expressions with the norm, the reverses, and some particular cases are considered.

MSC: 26D15

Keywords: Hardy-Hilbert-type inequality; weight function; equivalent form; reverse; operator

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = (\int_0^\infty f^p(x)\,dx)^{\frac{1}{p}} > 0$, $\|g\|_q > 0$, and we have the following Hardy-Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, then we have the following discrete analogy of (1) with the same best possible constant $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (*cf.* [1–5]).

If $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N} = \{1, 2, \ldots\})$,

$$U_m := \sum_{i=1}^m \mu_i, \qquad V_n := \sum_{j=1}^n \upsilon_j (m, n \in \mathbf{N}), \tag{3}$$

then we have the following Hardy-Hilbert-type inequality (*cf.* [1], Theorem 321, replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i = \upsilon_j = 1$ ($i, j \in \mathbf{N}$), inequality (4) reduces to (2).

Note The authors of [1] did not prove that (4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] gave an extension of (1) with the kernel $\frac{1}{(x+y)^\lambda}$ for $p = q = 2$. Following [6], Yang [5] gave some extensions of (1) and (2) as follows:

If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt \in \mathbf{R}_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{f; \|f\|_{p,\phi} := \left(\int_0^\infty \phi(x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then we have

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{5}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ keeps a finite value and $k_\lambda(x,y)x^{\lambda_1-1}$ ($k_\lambda(x,y)y^{\lambda_2-1}$) is decreasing with respect to $x > 0$ ($y > 0$), then, for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{a; \|a\|_{p,\phi} := \left(\sum_{n=1}^{\infty}\phi(n)|a_n|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

In 2015, by adding some conditions, Yang [7] gave an extension of (4) as follows:

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(U_m+V_n)^\lambda}\\ &\quad< B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}},\end{aligned} \tag{7}$$

where the constant $B(\lambda_1, \lambda_2)$ is still the best possible.

Some other results including multidimensional Hilbert-type inequalities are provided by [8–30].

About the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [31] gave a result with the kernel $\frac{1}{(1+nx)^{\lambda}}$ by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [32] gave the following half-discrete Hardy-Hilbert inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^{\infty} f(x)\left[\sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}}\right] dx < B(\lambda_1, \lambda_2)\|f\|_{p,\phi}\|a\|_{q,\psi}, \tag{8}$$

where $\lambda_1 > 0$, $0 < \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$. Zhong *et al.* ([17, 33, 34]) investigated several half-discrete Hilbert-type inequalities with particular kernels. Applying weight functions, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ are obtained as follows:

$$\int_0^{\infty} f(x)\sum_{n=1}^{\infty} k_{\lambda}(x,n)a_n\, dx < k(\lambda_1)\|f\|_{p,\phi}\|a\|_{q,\psi}, \tag{9}$$

which is an extension of (8) (*cf.* [35]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor are given by Yang [36]. In 2012-2014, Yang *et al.* published three books [37, 38] and [39] concerned with building the theory of half-discrete Hilbert-type inequalities.

In this paper, by applying weight functions, the technique of real analysis, and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of exponential function with a best possible constant factor expressed by the gamma function is given, which is similar to (7) and an extension of (9) in the following particular kernel:

$$k_0(x,n) = \frac{1}{e^{\alpha(\frac{n}{x})^{\gamma}}} \quad (\alpha > 0, 0 < \gamma \leq 1).$$

Furthermore, the more accurate equivalent forms, the operator expressions with the norm, the reverses, and some particular cases are considered.

2 An example and some lemmas

In the following, we agree that $\nu_n > 0$, $0 \leq \tau_n \leq \frac{\nu_n}{2}$ $(n \in \mathbf{N})$, $V_n = \sum_{i=1}^{n} \nu_i$, $\mu(t)$ is a positive continuous function in $\mathbf{R}_+ = (0, \infty)$,

$$U(0) := 0; \qquad U(x) := \int_0^x \mu(t)\, dt < \infty \quad \bigl(x \in (0,\infty)\bigr),$$

$\nu(t) := \nu_n$, $t \in (n - \frac{1}{2}, n + \frac{1}{2}]$ $(n \in \mathbf{N})$, and

$$V\left(\frac{1}{2}\right) := 0; \qquad V(y) := \int_{\frac{1}{2}}^{y} \nu(t)\, dt \quad \left(y \in \left(\frac{1}{2}, \infty\right)\right),$$

$p \neq 0,1$, $\frac{1}{p}+\frac{1}{q}=1$, $\delta \in \{-1,1\}$, $f(x), a_n \geq 0$ $(x \in \mathbf{R}_+, n \in \mathbf{N})$, $\|f\|_{p,\Phi_\delta} = (\int_0^\infty \Phi_\delta(x) f^p(x)\,dx)^{\frac{1}{p}}$, $\|a\|_{q,\widehat{\Psi}} = (\sum_{n=1}^\infty \widehat{\Psi}(n) b_n^q)^{\frac{1}{q}}$, where

$$\Phi_\delta(x) := \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} \quad (x \in \mathbf{R}_+), \qquad \widehat{\Psi}(n) := \frac{(V_n - \tau_n)^{q(1-\sigma)-1}}{\nu_n^{q-1}} \quad (n \in \mathbf{N}).$$

Example 1 For $\alpha > 0$, $0 < \gamma, \sigma \leq 1$, we set $h(t) = \frac{1}{e^{\alpha t^\gamma}}$ $(t \in \mathbf{R}_+)$.

(i) Setting $u = \alpha t^\gamma$, we find

$$k(\sigma) := \int_0^\infty \frac{t^{\sigma-1}}{e^{\alpha t^\gamma}}\,dt = \frac{1}{\gamma\alpha^{\sigma/\gamma}} \int_0^\infty e^{-u} u^{\frac{\sigma}{\gamma}-1}\,du = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}} \in \mathbf{R}_+, \tag{10}$$

where

$$\Gamma(y) := \int_0^\infty e^{-v} v^{y-1}\,dv \quad (y > 0)$$

is called the gamma function (*cf.* [40]).

(ii) We obtain, for $t > 0$, $\alpha > 0$, $0 < \gamma \leq 1$, $h(t) = \frac{1}{e^{\alpha t^\gamma}} > 0$, $h'(t) = -\alpha\gamma t^{\gamma-1}\frac{1}{e^{\alpha t^\gamma}} < 0$ and

$$h''(t) = -\alpha\gamma(\gamma-1)t^{\gamma-1}\frac{1}{e^{\alpha t^\gamma}} + \left(\alpha\gamma t^{\gamma-1}\right)^2 \frac{1}{e^{\alpha t^\gamma}} > 0.$$

(iii) If $g(u) > 0$, $g'(u) < 0$, $g''(u) > 0$, then we find that, for $y \in (n-\frac{1}{2}, n+\frac{1}{2})$, $g(V(y)) > 0$, $\frac{d}{dy}g(V(y)) = g'(V(y))\nu_n < 0$, and

$$\frac{d^2}{dy^2} g\big(V(y)\big) = g''\big(V(y)\big)\nu_n^2 > 0 \quad (n \in \mathbf{N});$$

For $g_1(u) > 0$, $g_1'(u) < 0$, $g_1''(u) > 0$, $g_2(u) > 0$, $g_2'(u) \leq 0$, $g_2''(u) \geq 0$ $(u > 0)$, we obtain $g_1(u)g_2(u) > 0$, $(g_1(u)g_2(u))' = g_1'(u)g_2(u) + g_1(u)g_2'(u) < 0$, and

$$\big(g_1(u)g_2(u)\big)'' = g_1''(u)g_2(u) + 2g_1'(u)g_2'(u) + g_1(u)g_2''(u) > 0 \quad (u > 0).$$

(iv) For $\alpha > 0$, $0 < \gamma, \sigma \leq 1$, $c > 0$, we have $h(cV(y))V^{\sigma-1}(y) > 0$, $\frac{d}{dy}(h(cV(y))V^{\sigma-1}(y)) < 0$, and

$$\frac{d^2}{dy^2}\big(h\big(cV(y)\big)V^{\sigma-1}(y)\big) > 0 \quad \left(y \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right), n \in \mathbf{N}\right).$$

Then by Hermite-Hadamard's inequality (*cf.* [41]), we have

$$h\big(cV(n)\big)V^{\sigma-1}(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h\big(cV(y)\big)V^{\sigma-1}(y)\,dy \quad (n \in \mathbf{N}). \tag{11}$$

Lemma 1 *If $g(t)$ (> 0) is a strictly decreasing continuous function in $(\frac{1}{2}, \infty)$, which is strictly convex satisfying $\int_{\frac{1}{2}}^\infty g(t)\,dt \in \mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)\,dt < \sum_{n=1}^\infty g(n) < \int_{\frac{1}{2}}^\infty g(t)\,dt. \tag{12}$$

Proof By Hermite-Hadamard's inequality and the decreasing property, we have

$$\int_{n}^{n+1} g(t)\,dt < \int_{n}^{n+1} g(n)\,dt = g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)\,dt \quad (n \in \mathbf{N}), \tag{13}$$

and, for $n_0 \in \mathbf{N}$, it follows that

$$\int_{1}^{n_0+1} g(t)\,dt < \sum_{n=1}^{n_0} g(n) < \sum_{n=1}^{n_0} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)\,dt = \int_{\frac{1}{2}}^{n_0+\frac{1}{2}} g(t)\,dt,$$

$$\int_{n_0+1}^{\infty} g(t)\,dt \leq \sum_{n=n_0+1}^{\infty} g(n) \leq \int_{n_0+\frac{1}{2}}^{\infty} g(t)\,dt < \infty.$$

Hence, choosing plus for the above two inequalities, we have (12). □

Lemma 2 *If* $\alpha > 0$, $0 < \gamma, \sigma \leq 1$, *define the following weight coefficients*:

$$\omega_\delta(\sigma, x) := \sum_{n=1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{U^{\delta\sigma}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}}, \quad x \in \mathbf{R}_+, \tag{14}$$

$$\varpi_\delta(\sigma, n) := \int_0^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{(V_n-\tau_n)^\sigma \mu(x)}{U^{1-\delta\sigma}(x)}\,dx, \quad n \in \mathbf{N}. \tag{15}$$

Then we have the following inequalities:

$$\omega_\delta(\sigma, x) < k(\sigma) \quad (x \in \mathbf{R}_+), \tag{16}$$

$$\varpi_\delta(\sigma, n) \leq k(\sigma) \quad (n \in \mathbf{N}), \tag{17}$$

where $k(\sigma)$ *is indicated by* (10).

Proof Since $V_n - \tau_n \geq \int_{\frac{1}{2}}^{n+\frac{1}{2}} \nu(t)\,dt - \frac{\nu_n}{2} = \int_{\frac{1}{2}}^{n} \nu(t)\,dt = V(n)$, and, for $t \in (n-\frac{1}{2}, n+\frac{1}{2})$, $\nu_n = V'(t)$, by (11) (for $c = U^\delta(x)$) and (12), we have

$$\begin{aligned} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V_n-\tau_n)^{1-\sigma}} &\leq \frac{1}{e^{\alpha U^{\delta\gamma}(x)V^\gamma(n)}} \frac{U^{\delta\sigma}(x)}{V^{1-\sigma}(n)} \\ &< \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{e^{\alpha U^{\delta\gamma}(x)V^\gamma(t)}} \frac{U^{\delta\sigma}(x)}{V^{1-\sigma}(t)}\,dt \quad (n \in \mathbf{N}), \end{aligned}$$

$$\begin{aligned} \omega_\delta(\sigma, x) &< \sum_{n=1}^{\infty} \nu_n \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{e^{\alpha U^{\delta\gamma}(x)V^\gamma(t)}} \frac{U^{\delta\sigma}(x)}{V^{1-\sigma}(t)}\,dt \\ &= \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{e^{\alpha U^{\delta\gamma}(x)V^\gamma(t)}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)}\,dt \\ &= \int_{\frac{1}{2}}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)V^\gamma(t)}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)}\,dt. \end{aligned}$$

Setting $u = U^{\delta}(x)V(t)$, by (10), we find

$$\begin{aligned}\omega_{\delta}(\sigma,x) &< \int_{U^{\delta}(x)V(\frac{1}{2})}^{U^{\delta}(x)V(\infty)} \frac{1}{e^{\alpha u^{\gamma}}} \frac{U^{\delta\sigma}(x)U^{-\delta}(x)}{(uU^{-\delta}(x))^{1-\sigma}}\,du \\ &\le \int_0^{\infty} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du = k(\sigma).\end{aligned}$$

Hence, (16) follows.

Setting $u = (V_n - \tau_n)U^{\delta}(x)$ in (15), we find $du = \delta(V_n - \tau_n)U^{\delta-1}(x)\mu(x)\,dx$ and

$$\varpi_{\delta}(\sigma,n) = \frac{1}{\delta}\int_{(V_n-\tau_n)U^{\delta}(0)}^{(V_n-\tau_n)U^{\delta}(\infty)} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du.$$

If $\delta = 1$, then

$$\varpi_1(\sigma,n) = \int_0^{(V_n-\tau_n)U(\infty)} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du \le \int_0^{\infty} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du;$$

if $\delta = -1$, then

$$\varpi_{-1}(\sigma,n) = -\int_{\infty}^{(V_n-\tau_n)U^{-1}(\infty)} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du \le \int_0^{\infty} \frac{1}{e^{\alpha u^{\gamma}}} u^{\sigma-1}\,du.$$

Hence, by (10), we have (17). □

Remark 1 (i) We do not need the condition of $\sigma \le 1$ in obtaining (17). (ii) If $U(\infty) = \infty$, then we have

$$\varpi_{\delta}(\sigma,n) = k(\sigma) \quad (n \in \mathbf{N}). \tag{18}$$

For example, we set $\mu(t) = \frac{1}{(1+t)^a}$ $(t > 0; 0 \le a \le 1)$, then for $x \ge 0$, we find

$$U(x) = \int_0^x \frac{dt}{(1+t)^a} = \begin{cases} \frac{(1+x)^{1-a}-1}{1-a}, & 0 \le a < 1, \\ \ln(1+x), & a = 1 \end{cases} < \infty,$$

$U(0) = 0$, and $U(\infty) = \int_0^{\infty} \frac{dt}{(1+t)^a} = \infty$.

Lemma 3 *If* $\alpha > 0$, $0 < \gamma, \sigma \le 1$, *there exists* $n_0 \in \mathbf{N}$, *such that* $\{v_n\}_{n=n_0}^{\infty}$ *is decreasing and* $V(\infty) = \infty$, *then*: (i) *for* $x \in \mathbf{R}_+$, *we have*

$$k(\sigma)\bigl(1 - \theta_{\delta}(\sigma,x)\bigr) < \omega_{\delta}(\sigma,x), \tag{19}$$

where

$$\theta_{\delta}(\sigma,x) := \frac{1}{k(\sigma)}\int_0^{U^{\delta}(x)V(n_0+1)} \frac{u^{\sigma-1}}{e^{\alpha u^{\gamma}}}\,du = O\bigl(\bigl(U(x)\bigr)^{\delta\sigma}\bigr) \in (0,1);$$

(ii) *for any $b > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{\nu_n}{(V_n - \tau_n)^{1+b}} = \frac{1}{b}\left(\frac{1}{\nu_1^b} + bO(1)\right). \tag{20}$$

Proof Since $V_n - \tau_n \le V_n \le V_{n+1} - \frac{\nu_{n+1}}{2} = V(n+1)$, and $\nu_n \ge V'(t)$ ($t \in (n, n+1)$; $n \ge n_0$), by (13), we find

$$\begin{aligned}
\omega_\delta(\sigma, x) &\ge \sum_{n=n_0}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x) V^\gamma(n+1)}} \frac{U^{\delta\sigma}(x)\nu_{n+1}}{V^{1-\sigma}(n+1)} \\
&= \sum_{n=n_0+1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x) V^\gamma(n)}} \frac{U^{\delta\sigma}(x)\nu_{n}}{V^{1-\sigma}(n)} \\
&> \sum_{n=n_0+1}^{\infty} \int_n^{n+1} \frac{1}{e^{\alpha U^{\delta\gamma}(x) V^\gamma(t)}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)}\,dt \\
&= \int_{n_0+1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x) V^\gamma(t)}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)}\,dt.
\end{aligned}$$

Setting $u = U^\delta(x)V(t)$, in view of $V(\infty) = \infty$, by (10), we find

$$\omega_\delta(\sigma, x) > \int_{U^\delta(x)V(n_0+1)}^{\infty} \frac{u^{\sigma-1}}{e^{\alpha u^\gamma}}\,du = k(\sigma) - \int_0^{U^\delta(x)V(n_0+1)} \frac{u^{\sigma-1}}{e^{\alpha u^\gamma}}\,du = k(\sigma)\bigl(1 - \theta_\delta(\sigma, x)\bigr).$$

We find

$$0 < \theta_\delta(\sigma, x) \le \frac{1}{k(\sigma)} \int_0^{U^\delta(x)V(n_0+1)} u^{\sigma-1}\,du = \frac{(U^\delta(x)V(n_0+1))^\sigma}{\sigma k(\sigma)} \quad (x \in \mathbf{R}_+),$$

and then (19) follows.

For $b > 0$, we find

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\nu_n}{(V_n - \tau_n)^{1+b}} &\le \sum_{n=1}^{\infty} \frac{\nu_n}{V^{1+b}(n)} = \frac{\nu_1}{V^{1+b}(1)} + \sum_{n=2}^{\infty} \frac{\nu_n}{V^{1+b}(n)} \\
&< \frac{2^{1+b}}{\nu_1^b} + \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{V'(x)\,dx}{V^{1+b}(x)} = \frac{2^{1+b}}{\nu_1^b} + \int_{\frac{3}{2}}^{\infty} \frac{V'(x)\,dx}{V^{1+b}(x)} \\
&= \frac{2^{1+b}}{\nu_1^b} + \frac{\nu_1^{-b}}{b} = \frac{1}{b}\left(\frac{1}{\nu_1^b} + b\frac{2^{1+b}}{\nu_1^b}\right),
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\nu_n}{(V_n - \tau_n)^{1+b}} &\ge \sum_{n=n_0}^{\infty} \frac{\nu_n}{(V_n - \tau_n)^{1+b}} \ge \sum_{n=n_0}^{\infty} \frac{\nu_{n+1}}{V^{1+b}(n+1)} \\
&= \sum_{n=n_0+1}^{\infty} \frac{\nu_n}{V^{1+b}(n)} > \sum_{n=n_0+1}^{\infty} \int_n^{n+1} \frac{V'(x)\,dx}{V^{1+b}(x)} = \int_{n_0+1}^{\infty} \frac{V'(x)\,dx}{V^{1+b}(x)} \\
&= \frac{1}{bV^b(n_0+1)} = \frac{1}{b}\left(\frac{1}{\nu_1^b} + b\frac{V^{-b}(n_0+1) - \nu_1^{-b}}{b}\right).
\end{aligned}$$

Since $\frac{V^{-b}(n_0+1) - \nu_1^{-b}}{b} \to \text{Constant}$ ($b \to 0^+$), we have (20). □

Note For example, $\nu_n = \frac{1}{n^a}$ ($n \in \mathbf{N}; 0 \le a \le 1$) satisfies the conditions of $\{\nu_n\}_{n=1}^{\infty}$ in Lemma 3 (for $n_0 = 1$).

3 Main results and operator expressions

Theorem 1 *If* $\alpha > 0$, $0 < \gamma$, $\sigma \le 1$, *then for* $p > 1$, $0 < \|f\|_{p,\Phi_\delta}$, $\|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities*:

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{a_n f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}} \|f\|_{p,\Phi_\delta} \|a\|_{q,\widehat{\Psi}}, \tag{21}$$

$$J_1 := \sum_{n=1}^{\infty} \frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}} \left[\int_0^{\infty} \frac{f(x)\,dx}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^p < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}} \|f\|_{p,\Phi_\delta}, \tag{22}$$

$$J_2 := \left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q dx\right\}^{\frac{1}{q}} < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}} \|a\|_{q,\widehat{\Psi}}. \tag{23}$$

Proof By Hölder's inequality with weight (*cf.* [41]), we have

$$\begin{aligned}
&\left[\int_0^{\infty} \frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]^p \\
&= \left[\int_0^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \frac{U^{\frac{1-\delta\sigma}{q}}(x) f(x)}{(V_n-\tau_n)^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)} \frac{(V_n-\tau_n)^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)}\,dx\right]^p \\
&\le \int_0^{\infty} \frac{(V_n-\tau_n)^{\gamma}}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \left[\frac{U^{\frac{p(1-\delta\sigma)}{q}}(x) f^p(x)}{(V_n-\tau_n)^{1-\sigma} \mu^{\frac{p}{q}}(x)}\right] dx \\
&\quad \times \left[\int_0^{\infty} \frac{(V_n-\tau_n)^{\gamma}}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \frac{(V_n-\tau_n)^{(1-\sigma)(p-1)}\mu(x)}{U^{1-\delta\sigma}(x)}\,dx\right]^{p-1} \\
&= \frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\tau_n)^{p\sigma-1}\nu_n} \int_0^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n f^p(x)}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)}\,dx.
\end{aligned} \tag{24}$$

In view of (17) and the Lebesgue term by term integration theorem (*cf.* [42]), we find

$$\begin{aligned}
J_1 &\le \big(k(\sigma)\big)^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&= \big(k(\sigma)\big)^{\frac{1}{q}} \left[\int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&= \big(k(\sigma)\big)^{\frac{1}{q}} \left[\int_0^{\infty} \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned} \tag{25}$$

Then by (16), we have (22).

By Hölder's inequality (*cf.* [41]), we have

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} \left[\frac{\nu_n^{\frac{1}{p}}}{(V_n-\tau_n)^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]\left[\frac{(V_n-\tau_n)^{\frac{1}{p}-\sigma} a_n}{\nu_n^{1/p}}\right] \\
&\le J_1 \|a\|_{q,\widehat{\Psi}}.
\end{aligned} \tag{26}$$

Then by (22), we have (21). On the other hand, assuming that (21) is valid, we set

$$a_n := \frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} f(x)\,dx\right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J_1^p = \|a\|_{q,\widehat{\Psi}}^q$. If $J_1 = 0$, then (22) is trivially valid; if $J_1 = \infty$, then (22) remains impossible. Suppose that $0 < J_1 < \infty$. By (21), we have

$$\|a\|_{q,\widehat{\Psi}}^q = J_1^p = I < k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}},$$
$$\|a\|_{q,\widehat{\Psi}}^{q-1} = J_1 < k(\sigma)\|f\|_{p,\Phi_\delta},$$

and then (22) follows, which is equivalent to (21).

Still by Hölder's inequality with weight (*cf.* [41]), we have

$$\begin{aligned}
&\left[\sum_{n=1}^\infty \frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\right]^q \\
&= \left[\sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \cdot \frac{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{\frac{1}{p}}}{(V_n-\tau_n)^{\frac{1-\sigma}{p}}} \cdot \frac{(V_n-\tau_n)^{\frac{1-\sigma}{p}} a_n}{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{1/p}}\right]^q \\
&\leq \left[\sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}}\right]^{q-1} \\
&\quad \times \sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{(V_n-\tau_n)^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q \\
&= \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q.
\end{aligned} \tag{27}$$

Then by (16) and the Lebesgue term by term integration theorem (*cf.* [42]), it follows that

$$\begin{aligned}
J_2 &< \big(k(\sigma)\big)^{\frac{1}{p}}\left[\int_0^\infty \sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q\,dx\right]^{\frac{1}{q}} \\
&= \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=1}^\infty \int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} \frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q\,dx\right]^{\frac{1}{q}} \\
&= \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=1}^\infty \varpi_\delta(\sigma,n)\frac{(V_n-\tau_n)^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{28}$$

Then by (17), we have (23).

By Hölder's inequality (*cf.* [41]), we have

$$\begin{aligned}
I &= \int_0^\infty \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right)\left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} a_n\right]dx \\
&\leq \|f\|_{p,\Phi_\delta} J_2.
\end{aligned} \tag{29}$$

Then by (23), we have (21). On the other hand, assuming that (23) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}a_n\right]^{q-1}, \quad x \in \mathbf{R}_+.$$

Then we find $J_2^q = \|f\|_{p,\Phi_\delta}^p$. If $J_2 = 0$, then (23) is trivially valid; if $J_2 = \infty$, then (23) keeps impossible. Suppose that $0 < J_2 < \infty$. By (21), we have

$$\|f\|_{p,\Phi_\delta}^p = J_2^q = I < k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}}, \qquad \|f\|_{p,\Phi_\delta}^{p-1} = J_2 < k(\sigma)\|a\|_{q,\widehat{\Psi}},$$

and then (23) follows, which is equivalent to (21).

Therefore, (21), (22), and (23) are equivalent. □

Theorem 2 *As regards the assumptions of Theorem* 1, *if there exists* $n_0 \in \mathbf{N}$, *such that* $\{\nu_n\}_{n=n_0}^{\infty}$ *is decreasing and* $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k(\sigma) = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$ *in* (21), (22), *and* (23) *is the best possible*.

Proof For $\varepsilon \in (0, q\sigma)$, we set $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ $(\in (0,1))$, and $\widetilde{f} = \widetilde{f}(x)$, $x \in \mathbf{R}_+$, $\widetilde{a} = \{\widetilde{a}_n\}_{n=1}^{\infty}$,

$$\widetilde{f}(x) = \begin{cases} U^{\delta(\widetilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0 < x^\delta \le 1, \\ 0, & x^\delta > 0, \end{cases} \tag{30}$$

$$\widetilde{a}_n = (V_n - \tau_n)^{\widetilde{\sigma}-1}\nu_n = (V_n - \tau_n)^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, \quad n \in \mathbf{N}. \tag{31}$$

Then for $\delta = \pm 1$, since $U(\infty) = \infty$, we find

$$\int_{\{x>0;0<x^\delta\le 1\}}\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx = \frac{1}{\varepsilon}U^{\delta\varepsilon}(1). \tag{32}$$

By (20), (32), and (19), we obtain

$$\begin{aligned} \|\widetilde{f}\|_{p,\Phi_\delta}\|\widetilde{a}\|_{q,\Psi} &= \left(\int_{\{x>0;0<x^\delta\le 1\}}\frac{\mu(x)\,dx}{U^{1-\delta\varepsilon}(x)}\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1+\varepsilon}}\right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{\nu_1^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{q}}, \end{aligned} \tag{33}$$

$$\begin{aligned} \widetilde{I} &:= \int_0^\infty \sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\widetilde{a}_n\widetilde{f}(x)\,dx \\ &= \int_{\{x>0;0<x^\delta\le 1\}}\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{(V_n-\tau_n)^{\widetilde{\sigma}-1}\nu_n\mu(x)}{U^{1-\delta(\widetilde{\sigma}+\varepsilon)}(x)}\,dx \\ &= \int_{\{x>0;0<x^\delta\le 1\}}\omega_\delta(\widetilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx \\ &\ge k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}}\big(1-\theta_\delta(\widetilde{\sigma},x)\big)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx \\ &= k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}}\big(1-O\big(\big(U(x)\big)^{\delta\widetilde{\sigma}}\big)\big)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx \end{aligned}$$

$$= k(\widetilde{\sigma})\left[\int_{\{x>0;0<x^\delta\le 1\}} \frac{\mu(x)\,dx}{U^{1-\delta\varepsilon}(x)} - \int_{\{x>0;0<x^\delta\le 1\}} O\left(\frac{\mu(x)}{U^{1-\delta(\sigma+\frac{\varepsilon}{p})}(x)}\right) dx\right]$$
$$= \frac{1}{\varepsilon}k\left(\sigma - \frac{\varepsilon}{q}\right)\left(U^{\delta\varepsilon}(1) - \varepsilon O_1(1)\right).$$

If there exists a positive constant $K \le k(\sigma)$, such that (21) is valid when replacing $k(\sigma)$ to K, then in particular, by Lebesgue term by term integration theorem, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{f}\|_{p,\Phi_\delta}\|\widetilde{a}\|_{q,\Psi}$, namely,

$$k\left(\sigma - \frac{\varepsilon}{q}\right)\left(U^{\delta\varepsilon}(1) - \varepsilon O_1(1)\right) < K\cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{\nu_1^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \le K$ ($\varepsilon \to 0^+$). Hence, $K = k(\sigma)$ is the best possible constant factor of (21).

The constant factor $k(\sigma)$ in (22) ((23)) is still the best possible. Otherwise, we would reach a contradiction by (26) ((29)) that the constant factor in (21) is not the best possible. □

For $p > 1$, we find $\widehat{\Psi}^{1-p}(n) = \frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}$ ($n \in \mathbf{N}$), $\Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}$ ($x \in \mathbf{R}_+$), and we define the following real normed spaces:

$$L_{p,\Phi_\delta}(\mathbf{R}_+) = \left\{f; f = f(x), x \in \mathbf{R}_+, \|f\|_{p,\Phi_\delta} < \infty\right\},$$
$$l_{q,\widehat{\Psi}} = \left\{a; a = \{a_n\}_{n=1}^\infty, \|a\|_{q,\widehat{\Psi}} < \infty\right\},$$
$$L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+) = \left\{h; h = h(x), x \in \mathbf{R}_+, \|h\|_{q,\Phi_\delta^{1-q}} < \infty\right\},$$
$$l_{p,\widehat{\Psi}^{1-p}} = \left\{c; c = \{c_n\}_{n=1}^\infty, \|c\|_{p,\widehat{\Psi}^{1-p}} < \infty\right\}.$$

Assuming that $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, setting

$$c = \{c_n\}_{n=1}^\infty, \qquad c_n := \int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} f(x)\,dx, \quad n \in \mathbf{N},$$

we can rewrite (22) as $\|c\|_{p,\widehat{\Psi}^{1-p}} < k(\sigma)\|f\|_{p,\Phi_\delta} < \infty$, namely, $c \in l_{p,\widehat{\Psi}^{1-p}}$.

Definition 1 Define a half-discrete Hardy-Hilbert-type operator $T_1 : L_{p,\Phi_\delta}(\mathbf{R}_+) \to l_{p,\widehat{\Psi}^{1-p}}$ as follows: For any $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = c \in l_{p,\widehat{\Psi}^{1-p}}$. Define the formal inner product of $T_1 f$ and $a = \{a_n\}_{n=1}^\infty \in l_{q,\widehat{\Psi}}$ as follows:

$$(T_1 f, a) := \sum_{n=1}^\infty \left[\int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}} f(x)\,dx\right] a_n. \tag{34}$$

Then we can rewrite (21) and (22) as follows:

$$(T_1 f, a) < k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}}, \tag{35}$$
$$\|T_1 f\|_{p,\widehat{\Psi}^{1-p}} < k(\sigma)\|f\|_{p,\Phi_\delta}. \tag{36}$$

Define the norm of operator T_1 as follows:

$$\|T_1\| := \sup_{f(\neq\theta)\in L_{p,\Phi_\delta}(\mathbf{R}_+)} \frac{\|T_1 f\|_{p,\widehat{\Psi}^{1-p}}}{\|f\|_{p,\Phi_\delta}}.$$

Then by (36), it follows that $\|T_1\| \leq k(\sigma)$. Since, by Theorem 2, the constant factor in (36) is the best possible, we have

$$\|T_1\| = k(\sigma) = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}. \tag{37}$$

Assuming that $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\widehat{\Psi}}$, setting

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} a_n, \quad x \in \mathbf{R}_+,$$

we can rewrite (23) as $\|h\|_{q,\Phi_\delta^{1-q}} < k(\sigma)\|a\|_{q,\widehat{\Psi}} < \infty$, namely, $h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$.

Definition 2 Define a half-discrete Hardy-Hilbert-type operator $T_2 : l_{q,\widehat{\Psi}} \to L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$ as follows: For any $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\widehat{\Psi}}$, there exists a unique representation $T_2 a = h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$. Define the formal inner product of $T_2 a$ and $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$ as follows:

$$(T_2 a, f) := \int_0^{\infty} \left[\sum_{n=1}^{\infty} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}} a_n\right] f(x)\,dx. \tag{38}$$

Then we can rewrite (21) and (23) as follows:

$$(T_2 a, f) < k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}}, \tag{39}$$

$$\|T_2 a\|_{q,\Phi_\delta^{1-q}} < k(\sigma)\|a\|_{q,\widehat{\Psi}}. \tag{40}$$

Define the norm of operator T_2 as follows:

$$\|T_2\| := \sup_{a(\neq\theta)\in l_{q,\widehat{\Psi}}} \frac{\|T_2 a\|_{q,\Phi_\delta^{1-q}}}{\|a\|_{q,\widehat{\Psi}}}.$$

Then by (40), we find $\|T_2\| \leq k(\sigma)$. Since, by Theorem 2, the constant factor in (40) is the best possible, we have

$$\|T_2\| = k(\sigma) = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}} = \|T_1\|. \tag{41}$$

4 Some equivalent reverses

In the following, we also set

$$\widetilde{\Phi}_\delta(x) := \bigl(1 - \theta_\delta(\sigma, x)\bigr)\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} \quad (x \in \mathbf{R}_+).$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $\|f\|_{p,\Phi_\delta}$, $\|f\|_{p,\widetilde{\Phi}_\delta}$, and $\|a\|_{q,\widehat{\Psi}}$.

Theorem 3 *As regards the assumptions of Theorem* 2, *for* $p < 0$, $0 < \|f\|_{p,\Phi_\delta}, \|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $k(\sigma) = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$:

$$I = \sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}}, \tag{42}$$

$$J_1 = \sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\,dx\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_\delta}, \tag{43}$$

$$J_2 = \left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\right]^q dx\right\}^{\frac{1}{q}} > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}. \tag{44}$$

Proof By the reverse Hölder inequality with weight (*cf.* [41]), since $p < 0$, in the similar way to obtaining (24) and (25), we have

$$\begin{aligned}&\left[\int_0^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}f(x)\,dx\right]^p\\&\quad\le\frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\tau_n)^{p\sigma-1}\nu_n}\int_0^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)}f^p(x)\,dx.\end{aligned}$$

Then by (18) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned}J_1&\ge\big(k(\sigma)\big)^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}\\&=\big(k(\sigma)\big)^{\frac{1}{q}}\left[\int_0^{\infty}\omega_\delta(\sigma,x)\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}.\end{aligned}$$

Then by (16), we have (43).

By the reverse Hölder inequality (*cf.* [41]), we have

$$\begin{aligned}I&=\sum_{n=1}^{\infty}\left[\frac{\nu_n^{\frac{1}{p}}}{(V_n-\tau_n)^{\frac{1}{p}-\sigma}}\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\,dx\right]\left[\frac{(V_n-\tau_n)^{\frac{1}{p}-\sigma}a_n}{\nu_n^{1/p}}\right]\\&\ge J_1\|a\|_{q,\widehat{\Psi}}.\end{aligned}\tag{45}$$

Then by (43), we have (42). On the other hand, assuming that (42) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = \|a\|_{q,\widehat{\Psi}}^q$. If $J_1 = \infty$, then (43) is trivially valid; if $J_1 = 0$, then (43) keeps impossible. Suppose that $0 < J_1 < \infty$. By (42), it follows that

$$\|a\|_{q,\widehat{\Psi}}^q = J_1^p = I > k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}},\qquad \|a\|_{q,\widehat{\Psi}}^{q-1} = J_1 > k(\sigma)\|f\|_{p,\Phi_\delta},$$

and then (43) follows, which is equivalent to (42).

Still by the reverse Hölder's inequality with weight (*cf.* [41]), since $0 < q < 1$, in the similar way to obtaining (27) and (28), we have

$$\begin{aligned}&\left[\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}a_n\right]^q\\&\quad\geq\frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)}\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q.\end{aligned}$$

Then by (16) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned}J_2&>\big(k(\sigma)\big)^{\frac{1}{p}}\left[\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q\,dx\right]^{\frac{1}{q}}\\&=\big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\varpi_\delta(\sigma,n)\frac{(V_n-\tau_n)^{q(1-\sigma)-1}}{\nu_n^{q-1}}a_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

Then by (18), we have (44).

By the reverse Hölder inequality (*cf.* [41]), we have

$$\begin{aligned}I&=\int_0^{\infty}\left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)}f(x)\right)\left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)}\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]dx\\&\geq\|f\|_{p,\Phi_\delta}J_2.\end{aligned}\tag{46}$$

Then by (44), we have (42). On the other hand, assuming that (44) is valid, we set $f(x)$ as in Theorem 1. Then we find $J_2^q=\|f\|_{p,\Phi_\delta}^p$. If $J_2=\infty$, then (44) is trivially valid; if $J_2=0$, then (44) remains impossible. Suppose that $0<J_2<\infty$. By (42), it follows that

$$\|f\|_{p,\Phi_\delta}^p=J_2^q=I>k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\widehat{\Psi}},\qquad\|f\|_{p,\Phi_\delta}^{p-1}=J_2>k(\sigma)\|a\|_{q,\widehat{\Psi}},$$

and then (44) follows, which is equivalent to (42).

Therefore, inequalities (42), (43), and (44) are equivalent.

For $\varepsilon\in(0,q\sigma)$, we set $\widetilde{\sigma}=\sigma-\frac{\varepsilon}{q}$ ($\in(0,1)$) and $\widetilde{f}=\widetilde{f}(x)$, $x\in\mathbf{R}_+$, $\widetilde{a}=\{\widetilde{a}_n\}_{n=1}^{\infty}$,

$$\begin{aligned}&\widetilde{f}(x)=\begin{cases}U^{\delta(\widetilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0<x^\delta\leq 1,\\ 0, & x^\delta>0,\end{cases}\\&\widetilde{a}_n=(V_n-\tau_n)^{\widetilde{\sigma}-1}\nu_n=(V_n-\tau_n)^{\sigma-\frac{\varepsilon}{q}-1}\nu_n,\quad n\in\mathbf{N}.\end{aligned}$$

By (20), (32), and (16), we obtain

$$\begin{aligned}&\|\widetilde{f}\|_{p,\Phi_\delta}\|\widetilde{a}\|_{q,\widehat{\Psi}}=\frac{1}{\varepsilon}U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{\nu_1^{\varepsilon}}+\varepsilon O(1)\right)^{\frac{1}{q}},\\&\widetilde{I}=\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\widetilde{a}_n\widetilde{f}(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx=\int_{\{x>0;0<x^\delta\leq 1\}}\omega_\delta(\widetilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx\\&\quad\leq k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\leq 1\}}\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}\,dx=\frac{1}{\varepsilon}k\left(\sigma-\frac{\varepsilon}{q}\right)U^{\delta\varepsilon}(1).\end{aligned}$$

If there exists a positive constant $K \geq k(\sigma)$, such that (42) is valid when replacing $k(\sigma)$ to K, then in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{f}\|_{p,\Phi_\delta}\|\widetilde{a}\|_{q,\widehat{\Psi}}$, namely,

$$k\left(\sigma - \frac{\varepsilon}{q}\right)U^{\delta\varepsilon}(1) > K \cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{\nu_1^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \geq K$ ($\varepsilon \to 0^+$). Hence, $K = k(\sigma)$ is the best possible constant factor of (42).

The constant factor $k(\sigma)$ in (43) ((44)) is still the best possible. Otherwise, we would reach a contradiction by (45) ((46)) that the constant factor in (42) is not the best possible. □

Theorem 4 *As regards the assumptions of Theorem* 2, *if* $0 < p < 1$, $0 < \|f\|_{p,\Phi_\delta}, \|a\|_{q,\widehat{\Psi}} < \infty$, *then we have the following equivalent inequalities with the best possible constant factor* $k(\sigma) = \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$:

$$I = \sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_\delta}\|a\|_{q,\widehat{\Psi}}, \tag{47}$$

$$J_1 = \sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)\,dx}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_\delta}, \tag{48}$$

$$\begin{aligned} J &:= \left\{\int_0^{\infty}\frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q dx\right\}^{\frac{1}{q}} \\ &> \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}. \end{aligned} \tag{49}$$

Proof By the reverse Hölder inequality with weight (*cf.* [41]), since $0 < p < 1$, in a similar way to obtaining (24) and (25), we have

$$\begin{aligned} &\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]^p \\ &\quad\geq \frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\tau_n)^{p\sigma-1}\nu_n}\int_0^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)}f^p(x)\,dx. \end{aligned}$$

In view of (18) and the Lebesgue term by term integration theorem, we find

$$\begin{aligned} J_1 &\geq (k(\sigma))^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{(V_n-\tau_n)^{1-\sigma}\mu^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}}\left[\int_0^{\infty}\omega_\delta(\sigma,x)\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}. \end{aligned}$$

Then by (19), we have (48).

By the reverse Hölder inequality (*cf.* [41]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\left[\frac{\nu_n^{1/p}}{(V_n-\tau_n)^{\frac{1}{p}-\sigma}}\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]\left[\frac{(V_n-\tau_n)^{\frac{1}{p}-\sigma}a_n}{\nu_n^{1/p}}\right] \\ &\geq J_1\|a\|_{q,\widehat{\Psi}}. \end{aligned} \tag{50}$$

Then by (48), we have (47). On the other hand, assuming that (47) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = \|a\|_{q,\widehat{\Psi}}^q$. If $J_1 = \infty$, then (48) is trivially valid; if $J_1 = 0$, then (48) remains impossible. Suppose that $0 < J_1 < \infty$. By (47), it follows that

$$\|a\|_{q,\widehat{\Psi}}^q = J_1^p = I > k(\sigma)\|f\|_{p,\widetilde{\Phi}_\delta}\|a\|_{q,\widehat{\Psi}}, \qquad \|a\|_{q,\widehat{\Psi}}^{q-1} = J_1 > k(\sigma)\|f\|_{p,\widetilde{\Phi}_\delta},$$

and then (48) follows, which is equivalent to (47).

Still by the reverse Hölder inequality with weight (*cf.* [41]), since $q < 0$, we have

$$\begin{aligned} &\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q \\ &\quad\leq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)}\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q. \end{aligned}$$

Then by (19) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J &> \big(k(\sigma)\big)^{\frac{1}{p}}\left[\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\frac{(V_n-\tau_n)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q\,dx\right]^{\frac{1}{q}} \\ &= \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\varpi_\delta(\sigma,n)\frac{(V_n-\tau_n)^{q(1-\sigma)-1}}{\nu_n^{q-1}}a_n^q\right]^{\frac{1}{q}}. \end{aligned}$$

Then by (18), we have (49).

By the reverse Hölder inequality (*cf.* [41]), we have

$$\begin{aligned} I &= \int_0^{\infty}\left[\big(1-\theta_\delta(\sigma,x)\big)^{\frac{1}{p}}\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)}f(x)\right] \\ &\quad\times\left[\frac{(1-\theta_\delta(\sigma,x))^{\frac{-1}{p}}\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)}\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]dx \\ &\geq \|f\|_{p,\widetilde{\Phi}_\delta}J. \end{aligned} \tag{51}$$

Then by (49), we have (47). On the other hand, assuming that (47) is valid, we set $f(x)$ as in Theorem 1. Then we find $J^q = \|f\|_{p,\widetilde{\Phi}_\delta}^p$. If $J = \infty$, then (49) is trivially valid; if $J = 0$, then (49) keeps impossible. Suppose that $0 < J < \infty$. By (47), it follows that

$$\|f\|_{p,\widetilde{\Phi}_\delta}^p = J^q = I > k(\sigma)\|f\|_{p,\widetilde{\Phi}_\delta}\|a\|_{q,\widehat{\Psi}}, \qquad \|f\|_{p,\widetilde{\Phi}_\delta}^{p-1} = J > k(\sigma)\|a\|_{q,\widehat{\Psi}},$$

and then (49) follows, which is equivalent to (47).

Therefore, inequalities (47), (48), and (49) are equivalent.

For $\varepsilon \in (0, p\sigma)$, we set $\widetilde{\sigma} = \sigma + \frac{\varepsilon}{p}$ and $\widetilde{f} = \widetilde{f}(x)$, $x \in \mathbf{R}_+$, $\widetilde{a} = \{\widetilde{a}_n\}_{n=1}^{\infty}$,

$$\widetilde{f}(x) = \begin{cases} U^{\delta\widetilde{\sigma}-1}(x)\mu(x), & 0 < x^\delta \le 1, \\ 0, & x^\delta > 0, \end{cases}$$

$$\widetilde{a}_n = (V_n - \tau_n)^{\widetilde{\sigma}-\varepsilon-1}\nu_n = (V_n - \tau_n)^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, \quad n \in \mathbf{N}.$$

By (19), (20), and (32), we obtain

$$\begin{aligned}
&\|\widetilde{f}\|_{p,\widetilde{\Phi}_\delta}\|\widetilde{a}\|_{q,\widehat{\Psi}} \\
&\quad = \left[\int_{\{x>0;0<x^\delta\le 1\}} \left(1 - O\left(\left(U(x)\right)^{\delta\sigma}\right)\right)\frac{\mu(x)\,dx}{U^{1-\delta\varepsilon}(x)}\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{\nu_n}{(V_n - \tau_n)^{1+\varepsilon}}\right]^{\frac{1}{q}} \\
&\quad = \frac{1}{\varepsilon}\left(U^{\delta\varepsilon}(1) - \varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\nu_1^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{q}}, \\
&\widetilde{I} = \sum_{n=1}^{\infty}\int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\widetilde{a}_n\widetilde{f}(x)\,dx \\
&\quad = \sum_{n=1}^{\infty}\left[\int_{\{x>0;0<x^\delta\le 1\}} \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\frac{(V_n-\tau_n)^{\widetilde{\sigma}}\mu(x)}{U^{1-\delta\widetilde{\sigma}}(x)}\,dx\right]\frac{\nu_n}{(V_n-\tau_n)^{1+\varepsilon}} \\
&\quad \le \sum_{n=1}^{\infty}\left[\int_0^\infty \frac{1}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^\gamma}}\frac{(V_n-\tau_n)^{\widetilde{\sigma}}\mu(x)}{U^{1-\delta\widetilde{\sigma}}(x)}\,dx\right]\frac{\nu_n}{(V_n-\beta)^{1+\varepsilon}} \\
&\quad = \sum_{n=1}^{\infty}\varpi_\delta(\widetilde{\sigma}, n)\frac{\nu_n}{(V_n-\tau_n)^{1+\varepsilon}} = k(\widetilde{\sigma})\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1+\varepsilon}} \\
&\quad = \frac{1}{\varepsilon}k\left(\sigma + \frac{\varepsilon}{p}\right)\left(\frac{1}{\nu_1^\varepsilon} + \varepsilon O(1)\right).
\end{aligned}$$

If there exists a positive constant $K \ge k(\sigma)$, such that (42) is valid when replacing $k(\sigma)$ to K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{f}\|_{p,\widetilde{\Phi}_\delta}\|\widetilde{a}\|_{q,\widehat{\Psi}}$, namely,

$$\begin{aligned}
&k\left(\sigma + \frac{\varepsilon}{p}\right)\left(\frac{1}{\nu_1^\varepsilon} + \varepsilon O(1)\right) \\
&\quad > K\left(U^{\delta\varepsilon}(1) - \varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\nu_1^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{q}}.
\end{aligned}$$

It follows that $k(\sigma) \ge K$ $(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (47).

The constant factor $k(\sigma)$ in (48) ((49)) is still the best possible. Otherwise, we would reach the contradiction by (50) ((51)) that the constant factor in (47) is not the best possible. □

5 Some corollaries and a remark

For $\delta = 1$ in Theorems 2-4, we have the following inequalities with the non-homogeneous kernel.

Corollary 1 *As regards the assumptions of Theorem* 2, (i) *for* $p > 1$, $0 < \|f\|_{p,\Phi_1}, \|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_1}\|a\|_{q,\widehat{\Psi}}, \tag{52}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]^p < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_1}, \tag{53}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q dx\right\}^{\frac{1}{q}} < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}; \tag{54}$$

(ii) *for* $p < 0$, $0 < \|f\|_{p,\Phi_1}, \|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_1}\|a\|_{q,\widehat{\Psi}}, \tag{55}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_1}, \tag{56}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q dx\right\}^{\frac{1}{q}} > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}; \tag{57}$$

(iii) *for* $0 < p < 1$, $0 < \|f\|_{p,\Phi_1}, \|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_1}\|a\|_{q,\widehat{\Psi}}, \tag{58}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\,dx\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_1}, \tag{59}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_1(\sigma,x))^{1-q}\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{a_n}{e^{\alpha U^{\gamma}(x)(V_n-\tau_n)^{\gamma}}}\right]^q dx\right\}^{\frac{1}{q}} > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}. \tag{60}$$

The above inequalities are with the best possible constant factor $\frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$.

For $\delta = -1$ in Theorems 2-4, we have the following inequalities with the homogeneous kernel of degree 0:

Corollary 2 *As regards the assumptions of Theorem* 2, (i) *for* $p > 1$, $0 < \|f\|_{p,\Phi_{-1}}, \|a\|_{q,\widehat{\Psi}} < \infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^{\gamma}}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_{-1}}\|a\|_{q,\widehat{\Psi}}, \tag{61}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^{\infty}\frac{f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^{\gamma}}}\,dx\right]^p < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_{-1}}, \tag{62}$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^\infty \frac{a_n}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\right]^q dx\right\}^{\frac{1}{q}} < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}; \tag{63}$$

(ii) *for* $p<0$, $0<\|f\|_{p,\Phi_{-1}}, \|a\|_{q,\widehat{\Psi}}<\infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^\infty \int_0^\infty \frac{a_n f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_{-1}}\|a\|_{q,\widehat{\Psi}}, \tag{64}$$

$$\sum_{n=1}^\infty \frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^\infty \frac{f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\,dx\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_{-1}}, \tag{65}$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^\infty \frac{a_n}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\right]^q dx\right\}^{\frac{1}{q}} > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}; \tag{66}$$

(iii) *for* $0<p<1$, $0<\|f\|_{p,\Phi_{-1}}, \|a\|_{q,\widehat{\Psi}}<\infty$, *we have the following equivalent inequalities*:

$$\sum_{n=1}^\infty \int_0^\infty \frac{a_n f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\,dx > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_{-1}}\|a\|_{q,\widehat{\Psi}}, \tag{67}$$

$$\sum_{n=1}^\infty \frac{\nu_n}{(V_n-\tau_n)^{1-p\sigma}}\left[\int_0^\infty \frac{f(x)}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\,dx\right]^p > \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\widetilde{\Phi}_{-1}}, \tag{68}$$

$$\left\{\int_0^\infty \frac{(1-\theta_{-1}(\sigma,x))^{1-q}\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^\infty \frac{a_n}{e^{\alpha(\frac{V_n-\tau_n}{U(x)})^\gamma}}\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|a\|_{q,\widehat{\Psi}}. \tag{69}$$

The above inequalities are with the best possible constant factor $\frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$.

Remark 2 (i) For $\tau_n=0$ ($n\in\mathbf{N}$) in (21), setting $\Psi(n):=\frac{V_n{}^{q(1-\sigma)-1}}{\nu_n^{q-1}}$ ($n\in\mathbf{N}$), we have the following inequality:

$$\sum_{n=1}^\infty \int_0^\infty \frac{a_n f(x)}{e^{\alpha(U^\delta(x)V_n)^\gamma}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\|f\|_{p,\Phi_\delta}\|a\|_{q,\Psi}. \tag{70}$$

Hence, (21) is a more accurate inequality of (70) for $0<\tau_n\le\frac{\nu_n}{2}$.

(ii) For $\mu(x)=\nu_n=1$ in (21), setting $0\le\tau\le\frac{1}{2}$, we have the following inequality with the best possible constant factor $\frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{a_n f(x)}{e^{\alpha[x^\delta(n-\tau)]^\gamma}}\,dx$$
$$< \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\left[\int_0^\infty x^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{71}$$

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha[x(n-\tau)]^{\gamma}}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}; \tag{72}$$

for $\delta = -1$, we have the following inequality with the homogeneous kernel of degree 0:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{e^{\alpha(\frac{n-\tau}{x})^{\gamma}}}\,dx < \frac{\Gamma(\sigma/\gamma)}{\gamma\alpha^{\sigma/\gamma}}\left[\int_0^{\infty}x^{p(1+\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{73}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. JL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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