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A multidimensional Hilbert-type integral inequality

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Abstract

By applying the method of weight functions and the technique of real analysis, a multidimensional Hilbert-type integral inequality with multi-parameters and the best possible constant factor related to the gamma function is given. The equivalent forms and the reverses are obtained. We also consider the operator expressions and a few particular results related to the kernels of non-homogeneous and homogeneous.

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1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = (\int_0^\infty f^p(x)\,dx)^{\frac{1}{p}} > 0$, $\|g\|_q > 0$. We have the following well-known Hardy-Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is best possible. If $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, then we still have the discrete variant of the above inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ as follows:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in the analysis and its applications (*cf.* [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) at $p = q = 2$ with the kernel $\frac{1}{(x+y)^\lambda}$. In 2009 and 2011, Yang [3, 4] gave some best extensions of (1) and (2) as follows.

If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}\,dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{f; \|f\|_{p,\phi} := \left(\int_0^\infty \phi(x)\big|f(x)\big|^p\,dx\right)^{\frac{1}{p}} < \infty\right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then we have the following inequality:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover, if $k_\lambda(x,y)$ stays finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{a; \|a\|_{p,\phi} := \left(\sum_{n=1}^\infty \phi(n)|a_n|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2).

In 2006, Hong [8] first published a multidimensional Hilbert integral inequality by using the transfer formula, which is an extension of (3). Some other related results are given by [9–22], which provided some new methods to study these kinds of inequalities.

In this paper, by using the transfer formula and applying the method of weight functions and the technique of real analysis, we give a multidimensional Hilbert-type integral inequality with multi-parameters and the best possible constant factor related to the gamma function. The equivalent forms and the reverses are obtained. Furthermore, we also consider the operator expressions and a few particular results related to the kernels of non-homogeneous and homogeneous.

2 Some lemmas

If $m, n \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we set

$$\|x\|_\alpha := \left(\sum_{k=1}^m |x_k|^\alpha\right)^{\frac{1}{\alpha}} \quad \big(x = (x_1, \ldots, x_m) \in \mathbf{R}^m\big),$$

$$\|y\|_\beta := \left(\sum_{k=1}^n |y_k|^\beta\right)^{\frac{1}{\beta}} \quad \big(y = (y_1, \ldots, y_n) \in \mathbf{R}^n\big).$$

Lemma 1 *If* $s \in \mathbf{N}$, $\gamma, M > 0$, $\Psi(u)$ *is a non-negative measurable function in* $(0,1]$, *and*

$$D_M := \left\{x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1\right\},$$

then we have the following transfer formula (*cf.* [6]):

$$\int\cdots\int_{D_M}\Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right)dx_1\cdots dx_s = \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_0^1\Psi(u)u^{\frac{s}{\gamma}-1}\,du, \tag{5}$$

where $\Gamma(\cdot)$ *is the gamma function defined by*

$$\Gamma(t) := \int_0^{\infty} e^{-v}v^{t-1}\,dv \quad (t > 0).$$

In view of (5), since $\mathbf{R}_+^s = \lim_{M\to\infty} D_M$, we have

$$\int\cdots\int_{\mathbf{R}_+^s}\Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right)dx_1\cdots dx_s = \lim_{M\to\infty}\frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_0^1\Psi(u)u^{\frac{s}{\gamma}-1}\,du. \tag{6}$$

By (6), (i) for

$$\left\{x\in\mathbf{R}_+^s;\|x\|_\gamma\ge 1\right\} = \lim_{M\to\infty}\left\{x\in\mathbf{R}_+^s;\frac{1}{M^\gamma}\le u=\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\le 1\right\},$$

setting $\Psi(u) = 0$ ($u \in (0, \frac{1}{M^\gamma})$), it follows that

$$\int\cdots\int_{\{x\in\mathbf{R}_+^s;\|x\|_\gamma\ge 1\}}\Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right)dx_1\cdots dx_s = \lim_{M\to\infty}\frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_{\frac{1}{M^\gamma}}^1\Psi(u)u^{\frac{s}{\gamma}-1}\,du; \tag{7}$$

(ii) for

$$\left\{x\in\mathbf{R}_+^s;\|x\|_\gamma\le 1\right\} = \lim_{M\to\infty}\left\{x\in\mathbf{R}_+^s;0<u=\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\le \frac{1}{M^\gamma}\right\},$$

setting $\Psi(u) = 0$ ($u \in (\frac{1}{M^\gamma}, \infty)$), we have

$$\int\cdots\int_{\{x\in\mathbf{R}_+^s;\|x\|_\gamma\le 1\}}\Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right)dx_1\cdots dx_s = \lim_{M\to\infty}\frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_0^{\frac{1}{M^\gamma}}\Psi(u)u^{\frac{s}{\gamma}-1}\,du. \tag{8}$$

Remark 1 For $\delta \in \{-1,1\}$, $s \in \mathbf{N}$, $\gamma, M > 0$, setting $E_\delta := \{u > 0; u^\delta \geq \frac{1}{M^{\delta\gamma}}\}$, in view of (7) and (8), it follows that

$$\begin{aligned} &\int \cdots \int_{\{x \in \mathbf{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1 \cdots dx_s \\ &\quad = \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} \Psi(u) u^{\frac{s}{\gamma}-1}\, du. \end{aligned} \tag{9}$$

Lemma 2 *For $\delta \in \{-1,1\}$, $s \in \mathbf{N}$, $\gamma, \varepsilon > 0$, we have*

$$\int_{\{x \in \mathbf{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \|x\|_\gamma^{-s-\delta\varepsilon}\, dx = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \tag{10}$$

Proof By (9), for $\delta \in \{-1,1\}$, it follows that

$$\begin{aligned} &\int_{\{x \in \mathbf{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \|x\|_\gamma^{-s-\delta\varepsilon}\, dx \\ &\quad = \int \cdots \int_{\{x \in \mathbf{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \left\{ M \left[\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right]^{\frac{1}{\gamma}} \right\}^{-s-\delta\varepsilon} dx_1 \cdots dx_s \\ &\quad = \lim_{M \to \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} \left(M u^{1/\gamma}\right)^{-s-\delta\varepsilon} u^{\frac{s}{\gamma}-1}\, du \\ &\quad = \lim_{M \to \infty} \frac{M^{-\delta\varepsilon} \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} u^{\frac{-\delta\varepsilon}{\gamma}-1}\, du \\ &\quad \overset{v=M^\gamma u}{=} \frac{M^{-\delta\varepsilon} \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\{v>0; v^\delta \geq 1\}} \left(M^{-\gamma} v\right)^{\frac{-\delta\varepsilon}{\gamma}-1} M^{-\gamma}\, dv \\ &\quad = \frac{\Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\{v>0; v^\delta \geq 1\}} v^{\frac{-\delta\varepsilon}{\gamma}-1}\, dv = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \end{aligned}$$

Hence, we have (10). □

Definition 1 For $m, n \in \mathbf{N}$, $\alpha, \beta, \lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = \lambda$, $\eta > -1$, $\delta \in \{-1,1\}$, $x = (x_1, \ldots, x_m) \in \mathbf{R}_+^m$, $y = (y_1, \ldots, y_n) \in \mathbf{R}_+^n$, we define two weight functions $\omega(\lambda_1, y)$ and $\varpi(\lambda_2, x)$ as follows:

$$\omega(\lambda_1, y) := \|y\|_\beta^{\lambda_2} \int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta / \|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{1}{\|x\|_\alpha^{m-\delta\lambda_1}}\, dx, \tag{11}$$

$$\varpi(\lambda_2, x) := \|x\|_\alpha^{\delta\lambda_1} \int_{\mathbf{R}_+^n} \frac{|\ln(\|y\|_\beta / \|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{1}{\|y\|_\beta^{n-\lambda_2}}\, dy. \tag{12}$$

By (6), we find

$$\begin{aligned} \omega(\lambda_1, y) &= \|y\|_\beta^{\lambda_2} \int_{\mathbf{R}_+^m} \frac{|\ln(\frac{\|y\|_\beta}{M^\delta [\sum_{i=1}^m (\frac{x_i}{M})^\alpha]^{\delta/\alpha}})|^\eta}{(\max\{M^\delta [\sum_{i=1}^m (\frac{x_i}{M})^\alpha]^{\delta/\alpha}, \|y\|_\beta\})^\lambda} \\ &\quad \times \frac{1}{M^{m-\delta\lambda_1} [\sum_{i=1}^m (\frac{x_i}{M})^\alpha]^{\frac{m-\delta\lambda_1}{\alpha}}}\, dx_1 \cdots dx_m \end{aligned}$$

$$= \|y\|_\beta^{\lambda_2} \lim_{M\to\infty} \frac{M^m\Gamma^m(\frac{1}{\alpha})}{\alpha^m\Gamma(\frac{m}{\alpha})}$$
$$\times \int_0^1 \frac{|\ln(\frac{\|y\|_\beta}{M^\delta u^{\delta/\alpha}})|^\eta}{(\max\{M^\delta u^{\delta/\alpha}, \|y\|_\beta\})^\lambda} \frac{u^{\frac{m}{\alpha}-1}\,du}{M^{m-\delta\lambda_1}u^{\frac{m-\delta\lambda_1}{\alpha}}}$$
$$= \|y\|_\beta^{\lambda_2} \lim_{M\to\infty} \frac{M^{\delta\lambda_1}\Gamma^m(\frac{1}{\alpha})}{\alpha^m\Gamma(\frac{m}{\alpha})} \int_0^1 \frac{|\ln(\frac{\|y\|_\beta}{M^\delta u^{\delta/\alpha}})|^\eta u^{\frac{\delta\lambda_1}{\alpha}-1}}{(\max\{M^\delta u^{\delta/\alpha}, \|y\|_\beta\})^\lambda}\,du$$
$$\overset{v=M\|y\|^{\frac{-1}{\delta}}u^{1/\alpha}}{=} \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_0^\infty \frac{|\ln v^\delta|^\eta v^{\delta\lambda_1-1}}{(\max\{v^\delta,1\})^\lambda}\,dv. \qquad (13)$$

Setting $t = v^{\delta_1}$ in (13), for $\delta = \pm 1$, by simplification, it follows that

$$\omega(\lambda_1, y) = \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_0^\infty \frac{|\ln t|^\eta t^{\lambda_1-1}}{(\max\{t,1\})^\lambda}\,dt$$
$$= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\left[\int_0^1 (-\ln t)^\eta t^{\lambda_1-1}\,dt + \int_1^\infty \frac{(\ln t)^\eta t^{\lambda_1-1}}{t^\lambda}\,dt\right]$$
$$= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_0^1 (-\ln t)^\eta \left(t^{\lambda_1-1} + t^{\lambda_2-1}\right) dt$$
$$\overset{u=-\ln t}{=} \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_\infty^0 u^\eta \left[e^{-u(\lambda_1-1)} + e^{-u(\lambda_2-1)}\right]\left(-e^{-u}\right) du$$
$$= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_0^\infty \left(e^{-\lambda_1 u} + e^{-\lambda_2 u}\right) u^\eta\,du$$
$$= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \left(\frac{1}{\lambda_1^\eta} + \frac{1}{\lambda_2^\eta}\right) \int_0^\infty e^{-v} v^{(\eta+1)-1}\,dv$$
$$= \frac{\Gamma^m(\frac{1}{\alpha})\Gamma(\eta+1)}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \left(\frac{1}{\lambda_1^\eta} + \frac{1}{\lambda_2^\eta}\right). \qquad (14)$$

Lemma 3 *For* $m, n \in \mathbf{N}$, $\alpha, \beta, \lambda_1, \lambda_2, \widetilde{\lambda}_1, \widetilde{\lambda}_2 > 0$, $\lambda_1 + \lambda_2 = \widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, $\eta > -1$, $\delta \in \{-1, 1\}$, *we have*

$$\omega(\lambda_1, y) = K_\alpha(\lambda_1) := \frac{\Gamma^m(\frac{1}{\alpha})\Gamma(\eta+1)}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \left(\frac{1}{\lambda_1^\eta} + \frac{1}{\lambda_2^\eta}\right) \quad \left(y \in \mathbf{R}_+^n\right), \qquad (15)$$
$$\varpi(\lambda_2, x) = K_\beta(\lambda_1) := \frac{\Gamma^n(\frac{1}{\beta})\Gamma(\eta+1)}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \left(\frac{1}{\lambda_1^\eta} + \frac{1}{\lambda_2^\eta}\right) \quad \left(x \in \mathbf{R}_+^m\right), \qquad (16)$$
$$w(\widetilde{\lambda}_1, y) := \|y\|_\beta^{\widetilde{\lambda}_2} \int_{\{x\in\mathbf{R}_+^m; \|x\|_\alpha^\delta \ge 1\}} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{1}{\|x\|_\alpha^{m-\delta\widetilde{\lambda}_1}}\,dx$$
$$= K_\alpha(\widetilde{\lambda}_1)\left(1 - \theta_{\widetilde{\lambda}_1}(y)\right), \qquad (17)$$
$$\theta_{\widetilde{\lambda}_1}(y) := \frac{\widetilde{\lambda}_1^{-\eta} + \widetilde{\lambda}_2^{-\eta}}{\Gamma(\eta+1)} \int_0^{\|y\|_\beta^{-1}} \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt = O\left(\|y\|_\beta^{-\frac{\widetilde{\lambda}_1}{2}}\right) \quad \left(y \in \mathbf{R}_+^n\right). \qquad (18)$$

Proof By (14), we have (15). By the same way, we can obtain (16).

In view of (9) and (13), we find

$$\begin{aligned} w(\widetilde{\lambda}_1, y) &= \|y\|_\beta^{\widetilde{\lambda}_2} \lim_{M\to\infty} \frac{M^{\delta\widetilde{\lambda}_1}\Gamma^m(\frac{1}{\alpha})}{\alpha^m\Gamma(\frac{m}{\alpha})} \int_{\{u>0;u^\delta\geq\frac{1}{M^{\delta\alpha}}\}} \frac{|\ln(\frac{\|y\|_\beta}{M^\delta u^{\delta/\alpha}})|^\eta u^{\frac{\delta\widetilde{\lambda}_1}{\alpha}-1}}{(\max\{M^\delta u^{\delta/\alpha}, \|y\|_\beta\})^\lambda}\,du \\ &\overset{v=M\|y\|^{\frac{-1}{\delta}} u^{\frac{1}{\alpha}}}{=} \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_{\{v>0;v^\delta\geq\|y\|_\beta^{-1}\}} \frac{|\ln v^\delta|^\eta v^{\delta\widetilde{\lambda}_1-1}}{(\max\{v^\delta,1\})^\lambda}\,dv \\ &\overset{t=v^\delta}{=} \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_{\|y\|_\beta^{-1}}^\infty \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt \\ &= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\left[\int_0^\infty \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}\,dt}{(\max\{t,1\})^\lambda} - \int_0^{\|y\|_\beta^{-1}} \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}\,dt}{(\max\{t,1\})^\lambda}\right] \\ &= K_\alpha(\widetilde{\lambda}_1)\bigl(1-\theta_{\widetilde{\lambda}_1}(y)\bigr). \end{aligned}$$

Setting $F(u) := \int_0^u \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt$ ($u\in(0,\infty)$), it follows that $F(u)$ is continuous in $(0,\infty)$. Since

$$\begin{aligned} &\lim_{u\to0^+} \frac{1}{u^{\widetilde{\lambda}_1/2}} \int_0^u \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt \\ &\quad= \lim_{u\to0^+} \frac{2}{\widetilde{\lambda}_1 u^{(\widetilde{\lambda}_1/2)-1}} \frac{|\ln u|^\eta u^{\widetilde{\lambda}_1-1}}{(\max\{u,1\})^\lambda} = \lim_{u\to0^+} \frac{2(-\ln u)^\eta u^{\widetilde{\lambda}_1/2}}{\widetilde{\lambda}_1} = 0, \\ &\lim_{u\to\infty} \frac{1}{u^{\widetilde{\lambda}_1/2}} \int_0^u \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt = 0, \end{aligned}$$

there exists a constant $L>0$ such that

$$0 < \int_0^u \frac{|\ln t|^\eta t^{\widetilde{\lambda}_1-1}}{(\max\{t,1\})^\lambda}\,dt \leq L u^{\frac{\widetilde{\lambda}_1}{2}} \quad \bigl(u\in(0,\infty)\bigr).$$

Then we have

$$0 < \theta_{\widetilde{\lambda}_1}(y) \leq \frac{\widetilde{\lambda}_1^{-\eta}+\widetilde{\lambda}_2^{-\eta}}{\Gamma(\eta+1)} L\|y\|_\beta^{-\frac{\widetilde{\lambda}_1}{2}},$$

namely, $\theta_{\widetilde{\lambda}_1}(y) = O(\|y\|_\beta^{-\frac{\widetilde{\lambda}_1}{2}})$ ($y\in\mathbf{R}_+^n$). Hence, we have (17) and (18). □

Lemma 4 *As the assumptions of Definition* 1, *if* $p\in\mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_m)\geq 0$, $g(y)=g(y_1,\ldots,y_n)\geq0$, *then* (i) *for* $p>1$, *we have the following inequality*:

$$\begin{aligned} J_1 &:= \left\{\int_{\mathbf{R}_+^n} \frac{\|y\|_\beta^{p\lambda_2-n}}{(\omega(\lambda_1,y))^{p-1}}\left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta f(x)}{(\max\{\|x\|_\alpha^\delta,\|y\|_\beta\})^\lambda}\,dx\right]^p dy\right\}^{\frac{1}{p}} \\ &\leq \left[\int_{\mathbf{R}_+^m} \varpi(\lambda_2,x)\|x\|_\alpha^{p(m-\delta\lambda_1)-m} f^p(x)\,dx\right]^{\frac{1}{p}}; \end{aligned} \tag{19}$$

(ii) *for* $0<p<1$, *or* $p<0$, *we have the reverse of* (19).

Proof (i) For $p > 1$, by Hölder's inequality with weight (*cf.* [23]), it follows that

$$\begin{aligned}
&\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} f(x)\,dx \\
&\quad= \int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \left[\frac{\|x\|_\alpha^{(m-\delta\lambda_1)/q}}{\|y\|_\beta^{(n-\lambda_2)/p}} f(x)\right]\left[\frac{\|y\|_\beta^{(n-\lambda_2)/p}}{\|x\|_\alpha^{(m-\delta\lambda_1)/q}}\right] dx \\
&\quad\le \left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{\|x\|_\alpha^{(m-\delta\lambda_1)(p-1)}}{\|y\|_\beta^{n-\lambda_2}} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&\qquad\times \left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{\|y\|_\beta^{(n-\lambda_2)(q-1)}}{\|x\|_\alpha^{m-\delta\lambda_1}}\,dx\right]^{\frac{1}{q}} \\
&\quad= \big(\omega(\lambda_1, y)\big)^{\frac{1}{q}} \|y\|_\beta^{\frac{n}{p}-\lambda_2} \\
&\qquad\times \left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{\|x\|_\alpha^{(m-\delta\lambda_1)(p-1)}}{\|y\|_\beta^{n-\lambda_2}} f^p(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned} \tag{20}$$

Then by Fubini's theorem (*cf.* [24]), we have

$$\begin{aligned}
J_1 &\le \left\{\int_{\mathbf{R}_+^m}\left[\int_{\mathbf{R}_+^{i_0}} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{\|x\|_\alpha^{(m-\delta\lambda_1)(p-1)}}{\|y\|_\beta^{n-\lambda_2}} f^p(x)\,dx\right] dy\right\}^{\frac{1}{p}} \\
&= \left\{\int_{\mathbf{R}_+^m}\left[\int_{\mathbf{R}_+^n} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} \frac{\|x\|_\alpha^{(m-\delta\lambda_1)(p-1)}}{\|y\|_\beta^{n-\lambda_2}}\,dy\right] f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&= \left[\int_{\mathbf{R}_+^m} \varpi(\lambda_2, x)\|x\|_\alpha^{p(m-\delta\lambda_1)-m} f^p(x)\,dx\right]^{\frac{1}{p}}.
\end{aligned} \tag{21}$$

Hence, (19) follows.

(ii) For $0 < p < 1$, or $p < 0$, by the reverse Hölder inequality with weight (*cf.* [23]), we obtain the reverse of (20). Then by Fubini's theorem, we still can obtain the reverse of (19). □

Lemma 5 *As the assumptions of Lemma* 4, *then* (i) *for* $p > 1$, *we have the following inequality equivalent to* (19):

$$\begin{aligned}
I &:= \int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} f(x)g(y)\,dx\,dy \\
&\le \left[\int_{\mathbf{R}_+^m} \varpi(\lambda_2, x)\|x\|_\alpha^{p(m-\delta\lambda_1)-m} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&\quad\times \left[\int_{\mathbf{R}_+^n} \omega(\lambda_1, y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy\right]^{\frac{1}{q}};
\end{aligned} \tag{22}$$

(ii) *for* $0 < p < 1$, *or* $p < 0$, *we have the reverse of* (22) *equivalent to the reverse of* (19).

Proof (i) For $p > 1$, by Hölder's inequality (*cf.* [23]), it follows that

$$\begin{aligned} I &= \int_{\mathbf{R}_+^n} \frac{\|y\|_\beta^{\frac{n}{q}-(n-\lambda_2)}}{(\omega(\lambda_1,y))^{\frac{1}{q}}} \left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta,\|y\|_\beta\})^\lambda} f(x)\,dx\right] \\ &\quad \times \left[(\omega(\lambda_1,y))^{\frac{1}{q}} \|y\|_\beta^{(n-\lambda_2)-\frac{n}{q}} g(y)\right] dy \\ &\leq J_1 \left[\int_{\mathbf{R}_+^n} \omega(\lambda_1,y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy\right]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

Then by (19) we have (22).

On the other hand, assuming that (22) is valid, we set

$$g(y) := \frac{\|y\|_\beta^{p\lambda_2-n}}{(\omega(\lambda_1,y))^{p-1}} \left[\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta f(x)}{(\max\{\|x\|_\alpha^\delta,\|y\|_\beta\})^\lambda}\,dx\right]^{p-1}, \quad y \in \mathbf{R}_+^n.$$

Then it follows that

$$J_1^p = \int_{\mathbf{R}_+^n} \omega(\lambda_1,y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy.$$

If $J_1 = 0$, then (19) is trivially valid; if $J_1 = \infty$, then by (21), (19) keeps the form of equality ($= \infty$). Suppose that $0 < J_1 < \infty$. By (22), we have

$$\begin{aligned} 0 &< \int_{\mathbf{R}_+^n} \omega(\lambda_1,y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy = J_1^p = I \\ &\leq \left[\int_{\mathbf{R}_+^m} \varpi(\lambda_2,x)\|x\|_\alpha^{p(m-\delta\lambda_1)-m} f^p(x)\,dx\right]^{\frac{1}{p}} \\ &\quad \times \left[\int_{\mathbf{R}_+^n} \omega(\lambda_1,y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy\right]^{\frac{1}{q}} < \infty. \end{aligned}$$

Dividing out J_1^{p-1} in the above inequality, it follows that

$$\begin{aligned} J_1 &= \left[\int_{\mathbf{R}_+^n} \omega(\lambda_1,y)\|y\|_\beta^{q(n-\lambda_2)-n} g^q(y)\,dy\right]^{\frac{1}{p}} \\ &\leq \left[\int_{\mathbf{R}_+^m} \varpi(\lambda_2,x)\|x\|_\alpha^{p(m-\delta\lambda_1)-m} f^p(x)\,dx\right]^{\frac{1}{p}}, \end{aligned}$$

and then (19) follows. Hence, (19) and (22) are equivalent.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we have the reverse of (22) equivalent to the reverse of (19). □

3 Main results and operator expressions

Setting functions

$$\Phi(x) := \|x\|_\alpha^{p(m-\delta\lambda_1)-m}, \qquad \Psi(y) := \|y\|_\beta^{q(n-\lambda_2)-n} \quad \left(x \in \mathbf{R}_+^m, y \in \mathbf{R}_+^n\right),$$

we have the following.

Theorem 1 *Suppose that* $m,n\in\mathbf{N}$, $\alpha,\beta,\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=\lambda$, $\eta>-1$, $\delta\in\{-1,1\}$, $p\in\mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_m)\geq 0$, $g(y)=g(y_1,\ldots,y_n)\geq 0$,

$$0<\|f\|_{p,\Phi}=\left[\int_{\mathbf{R}_+^m}\|x\|_\alpha^{p(m-\delta\lambda_1)-m}f^p(x)\,dx\right]^{\frac{1}{p}}<\infty,$$

$$0<\|g\|_{q,\Psi}=\left[\int_{\mathbf{R}_+^n}\|y\|_\beta^{q(n-\lambda_2)-n}g^q(y)\,dy\right]^{\frac{1}{q}}<\infty.$$

(i) *For* $p>1$, *we have the following equivalent inequalities with the best possible constant factor* $K(\lambda_1)$:

$$I=\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta,\|y\|_\beta\})^\lambda}f(x)g(y)\,dx\,dy<K(\lambda_1)\|f\|_{p,\Phi}\|g\|_{q,\Psi},\tag{24}$$

$$J:=\left\{\int_{\mathbf{R}_+^n}\|y\|_\beta^{p\lambda_2-n}\left[\int_{\mathbf{R}_+^m}\frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta,\|y\|_\beta\})^\lambda}f(x)\,dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$<K(\lambda_1)\|f\|_{p,\Phi},\tag{25}$$

where we define the constant factor as follows:

$$K(\lambda_1):=\left(K_\beta(\lambda_1)\right)^{\frac{1}{p}}\left(K_\alpha(\lambda_1)\right)^{\frac{1}{q}}$$
$$=\left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right)^{\frac{1}{q}}\left(\frac{1}{\lambda_1^\eta}+\frac{1}{\lambda_2^\eta}\right)\Gamma(\eta+1).$$

(ii) *For* $0<p<1$, *or* $p<0$, *we still have the equivalent reverses of* (24) *and* (25) *with the same best constant factor* $K(\lambda_1)$.

Proof (i) For $p>1$, by the conditions, we can prove that (20) takes the form of strict inequality. Otherwise, if (20) takes the form of equality for $y\in\mathbf{R}_+^n$, then there exist constants A and B, which are not all zero, satisfying

$$A\frac{\|x\|_\alpha^{(m-\delta\lambda_1)(p-1)}}{\|y\|_\beta^{n-\lambda_2}}f^p(x)=B\frac{\|y\|_\beta^{(n-\lambda_2)(q-1)}}{\|x\|_\alpha^{m-\delta\lambda_1}}\quad\text{a.e. in } x\in\mathbf{R}_+^m.\tag{26}$$

If $A=0$, then $B=0$, which is impossible; if $A\neq 0$, then (26) reduces to

$$\|x\|_\alpha^{p(m-\delta\lambda_1)-m}f^p(x)=\frac{B\|y\|_\beta^{q(n-\lambda_2)}}{A\|x\|_\alpha^m}\quad\text{a.e. in } x\in\mathbf{R}_+^{i_0},$$

which contradicts the fact that $0<\|f\|_{p,\Phi}<\infty$. In fact, by (9), it follows that $\int_{\mathbf{R}_+^m}\|x\|_\alpha^{-m}\,dx=\infty$. Hence, (20) takes the form of strict inequality. So does (19). By (15) and (16), we have (25).

In view of (23) (putting $\omega(\lambda_1,y)=1$), we still have

$$I\leq J\left[\int_{\mathbf{R}_+^n}\|y\|_\beta^{q(n-\lambda_2)-n}g^q(y)\,dy\right]^{\frac{1}{q}}.\tag{27}$$

Then by (27) and (25), we have (24). It is evident that by Lemma 5 and the assumptions, (24) and (25) are also equivalent.

For $0 < \varepsilon < \frac{p\lambda_1}{2}$, we set $\widetilde{f}(x)$, $\widetilde{g}(y)$ as follows:

$$\widetilde{f}(x) := \begin{cases} 0, & 0 < \|x\|_\alpha^\delta < 1, \\ \|x\|_\alpha^{\delta(\lambda_1 - \frac{\varepsilon}{p}) - m}, & \|x\|_\alpha^\delta \geq 1, \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} 0, & 0 < \|y\|_\beta < 1, \\ \|y\|_\beta^{(\lambda_2 - \frac{\varepsilon}{q}) - n}, & \|y\|_\beta \geq 1. \end{cases}$$

Then, for $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (\frac{\lambda_1}{2}, \lambda)$ $(\subset (0, \lambda))$, by (10) we find

$$\begin{aligned} \|\widetilde{f}\|_{p,\Phi}\|\widetilde{g}\|_{q,\Psi} &= \left(\int_{\{x\in\mathbf{R}_+^m;\|x\|_\alpha^\delta\geq 1\}} \|x\|_\alpha^{-m-\delta\varepsilon}\,dx\right)^{\frac{1}{p}} \left(\int_{\{y\in\mathbf{R}_+^n;\|y\|_\beta\geq 1\}} \|y\|_\beta^{-n-\varepsilon}\,dy\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left(\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right)^{\frac{1}{p}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} 0 &\leq \int_{\{y\in\mathbf{R}_+^n;\|y\|_\beta\geq 1\}} \|y\|_\beta^{-n-\varepsilon} O\big(\|y\|_\beta^{-\frac{\widetilde{\lambda}_1}{2}}\big)\,dy \\ &\leq L_1 \int_{\{y\in\mathbf{R}_+^n;\|y\|_\beta\geq 1\}} \|y\|_\beta^{-n-(\varepsilon+\frac{\widetilde{\lambda}_1}{2})}\,dy = \frac{L_1\Gamma^n(\frac{1}{\beta})}{(\varepsilon+\frac{\widetilde{\lambda}_1}{2})\beta^{n-1}\Gamma(\frac{n}{\beta})} \\ &\leq \frac{L_1\Gamma^n(\frac{1}{\beta})}{(\varepsilon+\frac{\lambda_1}{4})\beta^{n-1}\Gamma(\frac{n}{\beta})} < \infty \quad (L_1 > 0), \end{aligned}$$

and then by (17) and (18) it follows that

$$\begin{aligned} \widetilde{I} &:= \int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda}\widetilde{f}(x)\widetilde{g}(y)\,dx\,dy \\ &= \int_{\{y\in\mathbf{R}_+^n;\|y\|_\beta\geq 1\}} \|y\|_\beta^{-n-\varepsilon} w(\widetilde{\lambda}_1, y)\,dy \\ &= K_\alpha(\widetilde{\lambda}_1)\int_{\{y\in\mathbf{R}_+^n;\|y\|_\beta\geq 1\}} \|y\|_\beta^{-n-\varepsilon}\big(1 - O\big(\|y\|_\beta^{-\frac{\widetilde{\lambda}_1}{2}}\big)\big)\,dy \\ &= \frac{1}{\varepsilon}K_\alpha(\widetilde{\lambda}_1)\left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} - \varepsilon O_{\lambda_1}(1)\right). \end{aligned}$$

If there exists a constant $K \leq K(\lambda_1)$, such that (24) is valid when replacing $K(\lambda_1)$ by K, then in particular we have

$$\begin{aligned} &\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\Gamma(\eta+1)\left(\frac{1}{\widetilde{\lambda}_1^\eta} + \frac{1}{\widetilde{\lambda}_2^\eta}\right)\left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} - \varepsilon O_{\lambda_1}(1)\right) \\ &\quad\leq \varepsilon\widetilde{I} < \varepsilon K\|\widetilde{f}\|_{p,\Phi}\|\widetilde{g}\|_{q,\Psi} = K\left(\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{q}}, \end{aligned}$$

and then $K(\lambda_1) \leq K(\varepsilon \to 0^+)$. Hence $K = K(\lambda_1)$ is the best possible constant factor of (24).

By the equivalency, we can prove that the constant factor $K(\lambda_1)$ in (25) is best possible. Otherwise, we would reach a contradiction by (27) that the constant factor $K(\lambda_1)$ in (24) is not best possible.

(ii) For $0 < p < 1$, or $p < 0$, by the same way, we still can obtain the equivalent reverses of (24) and (25) with the same best constant factor. □

As the assumptions of Theorem 1, for $p > 1$, in view of $J < K(\lambda_1)\|f\|_{p,\Phi}$, we give the following definition.

Definition 2 We define a multidimensional Hilbert-type integral operator

$$T : \mathbf{L}_{p,\Phi}(\mathbf{R}_+^m) \to \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^n)$$

as follows:

For $f \in \mathbf{L}_{p,\Phi}(\mathbf{R}_+^m)$, there exists a unique representation $Tf \in \mathbf{L}_{p,\Psi^{1-p}}(\mathbf{R}_+^n)$, satisfying

$$(Tf)(y) := \int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} f(x)\,dx \quad (y \in \mathbf{R}_+^n). \tag{28}$$

For $g \in \mathbf{L}_{q,\Psi}(\mathbf{R}_+^n)$, we define the following formal inner product of Tf and g as follows:

$$(Tf, g) := \int_{\mathbf{R}_+^n} \int_{\mathbf{R}_+^m} \frac{|\ln(\|y\|_\beta/\|x\|_\alpha^\delta)|^\eta}{(\max\{\|x\|_\alpha^\delta, \|y\|_\beta\})^\lambda} f(x)g(y)\,dx\,dy. \tag{29}$$

Then by Theorem 1, for $p > 1$, $0 < \|f\|_{p,\Phi}, \|g\|_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(Tf, g) < K(\lambda_1)\|f\|_{p,\Phi}\|g\|_{q,\Psi}, \tag{30}$$

$$\|Tf\|_{p,\Psi^{1-p}} < K(\lambda_1)\|f\|_{p,\Phi}. \tag{31}$$

It follows that T is bounded with

$$\|T\| := \sup_{f(\neq\theta)\in\mathbf{L}_{p,\Phi}(\mathbf{R}_+^m)} \frac{\|Tf\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi}} \le K(\lambda_1).$$

Since the constant factor $K(\lambda_1)$ in (31) is best possible, we have

$$\|T\| = K(\lambda_1) = \left(\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} \left(\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right)^{\frac{1}{q}} \left(\frac{1}{\lambda_1^\eta} + \frac{1}{\lambda_2^\eta}\right)\Gamma(\eta+1). \tag{32}$$

4 Some corollaries

We also set functions

$$\widetilde{\Phi}(x) := \|x\|_\alpha^{p(m-\lambda_2)-m}, \qquad \widehat{\Phi}(x) := \|x\|_\alpha^{p(m-\lambda_1)-m} \quad (x \in \mathbf{R}_+^m).$$

For $\delta = -1$ in Theorem 1, setting $F(x) = \|x\|_\alpha^\lambda f(x)$, by simplification, we have the following.

Corollary 1 *Suppose that* $m,n\in\mathbf{N}$, $\alpha,\beta,\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=\lambda$, $\eta>-1$, $p\in\mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $F(x)=F(x_1,\ldots,x_m)\geq 0$, $g(y)=g(y_1,\ldots,y_n)\geq 0$, $0<\|F\|_{p,\widetilde{\Phi}},\|g\|_{q,\Psi}<\infty$. (i) *For* $p>1$, *we have the following equivalent inequalities with the non-homogeneous kernel and the best possible constant factor* $K(\lambda_1)$:

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\frac{|\ln(\|x\|_\alpha\|y\|_\beta)|^\eta}{(\max\{1,\|x\|_\alpha\|y\|_\beta\})^\lambda}F(x)g(y)\,dx\,dy<K(\lambda_1)\|F\|_{p,\widetilde{\Phi}}\|g\|_{q,\Psi},\tag{33}$$

$$\left[\int_{\mathbf{R}_+^n}\|y\|_\beta^{p\lambda_2-n}\left(\int_{\mathbf{R}_+^m}\frac{|\ln(\|x\|_\alpha\|y\|_\beta)|^\eta F(x)}{(\max\{1,\|x\|_\alpha\|y\|_\beta\})^\lambda}\,dx\right)^p dy\right]^{\frac{1}{p}}<K(\lambda_1)\|F\|_{p,\widetilde{\Phi}};\tag{34}$$

(ii) *for* $0<p<1$, *or* $p<0$, *we still have the equivalent reverses of* (33) *and* (34) *with the same best constant factor* $K(\lambda_1)$.

For $\delta=1$ in Theorem 1, we have the following.

Corollary 2 *Suppose that* $m,n\in\mathbf{N}$, $\alpha,\beta,\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=\lambda$, $\eta>-1$, $p\in\mathbf{R}\backslash\{0,1\}$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x)=f(x_1,\ldots,x_m)\geq 0$, $g(y)=g(y_1,\ldots,y_n)\geq 0$, $0<\|f\|_{p,\widehat{\Phi}},\|g\|_{q,\Psi}<\infty$. (i) *For* $p>1$, *we have the following equivalent inequalities with the homogeneous kernel of degree* $-\lambda$ *and the best possible constant factor* $K(\lambda_1)$:

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\frac{|\ln(\|y\|_\beta/\|x\|_\alpha)|^\eta}{(\max\{\|x\|_\alpha,\|y\|_\beta\})^\lambda}f(x)g(y)\,dx\,dy<K(\lambda_1)\|f\|_{p,\widehat{\Phi}}\|g\|_{q,\Psi},\tag{35}$$

$$\left\{\int_{\mathbf{R}_+^n}\|y\|_\beta^{p\lambda_2-n}\left[\int_{\mathbf{R}_+^m}\frac{|\ln(\|y\|_\beta/\|x\|_\alpha)|^\eta f(x)}{(\max\{\|x\|_\alpha,\|y\|_\beta\})^\lambda}\,dx\right]^p dy\right\}^{\frac{1}{p}}<K(\lambda_1)\|f\|_{p,\widehat{\Phi}};\tag{36}$$

(ii) *for* $0<p<1$, *or* $p<0$, *we still have the equivalent reverses of* (35) *and* (36) *with the same best constant factor* $K(\lambda_1)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. ZH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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