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Norms and essential norms of composition operators from H^∞ to general weighted Bloch spaces in the polydisk

Maofa Wang[1*] and Huanqin Wu[2]

* Correspondence: mfwang.math@whu.edu.cn
[1]School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, PR China
Full list of author information is available at the end of the article

Abstract

Let U^n be the unit polydisk of $\mathbb{C}^n$ and ϕ a holomorphic self-map of U^n. $H^\infty(U^n)$ and $\mathcal{B}^\alpha_{\log}(U^n)$ denote the space of bounded holomorphic functions and the space of general weighted Bloch functions defined on U^n, respectively, where $\alpha>0$. This paper gives some estimates of the norm and essential norm of the composition operator C_ϕ induced by ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. As applications, some characterizations of the boundedness and compactness of C_ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$ are obtained. Moreover, we also characterize the weak compactness of the composition operator C_ϕ.

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1 Introduction

Let D be a bounded homogeneous domain in $\mathbb{C}^n$, $H(D)$ the class of all holomorphic functions on D. For ϕ, a holomorphic self-map of D, the linear operator defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(D),$$

is called the composition operator with symbol ϕ. The study of composition operators is fundamental in the study of Banach and Hilbert spaces of holomorphic functions. We refer to the books [1] and [2] for an overview of some classical results on the theory of composition operators.

Let $K(z, z)$ be the Bergman kernel function of D, and the Bergman metric $H_z(u, u)$ in D is defined by

$$H_z(u,\ u) = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z_k}} u_j \overline{u_k},$$

where $z = (z_1, ..., z_n) \in D$ and $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$. A function $f \in H(D)$ is said to be a Bloch function if $\beta_f = \sup_{z\in D} Q_f(z)$ is finite, where

$$Q_f(z) = \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|(\nabla f)(z)u|}{H_z^{1/2}(u,u)},$$

$(\nabla f)(z)u = < \nabla f(z), \bar{u} > = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z)u_k$. By fixing a base point $z_0 \in D$, the Bloch space $\mathcal{B}(D)$ of all Bloch functions on D is a Banach space under the norm $\| f\|_{\mathcal{B}} = |f(z_0)| + \beta_f$[3]. For convenience, we assume the bounded homogeneous domain D to contain the origin and take $z_0 = 0$. In [3], Timoney proved that the space $H^\infty(D)$ of bounded holomorphic functions on a bounded homogeneous domain D is a subspace of $\mathcal{B}(D)$ and for each $f \in H^\infty(D)$, $\| f\|_{\mathcal{B}} \le C_D \| f\|_\infty$, where $C_D$is a constant depending only on the domain D and $\| f\|_\infty = \sup_{z\in D} |f(z)|$.

Let U^nbe the unit polydisk of $\mathbb{C}^n$. Timony [3] showed that $f \in \mathcal{B}(D)$ if and only if

$$\sup_{z\in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2) < \infty,$$

and $|f(0)| + \sup_{z\in U^n} \sum_{k=1}^{n} |\frac{\partial f}{\partial z_k}(z)|(1 - |z_k|^2)$ is equivalent to the Bloch norm $\| f\|_{\mathcal{B}}$. This characterization was the starting point for introducing α-Bloch spaces. For $\alpha > 0$, the α -Bloch space $\mathcal{B}^\alpha(U^n)$ is defined as follows.

$$\mathcal{B}^\alpha(U^n) = \left\{ f \in H(U^n) : \sup_{z\in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha < \infty \right\}.$$

Recently, Li and Liu [4] introduced the notation of general weighted Bloch spaces (Stević called these the logarithmic Bloch-type spaces in [5]) in polydisk. For $\alpha > 0$, a function $f \in H(U^n)$ is said to belong to the general weighted Bloch space $\mathcal{B}^\alpha_{\log}(U^n)$ if

$$\sup_{z\in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha \log \frac{2}{1 - |z_k|^2} < \infty.$$

It is easy to show that $\mathcal{B}^\alpha_{\log}(U^n)$ is a Banach space with the norm

$$\| f\|_{\mathcal{B}^\alpha_{\log}} = |f(0)| + \sup_{z\in U^n} \sum_{k=1}^{n} \left| \frac{\partial f}{\partial z_k}(z) \right| (1 - |z_k|^2)^\alpha \log \frac{2}{1 - |z_k|^2}.$$

Composition operators on various Bloch-type spaces have been studied extensively by many authors. For the unit disk $U \subset \mathbb{C}$, Madigan and Matheson [6] proved that $C_\phi$is always bounded on $\mathcal{B}(U)$. They also gave some sufficient and necessary conditions that $C_\phi$is compact on $\mathcal{B}(U)$. Since then, there were many authors generalizing the results in [6] to the unit ball, polydisk and other classical symmetric domains, see, for example, [7-17]. At the same time, there were also many papers dealing with the composition operators between Bloch-type spaces and bounded holomorphic function spaces, refer to [18,19] and the references therein for the details. Specially, Li and Liu [4] stated and proved the corresponding boundedness and compactness characterizations for $C_\phi$from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. But there is a little gap in the proof [[4], line 17, p. 1637]. In this paper, we apply methods developed by Montes-Rodriguez [9] to give some estimates of the norm and essential norm of $C_\phi$from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$. Recall that the essential norm $||T||_e$of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the space of compact operators from X to Y. Notice that $||T||_e = 0$ if and only if T is compact, so that estimates on $||T||_e$lead to conditions for T to be compact. For convenience, we define $||T||_e = ||T|| = \infty$ for any unbounded linear operator T. As an application of our estimates, we obtain the main results in [4] with new proofs. In addition, we also show the equivalence of the compactness and weak compactness of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one occurrence to the others.

2 The norm of C_ϕ

In this section, we give the following estimate of the norm of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$.

Theorem 1. *Let $\alpha > 0$ and $\phi = (\phi_1, ..., \phi_n)$ be a holomorphic self-map of the unit polydisk U^n, then*

$$\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2} \lesssim \| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|$$
$$\lesssim 1+\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}.$$

Here and in the sequel, the symbol $A \lesssim B$(or $B \gtrsim A$) means that $A \le CB$ for some positive constant C independent of A and B. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

Proof. For the lower estimate: $\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| \gtrsim \sup_{z\in U^n}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}$.
If $\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| = \infty$, then the result is trivially true. Now suppose $C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is bounded. For any fixed $w \in U^n$ and $k \in \{1, ..., n\}$, take the following test function

$$f(z) = \frac{(1-|(\varphi_k(w)|^2)^\alpha}{(1-\overline{(\varphi_k(w)}z_k)^\alpha},\ z\in U^n.$$

Then, $f \in H^\infty(U^n)$ and $||f||_\infty \le 2^\alpha$. Fix any $\delta \in (0, 1)$. If $|\phi_k(w)| \ge \delta$, then

$$\begin{aligned}
\infty &> \| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| \gtrsim \| C_\varphi f\|_{\mathcal{B}^\alpha_{\log}}\\
&\ge \sup_{z\in U^n}\sum_{j=1}^{n}\left|\frac{\partial(f\circ\varphi)}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2}\\
&= \sup_{z\in U^n}\sum_{j=1}^{n}\left|\frac{\partial f}{\partial w_k}(\varphi(z))\frac{\partial\varphi_k}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2}\\
&= \sup_{z\in U^n}\sum_{j=1}^{n}\left|\alpha\overline{\varphi_k(w)}\frac{(1-|(\varphi_k(w)^2|)^\alpha}{(1-\overline{\varphi_k(w)}\varphi_k(z))^{\alpha+1}}\frac{\partial\varphi_k}{\partial z_j}(z)\right|(1-|z_j|^2)^\alpha\log\frac{2}{1-|z_j|^2}\\
&\ge \sum_{j=1}^{n}\alpha\left|\varphi_k(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\log\frac{2}{1-|w_j|^2}\\
&\gtrsim \sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2}.
\end{aligned}$$

If $|\phi_k(w)| < \delta$, then

$$\sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|\frac{(1-|w_j|^2)^\alpha}{1-|(\varphi_k(w)|^2}\log\frac{2}{1-|w_j|^2} \lesssim \sum_{j=1}^{n}\left|\frac{\partial\varphi_k}{\partial z_j}(w)\right|(1-|w_j|^2)^\alpha\log\frac{2}{1-|w_j|^2}$$
$$\le \| C_\varphi z_k\|_{\mathcal{B}^\alpha_{\log}} \le \| C_\varphi :\ H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|\| z_k\|_\infty = \| C_\varphi :\ H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\| < \infty.$$

That is, for any $w \in U^n$,

$$\sum_{j=1}^{n} \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right| \frac{(1-|w_j|^2)^{\alpha}}{1-|(\varphi_k(w)|^2} \log \frac{2}{1-|w_j|^2} \lesssim \| C_\varphi : H^\infty(U^n) \to \mathcal{B}^{\alpha}_{\log}(U^n) \| .$$

Since $k \in \{1, ..., n\}$ is arbitrary, so

$$\sup_{w \in U^n} \sum_{k,j=1}^{n} \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right| \frac{(1-|w_j|^2)^{\alpha}}{1-|(\varphi_k(w)|^2} \log \frac{2}{1-|w_j|^2} \lesssim \| C_\varphi : H^\infty(U^n) \to \mathcal{B}^{\alpha}_{\log}(U^n) \| .$$

For the upper estimate: $\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^{\alpha}_{\log}(U^n) \| \lesssim 1 + \sup_{z \in U^n} \sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}$.
We also assume $\sup_{z \in U^n} \sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} < \infty$, since for the other case nothing needs to be proven. For any $f \in H^\infty(U^n)$,

$$\begin{aligned}
&\sum_{k=1}^{n} \left| \frac{\partial (f \circ \varphi)}{\partial z_k}(z) \right| (1-|z_k|^2)^{\alpha} \log \frac{2}{1-|z_k|^2} \\
&\le \sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{\partial f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1-|z_k|^2)^{\alpha} \log \frac{2}{1-|z_k|^2} \\
&\le \sum_{l=1}^{n} \left| \frac{\partial f}{\partial w_l}(\varphi(z)) \right| (1-|\varphi_l(z)|^2) \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \\
&\lesssim \left(\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \right) \| f \|_{\mathcal{B}} \\
&\lesssim \left(\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \right) \| f \|_{\infty},
\end{aligned}$$

where $\| f \|_{\mathcal{B}} \lesssim \| f \|_\infty$ is used in the last line above.
Since

$$\begin{aligned}
\| C_\varphi f \|_{\mathcal{B}^{\alpha}_{\log}} &= |f(\varphi(0))| + \sup_{z \in U^n} \sum_{k=1}^{n} \left| \frac{\partial (f \circ \varphi)}{\partial z_k}(z) \right| (1-|z_k|^2)^{\alpha} \log \frac{2}{1-|z_k|^2} \\
&\lesssim \left(1 + \sup_{z \in U^n} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \right) \| f \|_{\infty}.
\end{aligned}$$

Taking the supremum over all $f \in H^\infty(U^n)$ with $|| f ||_\infty \le 1$, we have

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^{\alpha}_{\log}(U^n) \| \lesssim 1 + \sup_{z \in U^n} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2},$$

which completes the proof.
The following corollary is obtained immediately from Theorem 1.
Corollary 2. *Let* $\phi = (\phi_1, ..., \phi_n)$ *be a holomorphic self-map of* U^n *and* $\alpha > 0$. *Then,* $C_\varphi : H^\infty(U^n) \to \mathcal{B}^{\alpha}_{\log}(U^n)$ *is bounded if and only if*

$$\sup_{z \in U^n} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^{\alpha}}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} < \infty.$$

3 The essential norm of C_ϕ

This section mainly gives the following estimate of the essential norm of C_ϕ from $H^\infty(U^n)$ to $\mathcal{B}^\alpha_{\log}(U^n)$.

Theorem 3. *Let* $\phi = (\phi_1, ..., \phi_n)$ *be a holomorphic self-map of* U^n *and* $\alpha > 0$, *then*

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)\|_e \sim \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k,l=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

Proof. For the lower estimate:

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k,l=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

It is trivial when $C_\phi$is unbounded. So we assume that $C_\phi$is bounded. By Corollary 2,

$$\sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} \lesssim 1, \quad \forall l = 1, \ldots, n. \tag{1}$$

Take $f_m(z) = z_1^m (m \geq 2)$, then $||f_m||_\infty = 1$ and $f_m(z)$ converge to zero uniformly on any compact subset of U^n. So $\| Kf_m\|_{\mathcal{B}^\alpha_{\log}} \to 0$ for any compact operator $K : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$. Then

$$\begin{aligned}
\| C_\varphi - K \| &\geq \limsup_{m\to\infty} \| (C_\varphi - K)f_m\|_{\mathcal{B}^\alpha_{\log}} = \limsup_{m\to\infty} \| C_\varphi f_m\|_{\mathcal{B}^\alpha_{\log}} \\
&\geq \limsup_{m\to\infty} \sup_{z\in U^n} \sum_{k=1}^n \left|\frac{\partial (f_m\circ\varphi)}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&\geq \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial (f_m\circ\varphi)}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&= \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|m\varphi_1^{m-1}(z)\frac{\partial \varphi_1}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \log\frac{2}{1-|z_k|^2} \\
&= \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2} m|(\varphi_1(z)|^{m-1}(1-|(\varphi_1(z)|^2) \\
&\geq \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2} \liminf_{m\to\infty}\min_{z\in A_m} m|\varphi_1(z)|^{m-1}(1-|(\varphi_1(z)|^2),
\end{aligned} \tag{2}$$

where $A_m = \{z \in U^n : r_m \leq |\phi_1(z)| \leq r_{m+1}\}$, $r_m = (\frac{m-1}{m+1})^{1/2}$. Since $y = mx^{m-1}(1 - x^2)$, $x \in [0, 1)$, is increasing on $[0, r_m]$ and decreasing on $[r_m, 1)$, $\min_{z\in A_m} m|\varphi_1(z)|^{m-1}(1-|\varphi_1(z)|^2) = (\frac{m}{m+2})^{\frac{m-1}{2}} \frac{2m}{m+2} \to \frac{2}{e}(\text{as } m\to\infty)$. From (2), we have

$$\| C_\varphi - K \| \gtrsim \limsup_{m\to\infty} \sup_{z\in A_m} \sum_{k=1}^n \left|\frac{\partial \varphi_1}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_1(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

It is from (1) that

$$\lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} = a_l < \infty, \quad \forall l = 1, \ldots, n.$$

Then, for any $\varepsilon > 0$, there is $\delta_0 \in (0, 1)$ such that

$$\sum_{k=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} > a_l - \varepsilon,$$

whenever *dist* (ϕ(z), ∂U^n) $<\delta_0$. Again $r_m \uparrow 1$, so for m large enough,

$$\sup_{z\in A_m}\sum_{k=1}^{n}\left|\frac{\partial\varphi_1}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_1(z)|^2}\log\frac{2}{1-|z_k|^2} > a_1-\varepsilon.$$

So $||C_\phi - K|| \gtrsim a_1 - \varepsilon$. Since K is arbitrary,

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim a_1-\varepsilon.$$

Similarly, considering the functions $f(z) = z_l^m$, $l = 2, \ldots, n$, we also have

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim a_l-\varepsilon.$$

Thus,

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \gtrsim \sum_{l=1}^{n} a_l-\varepsilon,$$

$$=\lim_{\delta\to 0}\sup_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}-\varepsilon.$$

Again ε is arbitrary, and we obtain the desired lower estimate.

For the upper estimate:

$$\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e \lesssim \lim_{\delta\to 0}\sup_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}.$$

If $\lim\limits_{\delta\to 0}\sup\limits_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}=\infty$, then the estimate is trivial too. Now we suppose $\lim\limits_{\delta\to 0}\sup\limits_{dist(\varphi(z),\partial U^n)<\delta}\sum_{k,l=1}^{n}|\frac{\partial\varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2}\log\frac{2}{1-|z_k|^2}=\infty$, then $C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is bounded by Corollary 2. Define the operators $K_m (m \geq 2)$ as follows

$$K_m f(z) = f\left(\frac{m-1}{m}z\right).$$

It is easy to see that K_m: $H^\infty(U^n) \to H^\infty(U^n)$ is compact since $K_m$maps every bounded sequence in $H^\infty(U^n)$ converging to zero on compact subsets of $U^n$to the sequence converging to zero in norm of $H^\infty(U^n)$. In addition, $||I - K_m: H^\infty(U^n) \to H^\infty(U^n)|| \leq 2$. Therefore, $C_\varphi K_m : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)$ is compact. Then

$$\begin{aligned}\| C_\varphi : H^\infty(U^n)\to\mathcal{B}^\alpha_{\log}(U^n)\|_e &\leq \| C_\varphi - C_\varphi K_m\| = \| C_\varphi(I-K_m)\|\\ &= \sup_{\|f\|_\infty\leq 1}\| C_\varphi(I-K_m)f\|_{\mathcal{B}^\alpha_{\log}} = \sup_{\|f\|_\infty\leq 1}|(I-K_m)f(\varphi(0))|\\ &+\sup_{\|f\|_\infty\leq 1}\sup_{z\in U^n}\sum_{k,l=1}^{n}\left|\frac{\partial(I-K_m)f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\right|(1-|z_k|^2)^\alpha\log\frac{2}{1-|z_k|^2} = I_1+I_2.\end{aligned}$$

Where $I_1 = \sup\limits_{\|f\|_\infty\leq 1}|(I-K_m)f(\varphi(0))|$ and $I_2 = \sup\limits_{\|f\|_\infty\leq 1}\sup\limits_{z\in U^n}\sum\limits_{k,l=1}^{n}|\frac{\partial(I-K_m)f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)|(1-|z_k|^2)^\alpha\log\frac{2}{1-|z_k|^2}$.

Fix $\delta \in (0, 1)$ and let $G_1 = \{z \in U^n: dist(\phi(z), \partial U^n) <\delta\}$, $G_2 = U^n \backslash G_1 = \{z \in U^n: dist(\phi(z), \partial U^n) \geq \delta\}$, which is a compact subset of U^n.

$$
\begin{aligned}
I_2 = &\sup_{\|f\|_\infty \le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1-|z_k|^2)^\alpha \log \frac{2}{1-|z_k|^2} \\
&+ \sup_{\|f\|_\infty \le 1} \sup_{z\in G_2} \sum_{k,l=1}^{n} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1-|z_k|^2)^\alpha \log \frac{2}{1-|z_k|^2} = J_1 + J_2.
\end{aligned}
$$

Where

$$
\begin{aligned}
J_1 = &\sup_{\|f\|_\infty \le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right| (1-|\varphi_l(z)|^2) \\
&\lesssim \sup_{\|f\|_\infty \le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| (I-K_m)f \|_{\mathcal{B}} \\
&\lesssim \sup_{\|f\|_\infty \le 1} \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| (I-K_m)f \|_{\infty} \\
&\le \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} \| I-K_m \| \\
&\lesssim \sup_{z\in G_1} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}.
\end{aligned}
$$

And

$$
\begin{aligned}
J_2 = &\sup_{\|f\|_\infty \le 1} \sup_{z\in G_2} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|(\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2} | \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) | (1-|(\varphi_l(z)|^2) \\
&\lesssim \sum_{l=1}^{n} \sup_{\|f\|_\infty \le 1} \sup_{z\in G_2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right|.
\end{aligned}
$$

It is clear that the sequence of operators $\{I - K_m\}_m$satisfies $\lim_{m\to\infty}(I-K_m)f = 0$ for each $f \in H(U^n)$, and the space $H(U^n)$ endowed with the compact open topology τ is a Fréchet space. Further, D_j: $(H(U^n), \tau) \to (H(U^n), \tau)$ defined by $D_j f = \frac{\partial f}{\partial z_j}$ is a contionuous linear operator. Therefore, by the Banach-Steinhaus theorem, the sequence $\{D_j^\circ (I - K_m)\}_m$converges to zero uniformly on compact subsets of $(H(U^n), \tau)$. Since, by Montel's normal theorem, the closed unit ball of $H^\infty(U^n)$ is a compact subset of $(H(U^n), \tau)$, we conclude that

$$
\lim_{m\to\infty} \sup_{\|f\|_\infty \le 1} \sup_{z\in G_2} \left| \frac{\partial (I-K_m)f}{\partial w_l}(\varphi(z)) \right| = 0, \; l = 1, \ldots, n.
$$

Thence, $J_2 \to 0$ (as $m \to \infty$).

Similarly, we know that $I_1 = \sup_{\|f\|_\infty \le 1} |(I-K_m)f(\varphi(0))| \to 0$, (as $m \to \infty$).

Consequently,

$$
\begin{aligned}
\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n) \|_e &\le \limsup_{m\to\infty} \| C_\varphi (I-K_m) \| \\
&\le \limsup_{m\to\infty} I_1 + \limsup_{m\to\infty} J_1 + \limsup_{m\to\infty} J_2 \\
&\lesssim \sup_{dist(\varphi(z), \partial U^n) < \delta} \sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log \frac{2}{1-|z_k|^2}.
\end{aligned}
$$

Thus

$$\| C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)\|_e \le \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial U^n)<\delta} \sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2}.$$

The proof is complete.

As an application, we have the following corollary.

Corollary 4. *Let* $\phi = (\phi_1, ..., \phi_n)$ *be a holomorphic self-map of* U^n *and* $\alpha > 0$. *Then the following are equivalent.*

(1) $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ *is compact.*

(2) $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ *is weakly compact.*

(3) $\lim_{\varphi(z)\to\partial U^n} \sum_{k,l=1}^{n} |\frac{\partial\varphi_l}{\partial z_k}(z)| \frac{(1-|z_k|^2)^\alpha}{1-|\varphi_l(z)|^2} \log\frac{2}{1-|z_k|^2} = 0.$

Proof. (1)⇒ (2) is obvious, and (3)⇒ (1) follows immediately from Theorem 3. So it suffices to prove (2)⇒ (3). Now assume that $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ is weakly compact. If (3) is not true, then there is a sequence $\{z^j\} \subset U^n$ and $\varepsilon_0 > 0$ such that $w^j = \phi(z^j) \to \partial U^n$ (as $j \to \infty$) together with

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z^j)\right| \frac{(1-|z_k^j|^2)^\alpha}{1-|(\varphi_l(z^j)|^2} \log\frac{2}{1-|z_k^j|^2} \ge \varepsilon_0, \tag{3}$$

for each $j \ge 1$. Since C_ϕ is weakly compact, C_ϕ is bounded. Then, by Corollary 2,

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z^j)\right| \frac{(1-|z_k^j|^2)^\alpha}{1-|\varphi_l(z^j)|^2} \log\frac{2}{1-|z_k^j|^2} \lesssim 1.$$

Extracting a subsequence of $\{z^j\}$, if needed, we may assume that $\lim_{j\to\infty} |\varphi_l(z^j)|$ exists for every l and

$$\left|\frac{\partial \varphi_l}{\partial z_k}(z^j)\right| \frac{(1-|z_k^j|^2)^\alpha}{1-|(\varphi_l(z^j)|^2} \log\frac{2}{1-|z_k^j|^2} \to a_{lk} \in [0,\ \infty), \quad (\text{as } j \to \infty).$$

From (3), there are k_0 and l_0 such that $a_{l_0 k_0} > 0$, i.e.,

$$\left|\frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j)\right| \frac{(1-|z_{k_0}^j|^2)^\alpha}{1-|(\varphi_{l_0}(z^j)|^2} \log\frac{2}{1-|z_{k_0}^j|^2} \to a_{l_0 k_0} > 0. \tag{4}$$

If $|w_{l_0}^j| \to 1$, define $f_j(z) = \dfrac{1-|w_{l_0}^j|^2}{1-z_{l_0}\overline{w_{l_0}^j}}$. Then, the sequence $\{f_j\}_j \subset H^\infty(U^n)$ is bounded and converges to zero uniformly on any compact subset of U^n. That is, $\{f_j\}$ weakly converges to zero in $H^\infty(U^n)$. Because $H^\infty(U^n)$ has Dunford-Pettis property (See Theorem 5.3 in [20] for $H^\infty(U)$, and note the proof there works also for $H^\infty(U^n)$), the weak compactness of $C_\varphi : H^\infty(U^n) \to \mathcal{B}^\alpha_{\log}(U^n)$ implies that $\| C_\varphi f_j\|_{\mathcal{B}^\alpha_{\log}} \to 0 (\text{as } j \to \infty)$. But this is impossible since using (4) we may estimate that for each $j \ge 1$,

$$\begin{aligned}
\| C_\varphi f_j \|_{\mathcal{B}^\alpha_{\log}} &\geq \sum_{k=1}^{n} \left| \frac{\partial (f_j \circ \varphi)}{\partial z_k}(z^j) \right| (1 - |z_k^j|^2)^\alpha \log \frac{2}{1 - |z_k^j|^2} \\
&= \sum_{k=1}^{n} \left| \frac{\partial f_j}{\partial w_{l_0}}(\varphi(z^j)) \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| (1 - |z_k^j|^2)^\alpha \log \frac{2}{1 - |z_k^j|^2} \\
&= |w_{l_0}^j| \sum_{k=1}^{n} \frac{(1 - |z_k^j|^2)^\alpha}{1 - |\varphi_{l_0}(z^j)|^2} \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z_k^j|^2} \\
&\geq |w_{l_0}^j| \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \frac{(1 - |z_{k_0}^j|^2)^\alpha}{1 - |\varphi_{l_0}(z^j)|^2} \log \frac{2}{1 - |z_{k_0}^j|^2} \to a_{l_0 k_0} > 0.
\end{aligned}$$

If $|w_{l_0}^j| \to \rho < 1$. Since $w^j \to \partial U^n$, there is $l_1 \in \{1, \ldots, n\} \setminus \{l_0\}$ such that $|w_{l_1}^j| \to 1$. If there exists k_1 such that

$$\left| \frac{\partial \varphi_{l_1}}{\partial z_{k_1}}(z^j) \right| \frac{(1 - |z_{k_1}^j|^2)^\alpha}{1 - |\varphi_{l_1}(z^j)|^2} \log \frac{2}{1 - |z_{k_1}^j|^2} \to a_{l_1 k_1} > 0,$$

then as in the last paragraph above we obtain the desired contradiction using the following test functions:

$$g_j(z) = \frac{1 - |w_{l_1}^j|^2}{1 - z_{l_1} \overline{w_{l_1}^j}}.$$

Thus, we may assume that

$$\left| \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| \frac{(1 - |z_k^j|^2)^\alpha}{1 - |(\varphi_{l_1}(z^j)|^2} \log \frac{2}{1 - |z_k^j|^2} \to 0, \ (\text{as } j \to \infty), \tag{5}$$

for each k. We now define the test functions $h_j$as follows

$$h_j(z) = (z_{l_0} + 2) \frac{1 - |w_{l_1}^j|^2}{1 - z_{l_1} \overline{w_{l_1}^j}}.$$

Then, $||h_j||_\infty \lesssim 1$ and $h_j$converge to zero uniformly on any compact subset of U^n. But for any j large enough

$$\begin{aligned}
\| C_\varphi h_j \|_{\mathcal{B}^\alpha_{\log}} &\geq \sum_{k=1}^{n} \left| \frac{\partial (h_j \circ \varphi)}{\partial z_k}(z^j) \right| (1 - |z_k^j|^2)^\alpha \log \frac{2}{1 - |z_k^j|^2} \\
&= \sum_{k=1}^{n} \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) + (w_{l_0}^j + 2) \overline{w_{l_1}^j} \frac{1}{1 - |w_{l_1}^j|^2} \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| |(1 - |z_k^j|^2)^\alpha \log \frac{2}{1 - |z_k^j|^2} \\
&\geq \sum_{k=1}^{n} (1 - |z_k^j|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z_k^j|^2} - \sum_{k=1}^{n} \left| w_{l_1}^j \right| |w_{l_0}^j + 2| \left| \frac{\partial \varphi_{l_1}}{\partial z_k}(z^j) \right| \frac{(1 - |z_k^j|^2)^\alpha}{1 - |w_{l_1}^j|^2} \log \frac{2}{1 - |z_k^j|^2} \\
&\gtrsim \sum_{k=1}^{n} (1 - |z_k^j|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_k}(z^j) \right| \log \frac{2}{1 - |z_k^j|^2} \\
&\gtrsim (1 - |z_{k_0}^j|^2)^\alpha \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \log \frac{2}{1 - |z_{k_0}^j|^2} \\
&\gtrsim \frac{(1 - |z_{k_0}^j|^2)^\alpha}{1 - |(\varphi_{l_0}(z^j)|^2} \left| \frac{\partial \varphi_{l_0}}{\partial z_{k_0}}(z^j) \right| \log \frac{2}{1 - |z_{k_0}^j|^2} \to a_{l_0 k_0} > 0,
\end{aligned}$$

the inequalities in the third and fourth lines above follow from (5), and the last line is due to $|\varphi_{l_0}(z^j)| \to \rho < 1$. This contradicts again $\| C_\varphi h_j \|_{\mathcal{B}^\alpha_{\log}} \to 0$, which completes the proof.

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Author details
[1]School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, PR China [2]Huaxia College, Wuhan University of Technology, Wuhan, 430070, PR China

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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