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A note on the strong law of large numbers for Markov chains indexed by irregular trees

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Abstract

In this paper, a kind of an infinite irregular tree is introduced. The strong law of large numbers and the Shannon-McMillan theorem for Markov chains indexed by an infinite irregular tree are established. The outcomes generalize some known results on regular trees and uniformly bounded degree trees.

Keywords: Markov chain; tree; strong law of large numbers; AEP

1 Introduction

By a tree T we mean an infinite, locally finite, connected graph with a distinguished vertex o called the *root* and without loops or cycles. We only consider trees without leaves. That is, the degree (the number of neighboring vertices) of each vertex (except o) is required to be at least 2.

Let T be an infinite tree with root o , the set of all vertices with distance n from the root is called the n th generation (or n th level) of T . We denote by $T^{(n)}$ the union of the first n generations of T , by $T_{(m)}^{(n)}$ the union from the m th to n th generations of T , by L_n the subgraph of T containing the vertices in the n th generation. For each vertex t , there is a unique path from o to t , and $|t|$ for the number of edges on this path. We denote the first predecessor of t by 1_t , the second predecessor of t by 2_t , and by n_t the n th predecessor of t . We also call t one of 1_t 's sons. For any two vertices s and t , denote by $s \leq t$, if s is on the unique path from the root o to t , denote by $s \wedge t$ the vertex farthest from o satisfying $s \wedge t \leq s$ and $s \wedge t \leq t$. $X^A = \{X_t, t \in A\}$ and denote by $|A|$ the number of vertices of A .

If each vertex on a tree T has $m + 1$ neighboring vertices, we call it a Bethe tree $T_{B,m}$; if the root has m neighbors and the other vertices have $m + 1$ neighbors on a tree T , we call it a Cayley tree $T_{C,m}$. Both the Bethe tree and the Cayley tree are called regular (or homogeneous) trees. If the degrees of all vertices on a tree T are uniformly bounded, then we call T a uniformly bounded degree tree (see [1] and [2]).

Definition 1 (see [3]) Let T be a locally finite, infinite tree, S be a finite state-space, $\{X_t, t \in T\}$ be a collection of S -valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let

$$\{p(x), x \in S\} \tag{1}$$

be a distribution on S , and

$$(P(y|x)), \quad x, y \in S \tag{2}$$

be a stochastic matrix on S^2 . If for any vertex t ,

$$\begin{aligned} & \mathbf{P}(X_t = y | X_{1_t} = x \text{ and } X_s \text{ for } t \wedge s \leq 1_t) \\ &= \mathbf{P}(X_t = y | X_{1_t} = x) = P(y|x) \quad \forall x, y \in S \end{aligned} \tag{3}$$

and

$$\mathbf{P}(X_0 = x) = p(x) \quad \forall x \in S,$$

then $\{X_t, t \in T\}$ will be called S -valued Markov chains indexed by an infinite tree T with initial distribution (1) and transition matrix (2), or called T -indexed Markov chains with state-space S .

Benjamini and Peres [3] gave the notion of tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [4] studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger [5], by using Pemantle’s result [6] and a combinatorial approach, studied the Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu [7] studied the strong law of large numbers and Shannon-McMillan theorems for Markov chains field on the Cayley tree. Yang [8] studied some strong limit theorems for homogeneous Markov chains indexed by a homogeneous tree and the strong law of large numbers and the asymptotic equipartition property (AEP) for finite homogeneous Markov chains indexed by a homogeneous tree. Yang and Ye [9] studied strong theorems for countable nonhomogeneous Markov chains indexed by a homogeneous tree and the strong law of large numbers and the AEP for finite nonhomogeneous Markov chains indexed by a homogeneous tree. Bao and Ye [10] studied the strong law of large numbers and asymptotic equipartition property for nonsymmetric Markov chain fields on Cayley trees. Takacs [1] studied the strong law of large numbers for the univariate functions of finite Markov chains indexed by an infinite tree with uniformly bounded degree. Huang and Yang [2] studied the strong law of large numbers for Markov chains indexed by uniformly bounded degree trees.

However, the degrees of the vertices in the tree models are uniformly bounded. What if the degrees of the vertices are not uniformly bounded? In this paper, we drop the uniformly bounded restriction. We mainly study the strong law of large numbers and AEP with a.e. convergence for finite Markov chains indexed by trees under the following assumption.

For any integer $N \geq 0$, let $d^0(t) := 1$ and denote

$$d^N(t) := |\sigma \in T : N_\sigma = t| \tag{4}$$

by the amount of t ’s N th descendants. Denote

$$O(n) = \left\{ c_n : 0 < \limsup_{n \rightarrow \infty} \frac{c_n}{n} \leq c, c \text{ is a constant} \right\}.$$

We assume that for enough large n and any given integer $N \geq 0$,

$$\max\{d^N(t) : t \in T^{(n)}\} \leq O\left(\ln \frac{|T^{(n+N)}|}{|T^{(n)}|}\right). \tag{5}$$

The following examples are used to explain assumption (5).

Example 1 Both the Bethe tree $T_{B,m}$ and the Cayley tree $T_{C,m}$ satisfy assumption (5). Actually, $\max\{d^N(t) : t \in T^{(n-N)}\}$ is a constant m^N , and $\ln(|T^{(n)}|/|T^{(n-N)}|) = N \ln m$.

Example 2 A uniformly bounded degree tree satisfies assumption (5). In fact, if the tree T is a uniformly bounded degree tree, then $\max\{d^N(t) : t \in T^{(n-N)}\}$ is no more than a constant a^N , and

$$\ln \frac{|T^{(n)}|}{|T^{(n-N)}|} \leq \ln \frac{|T^{(n-N)}| \times a^N}{|T^{(n-N)}|} = N \ln a$$

is also a constant.

Example 3 Define the lower growth rate of the tree to be $\text{gr } T = \liminf_{n \rightarrow \infty} |T^{(n)}|^{\frac{1}{n}}$ and the upper growth rate of the tree to be $\text{Gr } T = \limsup_{n \rightarrow \infty} |T^{(n)}|^{\frac{1}{n}}$.

If both the $\text{gr } T$ and $\text{Gr } T$ are finite, then

$$\ln \frac{|T^{(n+N)}|}{|T^{(n)}|} \leq \ln \frac{(\text{Gr } T)^{n+N}}{(\text{gr } T)^n} = n \ln \frac{\text{Gr } T}{\text{gr } T} + N \ln \text{Gr } T,$$

hence (5) implies that

$$\max\{d^N(t) : t \in T^{(n)}\} \leq O\left(\ln \frac{|T^{(n+N)}|}{|T^{(n)}|}\right) \leq O(n).$$

2 Some notations and lemmas

In the following, let $\delta_k(\cdot)$ be a Kronecker δ -function. For any given integer $N \geq 0$, denote

$$S_k^N(T^{(n)}) := \sum_{t \in T^{(n-N)}} \delta_k(X_t) d^N(t), \tag{6}$$

which can be considered as the number of k 's among the variables in $T^{(n-N)}$, weighted according to the number of N th descendants. By (6), we have

$$\sum_{k \in S} S_n^N(k) = |T^{(n-N)}| - 1. \tag{7}$$

Define

$$H_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t) \tag{8}$$

and

$$G_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}]. \tag{9}$$

Lemma 1 (see [2]) *Let T be an infinite tree with assumption (5) holds. Let $(X_t)_{t \in T}$ be a T -indexed Markov chain with state-space S defined as before, $\{g_t(x, y), t \in T\}$ be functions defined on S^2 . Let $L_o = \{o\}$, $\mathcal{F}_n = \sigma(X^{T^{(n)}})$,*

$$t_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t)}}{\prod_{t \in T^{(n)} \setminus \{o\}} E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]}, \tag{10}$$

where λ is a real number. Then $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ is a nonnegative martingale.

Lemma 2 (see [2]) *Under the assumption of Lemma 1, let $\{a_n, n \geq 1\}$ be a sequence of nonnegative random variables, $\alpha > 0$. Set*

$$B = \left\{ \lim_{n \rightarrow \infty} a_n = \infty \right\} \tag{11}$$

and

$$D(\alpha) = \left\{ \limsup_{n \rightarrow \infty} \frac{1}{a_n} \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t^2(X_{1_t}, X_t) e^{\alpha |g_t(X_{1_t}, X_t)|} | X_{1_t}] = M(\omega) < \infty \right\} \cap B. \tag{12}$$

Then

$$\lim_{n \rightarrow \infty} \frac{H_n(\omega) - G_n(\omega)}{a_n} = 0 \quad \text{a.e. on } D(\alpha). \tag{13}$$

3 Strong law of large numbers and Shannon-McMillan theorem

In this section, we study the strong law of large numbers and the Shannon-McMillan theorem for finite Markov chains indexed by an infinite tree with assumption (5) holds.

Theorem 1 *Let T be an infinite tree with assumption (5) holds. Then under the assumption of Lemma 1, for all $k \in S$ and $N \geq 0$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)}) P(k|l) \right\} = 0 \quad \text{a.e.} \tag{14}$$

Proof Let $g_t(x, y) = d^N(t) \delta_k(y)$, $a_n = |T^{(n+N)}|$. Since

$$\begin{aligned} G_n(\omega) &= \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}] \\ &= \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) \sum_{x_t \in S} \delta_k(x_t) P(x_t | X_{1_t}) \\ &= \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) P(k | X_{1_t}) \\ &= \sum_{l \in S} \sum_{t \in T^{(n)} \setminus \{o\}} \delta_l(X_{1_t}) d^N(t) P(k|l) \\ &= \sum_{l \in S} \sum_{t \in T^{(n-1)}} \delta_l(X_t) d^{N+1}(t) P(k|l) \\ &= \sum_{l \in S} S_l^{N+1}(T^{(n-1)}) P(k|l) \end{aligned} \tag{15}$$

and

$$H_n(\omega) = \sum_{t \in T^{(n)} \setminus \{o\}} g_t(X_{1_t}, X_t) = \sum_{t \in T^{(n)} \setminus \{o\}} d^N(t) \delta_k(X_t) = S_k^N(T^{(n)}) - \delta_k(X_o) d^N(o). \tag{16}$$

By Lemma 1, we know that $\{t_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\}$ is a nonnegative martingale. According to the Doob martingale convergence theorem, we have

$$\lim_n t_n(\lambda, \omega) = t(\lambda, \omega) < \infty \quad \text{a.e.} \tag{17}$$

We have by (17)

$$\limsup_{n \rightarrow \infty} \frac{\ln t_n(\lambda, \omega)}{|T^{(n+N)}|} \leq 0 \quad \text{a.e. } \omega \in B. \tag{18}$$

By (10), (16) and (18), we get

$$\limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ \lambda H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} \ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]] \right\} \leq 0 \quad \text{a.e. } \omega \in B. \tag{19}$$

Let $\lambda > 0$. Dividing two sides of (19) by λ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{a_n} \left\{ H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} \frac{\ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]]}{\lambda} \right\} \leq 0 \quad \text{a.e. } \omega \in B. \tag{20}$$

The case $\{d^N(t) : t \in T^{(n)}\}$ is uniformly bounded was considered in [2], we only consider the case $\{d^N(t) : t \in T^{(n)}\}$ is not uniformly bounded. By (18) and inequalities $\ln x \leq x - 1$ ($x > 0$), $0 \leq e^x - 1 - x \leq 2^{-1}x^2 e^{|x|}$, as $0 < \lambda \leq \alpha$, we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left[H_n(\omega) - \sum_{t \in T^{(n)} \setminus \{o\}} E[g_t(X_{1_t}, X_t) | X_{1_t}] \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} \left\{ \frac{\ln[E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}]]}{\lambda} - E[g_t(X_{1_t}, X_t) | X_{1_t}] \right\} \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} \left\{ \frac{E[e^{\lambda g_t(X_{1_t}, X_t)} | X_{1_t}] - 1}{\lambda} - E[g_t(X_{1_t}, X_t) | X_{1_t}] \right\} \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T^{(n)}} E[g_t^2(X_{1_t}, X_t) e^{\lambda |g_t(X_{1_t}, X_t)|} | X_{1_t}] \\ & = \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} E[(d^N(t) \delta_k(X_t))^2 e^{\lambda |d^N(t) \delta_k(X_t)|} | X_{1_t}] \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} [(d^N(t))^2 e^{\lambda d^N(t)} P(k | X_{1_t})] \\ & \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} [(d^N(t))^2 e^{\lambda d^N(t)}] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \sum_{t \in T_{(1)}^{(n)}} e^{2\lambda d^N(t)} \quad (\text{for enough large } d^N(t)) \\
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} \max\{e^{2\lambda d^N(t)}, t \in T_{(1)}^{(n)}\} \\
 &\leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} (e^{\max\{d^N(t), t \in T_{(1)}^{(n)}\}})^{2\lambda}. \tag{21}
 \end{aligned}$$

By (5), there exists a constant $\beta > 0$ such that

$$\max\{d^N(t), t \in T_{(1)}^{(n)}\} \leq \beta \ln \frac{|T^{(n+N)}|}{|T^{(n)}|},$$

hence,

$$(e^{\max\{d^N(t), t \in T_{(1)}^{(n)}\}})^{2\lambda} < \left(\frac{|T^{(n+N)}|}{|T^{(n)}|}\right)^{2\lambda\beta}. \tag{22}$$

Set $0 < \lambda < \frac{1}{2\beta}$, by (21) and (22) we have

$$\limsup_{n \rightarrow \infty} \frac{H_n(\omega) - G_n(\omega)}{|T^{(n+N)}|} \leq \frac{\lambda}{2} \limsup_{n \rightarrow \infty} \frac{|T^{(n)}| - 1}{|T^{(n+N)}|} \times \left(\frac{|T^{(n+N)}|}{|T^{(n)}|}\right)^{2\lambda\beta}. \tag{23}$$

Let $\lambda \rightarrow 0^+$ in (23), by (15) and (16) we have

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)})P(k|l) \right\} \leq 0 \quad \text{a.e.} \tag{24}$$

Let $-\frac{1}{2\beta} \leq \lambda \rightarrow 0^-$. By (19), we similarly get

$$\lim_{n \rightarrow \infty} \frac{1}{|T^{(n+N)}|} \left\{ S_k^N(T^{(n)}) - \sum_{l \in S} S_l^{N+1}(T^{(n-1)})P(k|l) \right\} \geq 0 \quad \text{a.e.} \tag{25}$$

Combining (24) and (25), we obtain (14) directly. \square

Let T be a tree, $(X_t)_{t \in T}$ be a stochastic process indexed by the tree T with state-space S . Denote

$$P(x^{T^{(n)}}) = P(X^{T^{(n)}} = x^{T^{(n)}}).$$

Let

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \ln P(X^{T^{(n)}}), \tag{26}$$

$f_n(\omega)$ will be called the entropy density of $X^{T^{(n)}}$. If $(X_t)_{t \in T}$ is a T -indexed Markov chain with state-space S defined by Definition 1, we have by (5)

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \left[\ln P(X_0) + \sum_{t \in T^{(n)} \setminus \{o\}} \ln P(X_t, X_{I_t}) \right]. \tag{27}$$

The convergence of $f_n(\omega)$ to a constant in a sense (L_1 convergence, convergence in probability, a.e. convergence) is called the Shannon-McMillan theorem or the entropy theorem or the AEP in information theory.

Theorem 2 *Let T be an infinite tree with assumption (5) holds. Let $k \in S$, and P be an ergodic stochastic matrix. Denote the unique stationary distribution of P by π . Let $(X_t)_{t \in T}$ be a T -indexed Markov chain with state-space S generated by P . Then, for given integer $N \geq 0$,*

$$\lim_{n \rightarrow \infty} \frac{S_k^N(T^{(n)})}{|T^{(n+N)}|} = \pi(k) \quad a.e. \tag{28}$$

Let $S_{l,k}(T^{(n)}) := |\{t \in T^{(n)} : (X_{1_t}, X_t) = (l, k)\}|$, then

$$\lim_{n \rightarrow \infty} \frac{S_{l,k}(T^{(n)})}{|T^{(n)}|} = \pi(l)P(k|l) \quad a.e. \tag{29}$$

Let $f_n(\omega)$ be defined as (27), then

$$\lim_{n \rightarrow \infty} f_n(\omega) = - \sum_{l \in S} \sum_{k \in S} \pi(l)P(k|l) \ln P(k|l) \quad a.e. \tag{30}$$

Proof The proofs of (28) and (29) are similar to those of Huang and Yang ([2], Theorem 2 and Corollary 3). Letting $g_t(x, y) = -\ln P(y|x)$ in Lemma 1, then

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(\omega) &= \lim_{n \rightarrow \infty} \frac{H_n(\omega)}{|T^{(n)}|} = - \lim_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{o\}} \ln P(X_{1_t}, X_t) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}} \sum_{l \in S} \sum_{k \in S} \delta_l(X_{1_t}) \delta_k(X_t) \ln P(k|l) \\ &= - \lim_{n \rightarrow \infty} \sum_{l \in S} \sum_{k \in S} \ln P(k|l) \frac{S_{l,k}(T^{(n)})}{|T^{(n)}|} \end{aligned}$$

by (29), (30) holds. □

Competing interests

The author declares that they have no competing interests.

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