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Algorithms of common solutions for a variational inequality, a split equilibrium problem and a hierarchical fixed point problem

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Dedicated to Professor Bingsheng He on the occasion of his sixty-fifth birthday

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Abstract

In this paper, we suggest and analyze an iterative scheme for finding an approximate element of the common set of solutions of a split equilibrium problem, a variational inequality problem and a hierarchical fixed point problem in a real Hilbert space. We also consider the strong convergence of the proposed method under some conditions. Results proved in this paper may be viewed as an improvement and refinement of the previously known results.

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1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed convex subset of H and D be a mapping from C into H. A classical variational inequality problem, denoted by $VI(D,C)$, is to find a vector $u \in C$ such that

$$\langle v-u, Du\rangle \geq 0, \quad \forall v \in C. \tag{1}$$

The solution of $VI(D,C)$ is denoted by Ω^*. It is easy to observe that

$$u^* \in \Omega^* \quad \Longleftrightarrow \quad u^* = P_C\big[u^* - \lambda Du^*\big], \quad \text{where } \lambda > 0.$$

This alternative formulation has played a significant part in developing various projection-type methods for solving variational inequalities. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems; see [1–29].

We introduce the following definitions which are useful in the following analysis.

Definition 1.1 The mapping $T: C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e) k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C;$$

(f) contraction on C if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. It is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\bigl\langle \bigl(x - T(x)\bigr) - \bigl(y - T(y)\bigr), T(y) - T(x)\bigr\rangle \leq \frac{1}{2}\bigl\|\bigl(T(x) - x\bigr) - \bigl(T(y) - y\bigr)\bigr\|^2 \tag{2}$$

and therefore we get, for all $(x, y) \in H_1 \times F(T)$,

$$\bigl\langle x - T(x), y - T(x)\bigr\rangle \leq \frac{1}{2}\bigl\|T(x) - x\bigr\|^2; \tag{3}$$

see, *e.g.*, [9], Theorem 1 and [10], Theorem 3.

A mapping $T : C \to H$ is called a k-strict pseudo-contraction if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad \forall x, y \in C. \tag{4}$$

The fixed point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{5}$$

We denote by $F(T)$ the set of solutions of (5). It is well known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings, then $F(T)$ is closed and convex and $P_{F(T)}$ is well defined (see [29]).

The equilibrium problem denoted by EP is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{6}$$

The solution set of (6) is denoted by $EP(F)$. Numerous problems in physics, optimization and economics reduce to finding a solution of (6); see [7, 12, 23, 24]. In 1997, Combettes and Hirstoaga [8] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty. Recently Plubtieng and Punpaeng [23] introduced an iterative method for finding the common element of the set $F(T) \cap \Omega^* \cap EP(F)$.

Recently, Censor *et al.* [4] introduced a new variational inequality problem which we call the split variational inequality problem (SVIP). Let H_1 and H_2 be two real Hilbert spaces. Given operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and nonempty, closed and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$, the SVIP is formulated as follows: Find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in C \tag{7}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad \langle g(y^*), y - y^* \rangle \geq 0 \quad \text{for all } y \in Q. \tag{8}$$

In [22], Moudafi introduced an iterative method which can be regarded as an extension of the method given by Censor *et al.* [4] for the following split monotone variational inclusions:

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*)$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in g(y^*) + B_2(y^*),$$

where $B_i : H_i \to 2^{H_i}$ is a set-valued mapping for $i = 1, 2$. Later Byrne *et al.* [3] generalized and extended the work of Censor *et al.* [4] and Moudafi [22].

Very recently, Kazmi and Rivzi [13] studied the following pair of equilibrium problems called a split equilibrium problem: Let $F_1 : C \times C \to R$ and $F_2 : Q \times Q \to R$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C \tag{9}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{10}$$

The solution set of SEP (9)-(10) is denoted by $\Lambda = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: Find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{11}$$

It is known that the hierarchical fixed point problem (11) links with some monotone variational inequalities and convex programming problems; see [11, 27]. Various methods have been proposed to solve the hierarchical fixed point problem; see Moudafi [21], Mainge and Moudafi in [15], Marino and Xu in [17] and Cianciaruso *et al.* [5]. In 2010, Yao *et al.* [27] introduced the following strong convergence iterative algorithm to solve problem (11):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C\big[\alpha_n f(x_n) + (1 - \alpha_n) T y_n\big], \quad \forall n \geq 0, \end{aligned} \tag{12}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, Yao *et al.* proved that the sequence $\{x_n\}$ generated by (12) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\big\langle (I - f)z, y - z \big\rangle \geq 0, \quad \forall y \in F(T). \tag{13}$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (12), the sequence $\{x_n\}$ generated by (12) converges strongly to a point $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I - f)z + (I - S)z, y - z \right\rangle \geq 0, \quad \forall y \in F(T). \tag{14}$$

Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings. In 2011, Gu *et al.* [11] introduced the following iterative algorithm:

$$\begin{aligned} y_n &= P_C\big[\beta_n S x_n + (1 - \beta_n) x_n\big], \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{15}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0, 1)$ and $\{\beta_n\}$ is a sequence in $(0, 1)$. Under some certain conditions on parameters, Gu *et al.* proved that the sequence $\{x_n\}$ generated by (15) converges strongly to $z \in \bigcap_{i=1}^{\infty} F(T_i)$, which is the unique solution of one of variational inequalities (13) and (14).

In this paper, motivated by the work of Censor *et al.* [4], Moudafi [22], Byrne *et al.* [3] Kazmi and Rivzi [13], Yao *et al.* [27] and Gu *et al.* [11] and by the recent work going on in this direction, we give an iterative method for finding an approximate element of the common set of solutions of (1), (9)-(10) and (11) for a strictly pseudo-contraction mapping in a real Hilbert space. We establish a strong convergence theorem based on this method. The presented method improves and generalizes many known results for solving equilibrium problems, variational inequality problems and hierarchical fixed point problems; see, *e.g.*, [5, 11, 15, 27] and relevant references cited therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of the projection of H onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities,*

$$\langle z - P_C[z], P_C[z] - v\rangle \geq 0, \quad \forall z \in H, v \in C; \tag{16}$$

$$\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2, \quad \forall u, v \in H; \tag{17}$$

$$\|P_C[u] - P_C[v]\| \leq \|u - v\|, \quad \forall u, v \in H; \tag{18}$$

$$\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2, \quad \forall z \in H, u \in C. \tag{19}$$

Assumption 2.1 [2] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x,x) = 0$, $\forall x \in C$;
(ii) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x, y \in C$;
(iii) For each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(iv) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;
(v) Fixed $r > 0$ and $z \in C$, there exists a bounded subset K of H_1 and $x \in C \cap K$ such that

$$F(y,x) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \forall y \in C \setminus K.$$

Lemma 2.2 [8] *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfies Assumption* 2.1. *For $r > 0$ and $\forall x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows*:

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) $T_r^{F_1}$ *is nonempty and single-valued*;
(ii) $T_r^{F_1}$ *is firmly nonexpansive, i.e.*,

$$\|T_r^{F_1}(x) - T_r^{F_1}(y)\|^2 \leq \langle T_r^{F_1}(x) - T_r^{F_1}(y), x - y\rangle, \quad \forall x, y \in H_1;$$

(iii) $F(T_r^{F_1}) = EP(F_1)$;
(iv) $EP(F_1)$ *is closed and convex.*

Assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfies Assumption 2.1. For $s > 0$ and $\forall u \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(u) = \left\{ v \in Q : F_2(v,w) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q \right\}.$$

Then $T_s^{F_2}$ satisfies conditions (i)-(iv) of Lemma 2.2. $F(T_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:

$$\text{Find } y^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \quad \forall y \in Q.$$

Lemma 2.3 [6] *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfies Assumption* 2.1, *and let $T_r^{F_1}$ be defined as in Lemma* 2.2. *Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\left\| T_{r_2}^{F_1}(y) - T_{r_1}^{F_1}(x) \right\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| T_{r_2}^{F_1}(y) - y \right\|.$$

Lemma 2.4 [28] *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudo-contraction, then*:

(i) *The mapping $I - T$ is demiclosed at* 0, *i.e., if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to* 0, *then $(I - T)x = 0$*;

(ii) *The set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.5 [16] *Let H be a real Hilbert space. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.6 [26] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ *and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [1] *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) *the weak w-limit set $w_w(x_n) \subset C$, where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$*;

(ii) *for each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.8 [29] *Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to C$ be a k-strict pseudo-contraction mapping. Define a mapping $V : C \to H$ by $Vx = \lambda x + (1 - \lambda)Tx$, $\forall x \in C$. Then, as $k \leq \lambda < 1$, V is a nonexpansive mapping such that $F(V) = F(T)$.*

Lemma 2.9 [11] *Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then*

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x'\rangle, \quad \forall x' \in F(T), \forall x \in C.$$

3 The proposed method and some properties

In this section, we suggest and analyze our method for finding common solutions of the variational inequality (1), the split equilibrium problem (9)-(10) and the hierarchical fixed point problem (11).

Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be an α-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in the first argument. Let $S : C \to H_1$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings such that $F(T) \cap \Omega^* \cap \Lambda \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping.

Algorithm 3.1 For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{aligned}
u_n &= T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr); \\
z_n &= P_C[u_n - \lambda_n D u_n]; \\
y_n &= P_C\bigl[\beta_n S x_n + (1 - \beta_n) z_n\bigr]; \\
x_{n+1} &= P_C\Biggl[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n\Biggr], \quad \forall n \geq 0,
\end{aligned} \tag{20}$$

where $V_i = k_i I + (1 - k_i)T_i$, $0 \leq k_i < 1$, $\{r_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$,
(e) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ and $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Lemma 3.1 *Let $x^* \in F(T) \cap \Omega^* \cap \Lambda$. Then $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded.*

Proof First, we show that the mapping $(I - \lambda_n D)$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned}
\bigl\|(I - \lambda_n D)x - (I - \lambda_n D)y\bigr\|^2 &= \bigl\|(x - y) - \lambda_n (Dx - Dy)\bigr\|^2 \\
&= \|x - y\|^2 - 2\lambda_n \langle x - y, Dx - Dy\rangle + \lambda_n^2 \|Dx - Dy\|^2 \\
&\leq \|x - y\|^2 - \lambda_n (2\alpha - \lambda_n)\|Dx - Dy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Let $x^* \in F(T) \cap \Omega^* \cap \Lambda$, we have $x^* = T_{r_n}^{F_1}(x^*)$ and $Ax^* = T_{r_n}^{F_2}(Ax^*)$. Then

$$\begin{aligned}
\bigl\|u_n - x^*\bigr\|^2 &= \bigl\|T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr) - x^*\bigr\|^2 \\
&= \bigl\|T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr) - T_{r_n}^{F_1}\bigl(x^*\bigr)\bigr\|^2 \\
&\leq \bigl\|x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n - x^*\bigr\|^2 \\
&= \bigl\|x_n - x^*\bigr\|^2 + \gamma^2 \bigl\|A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr\|^2 + 2\gamma\bigl\langle x_n - x^*, A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr\rangle \\
&= \bigl\|x_n - x^*\bigr\|^2 + \gamma^2\bigl\langle\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n, AA^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr\rangle \\
&\quad + 2\gamma\bigl\langle x_n - x^*, A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr\rangle.
\end{aligned} \tag{21}$$

From the definition of L, it follows that

$$\begin{aligned}\gamma^2\langle (T_{r_n}^{F_2}-I)Ax_n, AA^*(T_{r_n}^{F_2}-I)Ax_n\rangle &\le L\gamma^2\langle (T_{r_n}^{F_2}-I)Ax_n, (T_{r_n}^{F_2}-I)Ax_n\rangle \\ &= L\gamma^2\|(T_{r_n}^{F_2}-I)Ax_n\|^2. \end{aligned}\tag{22}$$

It follows from (3) that

$$\begin{aligned}&2\gamma\langle x_n-x^*, A^*(T_{r_n}^{F_2}-I)Ax_n\rangle \\ &\quad= 2\gamma\langle A(x_n-x^*), (T_{r_n}^{F_2}-I)Ax_n\rangle \\ &\quad= 2\gamma\langle A(x_n-x^*)+(T_{r_n}^{F_2}-I)Ax_n-(T_{r_n}^{F_2}-I)Ax_n, (T_{r_n}^{F_2}-I)Ax_n\rangle \\ &\quad= 2\gamma\left(\langle T_{r_n}^{F_2}Ax_n-Ax^*, (T_{r_n}^{F_2}-I)Ax_n\rangle - \|(T_{r_n}^{F_2}-I)Ax_n\|^2\right) \\ &\quad\le 2\gamma\left(\frac{1}{2}\|(T_{r_n}^{F_2}-I)Ax_n\|^2-\|(T_{r_n}^{F_2}-I)Ax_n\|^2\right) \\ &\quad= -\gamma\|(T_{r_n}^{F_2}-I)Ax_n\|^2. \end{aligned}\tag{23}$$

Applying (23) and (22) to (21) and from the definition of γ, we get

$$\|u_n-x^*\|^2 \le \|x_n-x^*\|^2+\gamma(L\gamma-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2 \le \|x_n-x^*\|^2. \tag{24}$$

Since the mapping D is α-inverse strongly monotone, we have

$$\begin{aligned}\|z_n-x^*\|^2 &= \|P_C[u_n-\lambda_n Du_n]-P_C[x^*-\lambda_n Dx^*]\|^2 \\ &\le \|u_n-x^*-\lambda_n(Du_n-Dx^*)\|^2 \\ &\le \|u_n-x^*\|^2-\lambda_n(2\alpha-\lambda_n)\|Du_n-Dx^*\|^2 \\ &\le \|u_n-x^*\|^2 \\ &\le \|x_n-x^*\|^2. \end{aligned}\tag{25}$$

Next, we prove that the sequence $\{x_n\}$ is bounded, without loss of generality, we can assume that $\beta_n \le \alpha_n$ for all $n \ge 1$. From Lemma 2.8, we have V_i is a nonexpansive mapping and $V_i x^* = x^*$. Since $\sum_{i=1}^n(\alpha_{i-1}-\alpha_i) = 1-\alpha_n$, we get

$$\begin{aligned}\|x_{n+1}-x^*\| &= \left\|P_C\left[\alpha_n f(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i y_n\right]-x^*\right\| \\ &\le \left\|\alpha_n f(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i y_n-x^*\right\| \\ &= \left\|\alpha_n f(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i y_n-\alpha_n x^*-\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_i x^*\right\| \\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\|V_i y_n-V_i x^*\| \\ &\le \alpha_n\|f(x_n)-f(x^*)\|+\alpha_n\|f(x^*)-x^*\|+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\|y_n-x^*\| \end{aligned}$$

$$
\begin{aligned}
&= \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)z_n - x^*\| \\
&\le \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1-\alpha_n)\big(\beta_n \|Sx_n - Sx^*\| + \beta_n \|Sx^* - x^*\| + (1-\beta_n)\|z_n - x^*\|\big) \\
&\le \alpha_n \rho \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&\quad + (1-\alpha_n)\big(\beta_n \|x_n - x^*\| + \beta_n \|Sx^* - x^*\| + (1-\beta_n)\|x_n - x^*\|\big) \\
&= \big(1-\alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1-\alpha_n)\beta_n \|Sx^* - x^*\| \\
&\le \big(1-\alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|Sx^* - x^*\| \\
&\le \big(1-\alpha_n(1-\rho)\big)\|x_n - x^*\| + \alpha_n \big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big) \\
&= \big(1-\alpha_n(1-\rho)\big)\|x_n - x^*\| + \frac{\alpha_n(1-\rho)}{1-\rho}\big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big) \\
&\le \max\left\{\|x_n - x^*\|, \frac{1}{1-\rho}\big(\|f(x^*) - x^*\| + \|Sx^* - x^*\|\big)\right\}. \qquad (26)
\end{aligned}
$$

By induction on n, we obtain $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \frac{1}{1-\rho}(\|f(x^*) - x^*\| + \|Sx^* - x^*\|)\}$, for $n \ge 0$ and $x_0 \in C$. Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. □

Lemma 3.2 *Let $x^* \in F(T) \cap \Omega^* \cap \Lambda$ and $\{x_n\}$ be the sequence generated by Algorithm* 3.1. *Then we have*

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(b) *The weak w-limit set $w_w(x_n) \subset F(T)$ ($w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$).*

Proof From the nonexpansivity of the mapping $(I - \lambda_n D)$ and P_C, we have

$$
\begin{aligned}
\|z_n - z_{n-1}\| &\le \big\|(u_n - \lambda_n Du_n) - (u_{n-1} - \lambda_{n-1} Du_{n-1})\big\| \\
&= \big\|(u_n - u_{n-1}) - \lambda_n(Du_n - Du_{n-1}) - (\lambda_n - \lambda_{n-1})Du_{n-1}\big\| \\
&\le \big\|(u_n - u_{n-1}) - \lambda_n(Du_n - Du_{n-1})\big\| + |\lambda_n - \lambda_{n-1}|\|Du_{n-1}\| \\
&\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Du_{n-1}\|. \qquad (27)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \big\|\beta_n Sx_n + (1-\beta_n)z_n - \big(\beta_{n-1} Sx_{n-1} + (1-\beta_{n-1})z_{n-1}\big)\big\| \\
&\quad = \big\|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n)z_{n-1}\big\| \\
&\quad \le \beta_n \|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \qquad (28)
\end{aligned}
$$

It follows from (27) and (28) that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\le \beta_n \|x_n - x_{n-1}\| + (1-\beta_n)\big\{\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Du_{n-1}\|\big\} \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big). \qquad (29)
\end{aligned}
$$

On the other hand, $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$ and $u_{n-1} = T_{r_{n-1}}^{F_1}(x_{n-1} + \gamma A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})$. It follows from Lemma 2.3 that

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&\quad \le \big\|x_n - x_{n-1} + \gamma\big(A^*\big(T_{r_n}^{F_2} - I\big)Ax_n - A^*\big(T_{r_{n-1}}^{F_2} - I\big)Ax_{n-1}\big)\big\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad \le \big\|x_n - x_{n-1} - \gamma A^*A(x_n - x_{n-1})\big\| + \gamma\|A\|\big\|T_{r_n}^{F_2}Ax_n - T_{r_{n-1}}^{F_2}Ax_{n-1}\big\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad \le \big(\|x_n - x_{n-1}\|^2 - 2\gamma\big\|A(x_n - x_{n-1})\big\|^2 + \gamma^2\|A\|^4\|x_n - x_{n-1}\|^2\big)^{\frac{1}{2}} \\
&\qquad + \gamma\|A\|\left(\big\|A(x_n - x_{n-1})\big\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\|\right) \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad \le \big(1 - 2\gamma\|A\|^2 + \gamma^2\|A\|^4\big)^{\frac{1}{2}}\|x_n - x_{n-1}\| + \gamma\|A\|^2\|x_n - x_{n-1}\| \\
&\qquad + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad = \big(1 - \gamma\|A\|^2\big)\|x_n - x_{n-1}\| + \gamma\|A\|^2\|x_n - x_{n-1}\| + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad = \|x_n - x_{n-1}\| + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\qquad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\quad = \|x_n - x_{n-1}\| + \left|\frac{r_n - r_{n-1}}{r_n}\right|\big(\gamma\|A\|\sigma_n + \chi_n\big),
\end{aligned}$$

where $\sigma_n := \|T_{r_n}^{F_2}Ax_n - Ax_n\|$ and $\chi_n := \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)\|$. Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$ for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \le \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big). \tag{30}$$

It follows from (29) and (30) that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\|x_n - x_{n-1}\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + |\lambda_n - \lambda_{n-1}| \|Du_{n-1}\| \Big\} + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big) \\
&= \|x_n - x_{n-1}\| + (1-\beta_n) \left\{ \frac{1}{\mu} |r_{n-1} - r_n| \big(\gamma \|A\| \sigma_n + \chi_n \big) + |\lambda_n - \lambda_{n-1}| \|Du_{n-1}\| \right\} \\
&\quad + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big). \qquad (31)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \left\| \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n - \left(\alpha_{n-1} f(x_{n-1}) + \sum_{i=1}^{n-1} (\alpha_{i-1} - \alpha_i) V_i y_{n-1} \right) \right\| \\
&\quad = \left\| \alpha_n \big(f(x_n) - f(x_{n-1}) \big) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - V_i y_{n-1}) \right. \\
&\qquad \left. + (\alpha_{n-1} - \alpha_n) V_n y_{n-1} \right\| \\
&\quad \le \alpha_n \big\| f(x_n) - f(x_{n-1}) \big\| + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|V_i y_n - V_i y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \|V_n y_{n-1}\| \big) \\
&\quad \le \alpha_n \rho \|x_n - x_{n-1}\| + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|y_n - y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \|V_n y_{n-1}\| \big) \\
&\quad = \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n) \|y_n - y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \|V_n y_{n-1}\| \big). \qquad (32)
\end{aligned}
$$

From (31) and (32), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n) \Big\{ \|x_n - x_{n-1}\| \\
&\qquad + (1-\beta_n) \left(\frac{1}{\mu} |r_{n-1} - r_n| \big(\gamma \|A\| \sigma_n + \chi_n \big) + |\lambda_n - \lambda_{n-1}| \|Du_{n-1}\| \right) \\
&\qquad + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big) \Big\} + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \|V_n y_{n-1}\| \big) \\
&\quad \le \big(1 - (1-\rho)\alpha_n \big) \|x_n - x_{n-1}\| + \frac{1}{\mu} |r_{n-1} - r_n| \big(\gamma \|A\| \sigma_n + \chi_n \big) \\
&\qquad + |\lambda_n - \lambda_{n-1}| \|Du_{n-1}\| + |\beta_n - \beta_{n-1}| \big(\|Sx_{n-1}\| + \|z_{n-1}\| \big) \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big(\big\| f(x_{n-1}) \big\| + \|V_n y_{n-1}\| \big) \\
&\quad \le \big(1 - (1-\rho)\alpha_n \big) \|x_n - x_{n-1}\| \\
&\qquad + M \left(\frac{1}{\mu} |r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| \right), \qquad (33)
\end{aligned}
$$

where

$$M = \max\Big\{\sup_{n\ge1}\big(\gamma\|A\|\sigma_n + \chi_n\big), \sup_{n\ge1}\|Du_{n-1}\|, \sup_{n\ge1}\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big),$$
$$\sup_{n\ge1}\big(\|f(x_{n-1})\| + \|V_n y_{n-1}\|\big)\Big\}.$$

Since $\{x_n\}$, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded, we deduce that $\{Ax_n\}$, $\{Du_{n-1}\}$, $\{Sx_{n-1}\}$, $\{f(x_{n-1})\}$ and $\{V_n y_{n-1}\}$ are bounded. We can conclude that $\sup_{n\ge1}(\gamma\|A\|\sigma_n + \chi_n) < \infty$, $\sup_{n\ge1}\|Du_{n-1}\| < \infty$, $\sup_{n\ge1}(\|Sx_{n-1}\| + \|z_{n-1}\|) < \infty$, $\sup_{n\ge1}(\|f(x_{n-1})\| + \|V_n y_{n-1}\|) < \infty$, and $M < \infty$.

It follows by conditions (a)-(e) of Algorithm 3.1 and Lemma 2.6 that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since $x^* \in F(T)\cap\Omega^*\cap\Lambda$ and $\alpha_n + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i) = 1$, by using (24) and (25), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \Big\|P_C\Big[\alpha_n f(x_n) + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)V_i y_n\Big] - x^*\Big\|^2 \\
&\le \Big\|\alpha_n f(x_n) + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)V_i y_n - x^*\Big\|^2 \\
&= \Big\|\alpha_n f(x_n) + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)V_i y_n - \alpha_n x^* - \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)V_i x^*\Big\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)\|V_i y_n - V_i x^*\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \sum_{i=1}^n(\alpha_{i-1} - \alpha_i)\|y_n - x^*\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\big(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2\big) \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\beta_n\|Sx_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\big\{\|x_n - x^*\|^2 + \gamma(L\gamma - 1)\big\|\big(T_{r_n}^{F_2} - I\big)Ax_n\big\|^2 \\
&\quad - \lambda_n(2\alpha - \lambda_n)\|Du_n - Dx^*\|^2\big\} \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n)\big\{\gamma(1 - L\gamma)\big\|\big(T_{r_n}^{F_2} - I\big)Ax_n\big\|^2 \\
&\quad + \lambda_n(2\alpha - \lambda_n)\|Du_n - Dx^*\|^2\big\}.
\end{aligned} \tag{34}$$

Then, from the above inequality, we get

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\big\{\gamma(1 - L\gamma)\big\|\big(T_{r_n}^{F_2} - I\big)Ax_n\big\|^2 + \lambda_n(2\alpha - \lambda_n)\|Du_n - Dx^*\|^2\big\} \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad \le \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big)\|x_{n+1} - x_n\|.
\end{aligned}$$

Since $\gamma(1-L\gamma)>0$, $\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<2\alpha$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$ and $\beta_n\to 0$, we obtain

$$\lim_{n\to\infty}\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|=0 \tag{35}$$

and

$$\lim_{n\to\infty}\big\|Du_n-Dx^*\big\|=0.$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\big\|u_n-x^*\big\|^2 &= \big\|T_{r_n}^{F_1}\big(x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big)-T_{r_n}^{F_1}\big(x^*\big)\big\|^2\\
&\le \big\langle u_n-x^*,x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big\rangle\\
&= \frac{1}{2}\big\{\big\|u_n-x^*\big\|^2+\big\|x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big\|^2\\
&\quad -\big\|u_n-x^*-\big[x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big]\big\|^2\big\}\\
&= \frac{1}{2}\big\{\big\|u_n-x^*\big\|^2+\big\|x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big\|^2\\
&\quad -\big\|u_n-x_n-\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2\big\}\\
&\le \frac{1}{2}\big\{\big\|u_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2-\big\|u_n-x_n-\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2\big\}\\
&= \frac{1}{2}\big\{\big\|u_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2\\
&\quad -\big[\big\|u_n-x_n\big\|^2+\gamma^2\big\|A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2-2\gamma\big\langle u_n-x_n,A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\rangle\big]\big\},
\end{aligned}$$

where the last inequality follows from (21) and (24). Hence, we get

$$\big\|u_n-x^*\big\|^2\le\big\|x_n-x^*\big\|^2-\|u_n-x_n\|^2+2\gamma\|Au_n-Ax_n\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|.$$

From (34), (25) and the above inequality, we have

$$\begin{aligned}
\big\|x_{n+1}-x^*\big\|^2 &\le \alpha_n\big\|f(x_n)-x^*\big\|^2+(1-\alpha_n)\big(\beta_n\big\|Sx_n-x^*\big\|^2+(1-\beta_n)\big\|z_n-x^*\big\|^2\big)\\
&\le \alpha_n\big\|f(x_n)-x^*\big\|^2+(1-\alpha_n)\big(\beta_n\big\|Sx_n-x^*\big\|^2+(1-\beta_n)\big\|u_n-x^*\big\|^2\big)\\
&\le \alpha_n\big\|f(x_n)-x^*\big\|^2+(1-\alpha_n)\big\{\beta_n\big\|Sx_n-x^*\big\|^2\\
&\quad +(1-\beta_n)\big(\big\|x_n-x^*\big\|^2-\|u_n-x_n\|^2+2\gamma\|Au_n-Ax_n\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|\big)\big\}\\
&\le \alpha_n\big\|f(x_n)-x^*\big\|^2+\beta_n\big\|Sx_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2\\
&\quad -(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2+2\gamma\|Au_n-Ax_n\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\|u_n-x_n\|^2\\
&\quad\le \alpha_n\big\|f(x_n)-x^*\big\|^2+\beta_n\big\|Sx_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2-\big\|x_{n+1}-x^*\big\|^2
\end{aligned}$$

$$\begin{aligned}
&+ 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\| \\
&\leq \alpha_n \|f(x_n) - x^*\|^2 + \beta_n \|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+ 2\gamma \|Au_n - Ax_n\| \|(T_{r_n}^{F_2} - I)Ax_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$ and $\lim_{n\to\infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{36}$$

From (17), we get

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C[u_n - \lambda_n Du_n] - P_C[x^* - \lambda_n Dx^*]\|^2 \\
&\leq \langle z_n - x^*, (u_n - \lambda_n Du_n) - (x^* - \lambda_n Dx^*)\rangle \\
&= \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^* - \lambda_n(Du_n - Dx^*)\|^2 \\
&\quad - \|u_n - x^* - \lambda_n(Du_n - Dx^*) - (z_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n - \lambda_n(Du_n - Dx^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Du_n - Dx^*\rangle\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\|\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\| \\
&\leq \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\|.
\end{aligned}$$

From (34) and the above inequality, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)(\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2) \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + (1 - \alpha_n)\{\beta_n\|Sx_n - x^*\|^2 \\
&\quad + (1 - \beta_n)(\|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\|)\} \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - (1 - \alpha_n)(1 - \beta_n)\|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&(1 - \alpha_n)(1 - \beta_n)\|u_n - z_n\|^2 \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&+ 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\| \\
&\leq \alpha_n\|f(x_n) - x^*\|^2 + \beta_n\|Sx_n - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+ 2\lambda_n\|u_n - z_n\|\|Du_n - Dx^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$ and $\lim_{n\to\infty}\|Du_n-Dx^*\|=0$, we obtain

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{37}$$

It follows from (36) and (37) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{38}$$

Now, let $z\in F(T)\cap\Omega^*\cap\Lambda$, since for each $i\geq 1$, $V_ix_n\in C$ and $\alpha_n+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)=1$, we have $\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\in C$, and

$$\begin{aligned}
&\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)(x_n-V_ix_n)\\
&\quad=P_C\Bigg[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\Bigg]+(1-\alpha_n)x_n\\
&\qquad-\Bigg(\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\Bigg)+\alpha_nz-x_{n+1}\\
&\quad=P_C\Bigg[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\Bigg]+\alpha_n(z-x_{n+1})\\
&\qquad-P_C\Bigg[\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\Bigg]+(1-\alpha_n)(x_n-x_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n,x_n-x^*\rangle\\
&\quad=\Bigg\langle P_C\Bigg[\alpha_nf(x_n)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_iy_n\Bigg]-P_C\Bigg[\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)V_ix_n+\alpha_nz\Bigg],x_n-x^*\Bigg\rangle\\
&\qquad+\alpha_n\langle z-x_{n+1},x_n-x^*\rangle+(1-\alpha_n)\langle x_n-x_{n+1},x_n-x^*\rangle\\
&\quad\leq\Bigg\|\alpha_n\big(f(x_n)-z\big)+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)(V_iy_n-V_ix_n)\Bigg\|\big\|x_n-x^*\big\|\\
&\qquad+\alpha_n\|z-x_{n+1}\|\big\|x_n-x^*\big\|+(1-\alpha_n)\|x_n-x_{n+1}\|\big\|x_n-x^*\big\|\\
&\quad\leq\alpha_n\big\|f(x_n)-z\big\|\big\|x_n-x^*\big\|+\sum_{i=1}^n(\alpha_{i-1}-\alpha_i)\|y_n-x_n\|\big\|x_n-x^*\big\|\\
&\qquad+\alpha_n\|z-x_{n+1}\|\big\|x_n-x^*\big\|+(1-\alpha_n)\|x_n-x_{n+1}\|\big\|x_n-x^*\big\|\\
&\quad=\alpha_n\big\|f(x_n)-z\big\|\big\|x_n-x^*\big\|+(1-\alpha_n)\|y_n-x_n\|\big\|x_n-x^*\big\|\\
&\qquad+\alpha_n\|z-x_{n+1}\|\big\|x_n-x^*\big\|+(1-\alpha_n)\|x_n-x_{n+1}\|\big\|x_n-x^*\big\|\\
&\quad\leq\alpha_n\big\|f(x_n)-z\big\|\big\|x_n-x^*\big\|\\
&\qquad+(1-\alpha_n)\big\|\beta_nSx_n+(1-\beta_n)z_n-x_n\big\|\big\|x_n-x^*\big\|+\alpha_n\|z-x_{n+1}\|\big\|x_n-x^*\big\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\alpha_n)\|x_n - x_{n+1}\| \|x_n - x^*\| \\
&\le \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + (1-\alpha_n)\beta_n \|Sx_n - x_n\| \|x_n - x^*\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|z_n - x_n\| \|x_n - x^*\| \\
&\quad + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| + (1-\alpha_n)\|x_n - x_{n+1}\| \|x_n - x^*\|.
\end{aligned}
$$

From Lemma 2.9 and the above inequality, we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2 \\
&\le \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle x_n - V_i x_n, x_n - x^*\rangle \\
&\le \alpha_n \|f(x_n) - z\| \|x_n - x^*\| + (1-\alpha_n)\beta_n \|Sx_n - x_n\| \|x_n - x^*\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|z_n - x_n\| \|x_n - x^*\| + \alpha_n \|z - x_{n+1}\| \|x_n - x^*\| \\
&\quad + (1-\alpha_n)\|x_n - x_{n+1}\| \|x_n - x^*\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$
\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2 = 0.
$$

Since $(\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2 \le \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2$ and $\{\alpha_n\}$ is strictly decreasing, we have

$$
\lim_{n\to\infty}\|x_n - V_i x_n\| = 0.
$$

Hence, we obtain

$$
\lim_{n\to\infty}\|x_n - T_i x_n\| = \lim_{n\to\infty}\frac{\|x_n - V_i x_n\|}{(1-k_i)} = 0, \quad \forall i \ge 1.
$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n \rightharpoonup w \in C$. It follows from Lemma 2.4 that $w \in F(T)$. Therefore $w_w(x_n) \subset F(T)$. □

Theorem 3.1 *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $z = P_{\Omega^* \cap \Lambda \cap F(T)} f(z)$, *which is the unique solution of the variational inequality*

$$
\langle (I - f)z, x - z\rangle \ge 0, \quad \forall x \in \Omega^* \cap \Lambda \cap F(T), \tag{39}
$$

which is the optimality condition for a minimization problem

$$
\min_{x\in\Upsilon}\left\{\frac{1}{2}\|x\|^2 - h(x)\right\},
$$

where h *is a potential function for* f *(i.e.,* $h'(x) = f(x)$ *for* $x \in H$*) and* $\Upsilon = \Omega^* \cap \Lambda \cap F(T)$.

Proof Since $\{x_n\}$ is bounded $x_n \rightharpoonup w$ and from Lemma 3.2, we have $w \in F(T)$. Next, we show that $w \in EP(F_1)$. Since $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\big\langle y - u_n, \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big\rangle \geq 0, \quad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$-\frac{1}{r_n}\big\langle y - u_n, \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n), \quad \forall y \in C$$

and

$$-\frac{1}{r_{n_k}}\big\langle y - u_{n_k}, \gamma A^*\big(T_{r_{n_k}}^{F_2} - I\big)Ax_{n_k}\big\rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \geq F_1(y, u_{n_k}), \quad \forall y \in C. \tag{40}$$

Since $\lim_{n\to\infty}\|u_n - x_n\| = 0$, $\lim_{n\to\infty}\|(T_{r_n}^{F_2} - I)Ax_n\| = 0$ and $x_n \rightharpoonup w$, it is easy to observe that $u_{n_k} \to w$. It follows by Assumption 2.1(iv) that $F_1(y, w) \leq 0$, $\forall y \in C$.

For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)w$, we have $y_t \in C$. Then, from Assumption 2.1(i) and (iv), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) \leq tF_1(y_t, y) + (1 - t)F_1(y_t, w) \\ &\leq tF_1(y_t, y). \end{aligned}$$

Therefore $F_1(y_t, y) \geq 0$. From Assumption 2.1(iii), we have $F_1(w, y) \geq 0$, which implies that $w \in EP(F_1)$.

Next, we show that $Aw \in EP(F_2)$. Since $\{x_n\}$ is bounded and $x_n \rightharpoonup w$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w$ and since A is a bounded linear operator so that $Ax_{n_k} \rightharpoonup Aw$. Now set $v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows from (35) that $\lim_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{F_2}Ax_{n_k}$. Therefore from the definition of $T_{r_{n_k}}^{F_2}$, we have

$$F_2(Ax_{n_k} - v_{n_k}, y) + \frac{1}{r_{n_k}}\big\langle y - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k}\big\rangle \geq 0, \quad \forall y \in C.$$

Since F_2 is upper semicontinuous in the first argument, taking lim sup to the above inequality as $k \to \infty$ and using Assumption 2.1(iv), we obtain

$$F_2(Aw, y) \geq 0, \quad \forall y \in C,$$

which implies that $Aw \in EP(F_2)$ and hence $w \in \Lambda$.

Furthermore, we show that $w \in \Omega^*$. Let

$$Tv = \begin{cases} Dv + N_C v, & \forall v \in C, \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $N_C v := \{w \in H : \langle w, v - u\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$. Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega^*$ (see [25]). Let $G(T)$ denote the graph of T and let $(v,u) \in G(T)$. Since $u - Dv \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, u - Dv\rangle \geq 0. \tag{41}$$

On the other hand, it follows from $z_n = P_C[u_n - \lambda_n Du_n]$ and $v \in C$ that

$$\bigl\langle v - z_n, z_n - (u_n - \lambda_n Du_n)\bigr\rangle \geq 0$$

and

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Du_n\right\rangle \geq 0.$$

Therefore, from (41) and inverse strong monotonicity of D, we have

$$\begin{aligned}
\langle v - z_{n_k}, u\rangle &\geq \langle v - z_{n_k}, Dv\rangle \\
&\geq \langle v - z_{n_k}, Dv\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}} + Du_{n_k}\right\rangle \\
&\geq \langle v - z_{n_k}, Dv - Dz_{n_k}\rangle + \langle v - z_{n_k}, Dz_{n_k} - Du_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle \\
&\geq \langle v - z_{n_k}, Dz_{n_k} - Du_{n_k}\rangle - \left\langle v - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\lambda_{n_k}}\right\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - z_n\| = 0$ and $u_{n_k} \to w$, it is easy to observe that $z_{n_k} \to w$. Hence, we obtain $\langle v - w, u\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \Omega^*$. Thus we have

$$w \in \Omega^* \cap \Lambda \cap F(T).$$

Since Ω^*, Λ and $F(T)$ are convex, then $\Omega^* \cap \Lambda \cap F(T)$ is convex. Next, we claim that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{\Omega^*\cap\Lambda\cap F(T)} f(z)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle f(z) - z, x_n - z\bigr\rangle = \limsup_{k\to\infty}\bigl\langle f(z) - z, x_{n_k} - z\bigr\rangle = \bigl\langle f(z) - z, w - z\bigr\rangle \leq 0.$$

Next, we show that $x_n \to z$. From (16), we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\langle x_{n+1} - \alpha_n f(x_n) - \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n, x_{n+1} - z\right\rangle \\
&\quad + \left\langle \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n - z, x_{n+1} - z\right\rangle \\
&\leq \left\langle \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n - z, x_{n+1} - z\right\rangle \\
&\leq \alpha_n\bigl\langle f(x_n) - f(z), x_{n+1} - z\bigr\rangle + \alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle V_i y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n \big\|f(x_n)-f(z)\big\| \|x_{n+1}-z\| + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|V_i y_n - z\| \|x_{n+1}-z\| \\
&\le \alpha_n \rho\|x_n-z\| \|x_{n+1}-z\| + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - z\| \|x_{n+1}-z\| \\
&\le \alpha_n \rho\|x_n-z\| \|x_{n+1}-z\| + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+(1-\alpha_n)\big\{\beta_n\|Sx_n-Sz\| + \beta_n\|Sz-z\| + (1-\beta_n)\|z_n-z\|\big\}\|x_{n+1}-z\| \\
&\le \alpha_n \rho\|x_n-z\| \|x_{n+1}-z\| + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+(1-\alpha_n)\big\{\beta_n\|x_n-z\| + \beta_n\|Sz-z\| + (1-\beta_n)\|x_n-z\|\big\}\|x_{n+1}-z\| \\
&\le \big(1-\alpha_n(1-\rho)\big)\|x_n-z\| \|x_{n+1}-z\| + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+(1-\alpha_n)\beta_n\|Sz-z\| \|x_{n+1}-z\| \\
&\le \frac{1-\alpha_n(1-\rho)}{2}\big(\|x_n-z\|^2 + \|x_{n+1}-z\|^2\big) + \alpha_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+(1-\alpha_n)\beta_n\|Sz-z\| \|x_{n+1}-z\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\le \left(1-\frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n-z\|^2 + \frac{2\alpha_n}{1+\alpha_n(1-\rho)}\big\langle f(z)-z, x_{n+1}-z\big\rangle \\
&\quad+\frac{2(1-\alpha_n)\beta_n}{1+\alpha_n(1-\rho)}\|Sz-z\| \|x_{n+1}-z\| \\
&\le \left(1-\frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\right)\|x_n-z\|^2 + \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\left\{\frac{1}{1-\rho}\big\langle f(z)-z, x_{n+1}-z\big\rangle\right. \\
&\quad\left.+\frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\| \|x_{n+1}-z\|\right\}.
\end{aligned}
$$

Let $\gamma_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}$ and $\delta_n = \frac{2\alpha_n(1-\rho)}{1+\alpha_n(1-\rho)}\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\| \|x_{n+1}-z\|\}$.

Since

$$
\sum_{n=1}^{\infty}\alpha_n = \infty, \qquad 1+\alpha_n(1-\rho)\le 2 \quad \text{and}
$$

$$
\limsup_{n\to\infty}\left\{\frac{1}{1-\rho}\big\langle f(z)-z, x_{n+1}-z\big\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)}\|Sz-z\| \|x_{n+1}-z\|\right\} \le 0,
$$

it follows that

$$
\sum_{n=1}^{\infty}\gamma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0.
$$

Thus all the conditions of Lemma 2.6 are satisfied. Hence we deduce that $x_n \to z$.

$P_{\Omega^*\cap\Lambda\cap F(T)}f$ is a contraction, there exists a unique $z \in C$ such that $z = P_{\Omega^*\cap\Lambda\cap F(T)}f(z)$. From (16), it follows that z is the unique solution of problem (39). This completes the proof. □

Theorem 3.2 *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be an α-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in the first argument. Let $S : C \to H_1$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings such that $F(T) \cap \Omega^* \cap \Lambda \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$\begin{aligned} u_n &= T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr); \\ z_n &= P_C[u_n - \lambda_n A u_n]; \\ y_n &= \beta_n S x_n + (1-\beta_n) z_n; \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{42}$$

where $V_i = k_i I + (1-k_i)T_i$, $0 \leq k_i < 1$, $\{r_n\} \subset (0,\infty)$, $\{\lambda_n\} \subset (0,2\alpha)$ and $\gamma \in (0,1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0,\infty)$,
(c) $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\lim_{n\to\infty} \frac{\frac{1}{\mu}|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|}{\alpha_n \beta_n} = 0$,
(e) *there exists a constant $K > 0$ such that* $\frac{1}{\alpha_n}\bigl|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\bigr| \leq K$,
(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$,
(g) $\liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_{n-1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ generated by Algorithm (42) *converges strongly to $x^* \in \Omega^* \cap \Lambda \cap F(T)$, which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \Omega^* \cap \Lambda \cap F(T). \tag{43}$$

Proof From $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$, without loss of generality, we can assume that $\beta_n \leq (1+\tau)\alpha_n$ for all $n \geq 1$. Hence $\beta_n \to 0$. By a similar argument as that in Lemmas 3.1 and 3.2, we can deduce that $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - z_n\| = 0$ (see (38)) and $(I - V_i)x_n \to 0$. Then we have

$$\|y_n - x_n\| \leq \beta_n \|x_n - Sx_n\| + (1-\beta_n)\|x_n - z_n\| \to 0 \quad \text{as } n \to \infty. \tag{44}$$

It follows that for all $i \geq 1$,

$$\|y_n - V_i x_n\| \leq \|y_n - x_n\| + \|x_n - V_i x_n\| \to 0 \quad \text{as } n \to \infty. \tag{45}$$

From (44) and (45), we have

$$\|y_n - V_i y_n\| \leq \|y_n - V_i x_n\| + \|V_i x_n - V_i y_n\| \leq \|y_n - V_i x_n\| + \|y_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Set $w_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) V_i y_n$. From (32) and (33), we obtain

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|w_n - w_{n-1}\|}{\beta_n} \\
&\leq \bigl(1 - (1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_n} \\
&\quad + M\left(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&= \bigl(1 - (1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \bigl(1 - (1-\rho)\alpha_n\bigr)\|x_n - x_{n-1}\|\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) \\
&\quad + M\left(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq \bigl(1 - (1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\
&\quad + M\left(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq \bigl(1 - (1-\rho)\alpha_n\bigr)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right) \\
&\leq \bigl(1 - (1-\rho)\alpha_n\bigr)\frac{\|w_{n-1} - w_{n-2}\|}{\beta_{n-1}} + \alpha_n K\|x_n - x_{n-1}\| \\
&\quad + M\left(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n}\right).
\end{aligned}$$

Let $\gamma_n = (1-\rho)\alpha_n$ and $\delta_n = \alpha_n K\|x_n - x_{n-1}\| + M(\frac{1}{\mu}\frac{|r_n - r_{n-1}|}{\beta_n} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_n})$. From conditions (a) and (d), we have

$$\sum_{n=1}^{\infty} \gamma_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0.$$

By Lemma 2.6, we obtain

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0, \qquad \lim_{n\to\infty} \frac{\|w_{n+1} - w_n\|}{\beta_n} = \lim_{n\to\infty} \frac{\|w_{n+1} - w_n\|}{\alpha_n} = 0.$$

From (42), we have

$$x_{n+1} = P_C[w_n] - w_n + \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - y_n) + (1 - \alpha_n) y_n.$$

Hence it follows that

$$\begin{aligned} x_n - x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n x_n \\ &\quad - \left(P_C[w_n] - w_n + \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - y_n) + (1 - \alpha_n) y_n \right) \\ &= (1 - \alpha_n) \big[\beta_n (x_n - S x_n) + (1 - \beta_n)(x_n - z_n) \big] + \big(w_n - P_C[w_n] \big) \\ &\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(y_n - V_i y_n) + \alpha_n \big(x_n - f(x_n) \big), \end{aligned}$$

and hence

$$\begin{aligned} \frac{x_n - x_{n+1}}{(1 - \alpha_n)\beta_n} &= x_n - S x_n + \frac{(1 - \beta_n)}{\beta_n} (x_n - z_n) + \frac{1}{(1 - \alpha_n)\beta_n} \big(w_n - P_C[w_n] \big) \\ &\quad + \frac{1}{(1 - \alpha_n)\beta_n} \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(y_n - V_i y_n) + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \big(x_n - f(x_n) \big). \end{aligned}$$

Let $v_n = \frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n}$. For any $z \in \Omega^* \cap \Lambda \cap F(T)$, we have

$$\begin{aligned} \langle v_n, x_n - z \rangle &= \frac{1}{(1 - \alpha_n)\beta_n} \big\langle w_n - P_C[w_n], P_C[w_{n-1}] - z \big\rangle + \frac{\alpha_n}{(1 - \alpha_n)\beta_n} \big\langle (I - f) x_n, x_n - z \big\rangle \\ &\quad + \langle x_n - S x_n, x_n - z \rangle + \frac{(1 - \beta_n)}{\beta_n} \langle x_n - z_n, x_n - z \rangle \\ &\quad + \frac{1}{(1 - \alpha_n)\beta_n} \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle y_n - V_i y_n, x_n - z \rangle. \end{aligned} \tag{46}$$

Since S is a nonexpansive mapping, f is a ρ-contraction mapping and V_i is a k_i-strict pseudo-contraction mapping. Then $(I - S)$ and $(I - V_i)$ are monotone and f is strongly monotone with a coefficient $(1 - \rho)$. We can deduce

$$\begin{aligned} \langle x_n - S x_n, x_n - z \rangle &= \big\langle (I - S) x_n - (I - S) z, x_n - z \big\rangle + \big\langle (I - S) z, x_n - z \big\rangle \\ &\geq \big\langle (I - S) z, x_n - z \big\rangle, \end{aligned} \tag{47}$$

$$\begin{aligned} \big\langle (I - f) x_n, x_n - z \big\rangle &= \big\langle (I - f) x_n - (I - f) z, x_n - z \big\rangle + \big\langle (I - f) z, x_n - z \big\rangle \\ &\geq (1 - \rho) \|x_n - z\|^2 + \big\langle (I - f) z, x_n - z \big\rangle, \end{aligned} \tag{48}$$

$$\begin{aligned} \big\langle (I - V_i) y_n, x_n - z \big\rangle &= \big\langle (I - V_i) y_n - (I - V_i) z, x_n - y_n \big\rangle + \big\langle (I - V_i) y_n - (I - V_i) z, y_n - z \big\rangle \\ &\geq \big\langle (I - V_i) y_n - (I - V_i) z, x_n - y_n \big\rangle \\ &= \big\langle (I - V_i) y_n, x_n - y_n \big\rangle \\ &= \big\langle (I - V_i) y_n, \beta_n (x_n - S x_n) + (1 - \beta_n)(x_n - z_n) \big\rangle. \end{aligned} \tag{49}$$

From (16), we get

$$\begin{aligned}
&\langle w_n - P_C[w_n], P_C[w_{n-1}] - z\rangle \\
&\quad = \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \langle w_n - P_C[w_n], P_C[w_n] - z\rangle \\
&\quad \geq \langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle.
\end{aligned}$$

Then, from (46)-(49), we have

$$\begin{aligned}
\langle v_n, x_n - z\rangle \geq{} & \frac{1}{(1-\alpha_n)\beta_n}\langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\
& + \langle (I-S)z, x_n - z\rangle + \frac{(1-\beta_n)}{\beta_n}\langle x_n - z_n, x_n - z\rangle \\
& + \frac{(1-\beta_n)}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - z_n\rangle \\
& + \frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|x_n - z\|^2 \leq{} & \frac{1}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\|\|w_{n-1} - w_n\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\
& + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\bigl(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\bigr) \\
& - \frac{(1-\beta_n)(1-\alpha_n)}{(1-\rho)\alpha_n}\langle x_n - z_n, x_n - z\rangle \\
& - \frac{(1-\beta_n)}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - z_n\rangle \\
& - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle \\
\leq{} & \frac{\|w_{n-1} - w_n\|}{(1-\rho)\alpha_n}\|w_n - P_C[w_n]\| - \frac{1}{(1-\rho)}\langle (I-f)z, x_n - z\rangle \\
& + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\bigl(\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\bigr) \\
& + \frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\|x_n - z_n\|\|x_n - z\| \\
& + \frac{1}{(1-\rho)}\frac{(1-\beta_n)}{\beta_n}\frac{\beta_n}{\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|(I-V_i)y_n\|\|x_n - z_n\| \\
& - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle.
\end{aligned}$$

By condition (e) of Theorem 3.2, there exists a constant $N > 0$ such that $\frac{1-\beta_n}{\beta_n} \leq N$. Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $v_n \to 0$, $(I-V_i)y_n \to 0$ and $\frac{\|w_{n-1}-w_n\|}{\alpha_n} \to 0$ as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. Since $\{x_n\}$ is bounded, by

Lemma 3.2 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x^* \in F(T)$, and by some similar arguments in Theorem 3.1, we can show that $x^* \in \Omega^* \cap \Lambda \cap F(T)$.

From (46)-(49), it follows that for any $z \in \Omega^* \cap \Lambda \cap F(T)$,

$$\begin{aligned}
&\langle (I-f)x_{n_k}, x_{n_k} - z\rangle \\
&\quad = \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\langle w_{n_k} - P_C[w_{n_k}], P_C[w_{n_k-1}] - z\rangle \\
&\qquad - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle - \frac{(1-\alpha_{n_k})(1-\beta_{n_k})}{\alpha_{n_k}}\langle x_{n_k} - z_{n_k}, x_{n_k} - z\rangle \\
&\qquad - \frac{1}{\alpha_{n_k}}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle y_{n_k} - V_i y_{n_k}, x_{n_k} - z\rangle \\
&\quad \le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle + \frac{1}{\alpha_{n_k}}\big\| w_{n_k} - P_C[w_{n_k}]\big\| \| w_{n_k-1} - w_{n_k}\| \\
&\qquad - \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle + \frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\| x_{n_k} - z_{n_k}\| \| x_{n_k} - z\| \\
&\qquad + \frac{(1-\beta_{n_k})}{\beta_{n_k}}\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1} - \alpha_i)\big\|(I - V_i)y_{n_k}\big\| \| x_{n_k} - z_{n_k}\| \\
&\qquad - \frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1} - \alpha_i)\big\langle (I - V_i)y_{n_k}, x_{n_k} - Sx_{n_k}\big\rangle.
\end{aligned} \tag{50}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, $v_n \to 0$, $(I - V_i)y_n \to 0$ and $\frac{\|w_{n-1} - w_n\|}{\alpha_n} \to 0$, letting $k \to \infty$ in (50), we obtain

$$\big\langle (I-f)x^*, x^* - z\big\rangle \le -\tau\big\langle x^* - Sx^*, x^* - z\big\rangle,$$

i.e.,

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, z - x^*\right\rangle \ge 0.$$

In the following, we show that (43) has a unique solution. Assume that x' is another solution. Then we have

$$\big\langle (I-f)x', x' - x^*\big\rangle \le -\tau\big\langle x' - Sx', x' - x^*\big\rangle, \tag{51}$$

$$\big\langle (I-f)x^*, x^* - x'\big\rangle \le -\tau\big\langle x^* - Sx^*, x^* - x'\big\rangle. \tag{52}$$

Adding (51) and (52), we get

$$\begin{aligned}
(1-\rho)\big\|x' - x^*\big\|^2 &\le \big\langle (I-f)x' - (I-f)x^*, x' - x^*\big\rangle \\
&\le -\tau\big\langle (I-S)x' - (I-S)x^*, x' - x^*\big\rangle \\
&\le 0.
\end{aligned}$$

Then $x' = x^*$. Since (43) has a unique solution, it follows that $w_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $\{x_n\} \to x^*$. This completes the proof. □

4 Applications

In this section, we obtain the following results by using a special case of the proposed method. The first result can be viewed as an extension and improvement of the method of Gu *et al.* [11] for finding an approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 4.1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be an α-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in the first argument. Let $S : C \to H_1$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings such that $F(T) \cap \Omega^* \cap \Lambda \neq \emptyset$, where $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$\begin{aligned} & u_n = T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr); \\ & y_n = \beta_n S x_n + (1 - \beta_n) u_n; \\ & x_{n+1} = P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{53}$$

where $\{r_n\} \subset (0, \infty)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0, \infty)$,
(c) $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\lim_{n\to\infty} \frac{\frac{1}{\mu}|r_n - r_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|}{\alpha_n \beta_n} = 0$,
(e) *there exists a constant $K > 0$ such that* $\frac{1}{\alpha_n} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right| \leq K$,
(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ generated by Algorithm (53) *converges strongly to $x^* \in \Lambda \cap F(T)$, which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \Lambda \cap F(T).$$

Proof Put $\lambda_n = 0$ and $k_i = 0$, $\forall i \geq 1$ in Theorem 3.2. Then conclusion of Corollary 4.1 is obtained. □

The following result can be viewed as an extension and improvement of the method of Yao *et al.* [27] for finding an approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 4.2 *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be an α-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption* 2.1 *and F_2 is upper semicontinuous in the first argument. Let $S : C \to H_1$ be a nonexpansive mapping and $T : C \to C$ be a k-strict pseudo-contraction mapping such that $F(T) \cap \Lambda \neq \emptyset$. Let f be a ρ-contraction mapping. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by*

$$\begin{aligned} u_n &= T_{r_n}^{F_1}\bigl(x_n + \gamma A^*\bigl(T_{r_n}^{F_2} - I\bigr)Ax_n\bigr); \\ y_n &= \beta_n S x_n + (1 - \beta_n)u_n; \\ x_{n+1} &= P_C\bigl[\alpha_n f(x_n) + (1 - \alpha_n)Ty_n\bigr], \quad \forall n \geq 0, \end{aligned} \tag{54}$$

where $\{r_n\} \subset (0,\infty)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ *and $\{\beta_n\}$ is a sequence in* $(0,1)$ *satisfying the following conditions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0,\infty)$,
(c) $\sum_{n=1}^{\infty} (\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\lim_{n\to\infty} \frac{\frac{1}{\mu}|r_n - r_{n-1}| + |\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n|}{\alpha_n \beta_n} = 0$,
(e) *there exists a constant $K > 0$ such that* $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \leq K$,
(f) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Then the sequence $\{x_n\}$ generated by Algorithm (54) *converges strongly to $x^* \in \Lambda \cap F(T)$, which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \Lambda \cap F(T).$$

Proof Put $\lambda_n = 0$, $k_i = 0$ and $T_i = T$, $\forall i \geq 1$ in Theorem 3.2. Then conclusion of Corollary 4.2 is obtained. □

Competing interests
The author declares that they have no competing interests.

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