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Strong convergence theorems on a viscosity approximation method for a finite family of pseudo-contractive mappings in Banach spaces

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Abstract

In this paper, a new viscosity iterative process, which converges strongly to a common element of the set of fixed points of a finite family of pseudo-contractive mappings more general than non-expansive mappings, is introduced in Banach spaces. Strong convergence theorems are obtained under milder conditions. The results presented in this paper extend and unify most of the results that have been proposed for this class of nonlinear mappings.

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1 Introduction

Let E be a real Banach space with dual E^*. A normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that E is smooth if and only if J is single-valued, and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E.

Let C be a closed convex subset of the Banach space E. A mapping $T : C \to C$ is called non-expansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping T is said to be pseudo-contractive if for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2. \tag{1.3}$$

A mapping T is said to be κ-strictly pseudo-contractive if for any $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $0 \leq \kappa \leq 1$ such that

$$\langle x - y - (Tx - Ty), j(x-y) \rangle \geq \kappa \| (I - T)x - (I - T)y \|^2. \tag{1.4}$$

Clearly, the class of pseudo-contractive mappings includes the class of strict pseudo-contractive mappings and non-expansive mappings. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

An operator $A : C \to E$ is called accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0, \quad \forall x, y \in C. \tag{1.5}$$

We observe that A is accretive if and only if $T := I - A$ is pseudo-contractive, where I is the identity mapping on C, and thus a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$, is a fixed point of T. It is now well known that if A is accretive, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts have been devoted to iterative methods for approximating fixed points of T when T is pseudo-contractive (see, *e.g.*, [1–3] and the references contained therein).

A mapping $f : C \to C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0,1)$ such that

$$\| f(x) - f(y) \| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{1.6}$$

For finding an element of the set of fixed points of non-expansive mappings, Halpern [4] was the first to study the convergence of the scheme in 1967:

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T(x_n). \tag{1.7}$$

Viscosity approximation methods are very important because they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations. In a Hilbert space, many authors have studied fixed point problems for pseudo-contractive mappings by the viscosity approximation methods and obtained a series of good results (see [1–3, 5–18]).

In 2000, Moudafi [19] introduced viscosity approximation methods and proved the strong convergence of the following iterative algorithm in a Hilbert space under some suitable conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n). \tag{1.8}$$

Moudafi [19] generalized Halpern's theorems in the direction of viscosity approximations.

In 2008, Yao *et al.* [6] proposed the following modified Mann iterations for non-expansive mappings:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.9}$$

and obtained strong convergence theorems for a common fixed point of non-expansive mappings.

Recently, Zegeye [20] introduced the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{r_n} F_{r_n} x_n, \tag{1.10}$$

where T_{r_n}, F_{r_n} are non-expansive mappings, and obtained a strong convergence theorem but still in a Hilbert space.

On the other hand, for obtaining strong convergence theorems for a family of finite non-expansive mappings, Takahashi [12] defined the following mapping W_n:

$$\begin{cases} U_{n,1} = \alpha_{n,1} T_1 + (1 - \alpha_{n,1}) I, \\ U_{n,2} = \alpha_{n,2} T_2 U_{n,1} + (1 - \alpha_{n,2}) I, \\ \cdots \\ W_n := U_{n,m} = \alpha_{n,m} T_m U_{n,m-1} + (1 - \alpha_{n,m}) I, \end{cases} \tag{1.11}$$

where $\{T_i, i = 1, 2, \ldots, m\}$ are non-expansive mappings, and $F(W_n) = \bigcap_{i=1}^m F(T_i)$.

Our concern now is the following: Is it possible to construct a new sequence in Banach spaces which converges strongly to a common element of fixed points of a finite family of pseudo-contractive mappings?

In this paper, motivated and inspired by the above results, we introduce a new iteration scheme in Banach spaces which converges strongly to a common element of the set of fixed points of continuous pseudo-contractive mappings more general than non-expansive mappings. This provides affirmative answer to the above concern. Our theorems extend and unify most of the results that have been proposed for this class of nonlinear mappings.

2 Preliminaries

Let E be a real Banach space with dual E^*, C be a closed convex subset of E. Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The space E is said to have a Gâteaux differentiable norm if the limit $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in S$ and in this case E is said to be smooth. E is said to be uniformly Gâteaux differentiable if for each $y \in S$, the limit above is uniformly attained for $x \in S$.

In the proof of our main results, we also need the following definitions and results.

Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n; n \in N\} \le \mu(a) \le \sup\{a_n; n \in N\}, \quad \forall a = (a_1, a_2, \ldots) \in l^\infty. \tag{2.1}$$

According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a Banach limit if $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, \ldots) \in l^\infty$.

Define a map $\varphi : E \to \mathbb{R}$ by $\varphi(y) = \mu_n \|x_n - y\|^2$, $\{x_n\} \subset E$ is an arbitrary bounded sequence, then $\varphi(y)$ is convex and continuous, and $\varphi(y) \to \infty$ as $\|y\| \to \infty$. If E is reflexive, there exists $z \in C$ such that $\varphi(z) = \inf_{y \in C} \varphi(y)$ (see [21]). So the set

$$C_{\min} = \left\{ z \in C; \varphi(z) = \inf_{y \in C} \varphi(y) \right\} \neq \varnothing. \tag{2.2}$$

Clearly, $C_{\min}$ is a closed convex subset of E.

In the sequel, we shall use the following lemmas.

Lemma 2.1 [7, 22] *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E, and let μ_n be a Banach limit and $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n \langle y - z, J(x_n - z) \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.2 [6, 7] *Let α be a real number and $(x_0, x_1, \ldots) \in l^\infty$ for all Banach limits satisfying $\mu_n x_n \leq \alpha$. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$, then $\limsup_{n\to\infty} x_n \leq \alpha$.*

Lemma 2.3 [8] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \theta_n) a_n + \sigma_n, \quad n \geq 0,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\sigma_n}{\theta_n} \leq 0$ *or* $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [10] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space, and let $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \geq 0,$$

and

$$\lim_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5 [23] *Let E be a real Banach space with dual E^*, $J : E \to 2^{E^*}$ be the generalized duality pairing, then, $\forall x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle, \quad \forall j(x+y) \in J(x+y).$$

Moreover, by a similar argument as in the proof of Lemmas 3.1 and 3.2 of [24], we get the following lemmas.

Lemma 2.6 *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $T : C \to E$ be a continuous pseudo-contractive mapping. Then, for $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$\langle j(y-z), Tz\rangle - \frac{1}{r}\langle j(y-z), (1+r)z - x\rangle \leq 0, \quad \forall y \in C.$$

Proof Let $x \in C$ and $r > 0$. Let $A := I - T$, clearly A is a continuous accretive mapping. Thus, by a similar argument as in [24], the lemma holds. □

Lemma 2.7 *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $T : C \to E$ be a continuous pseudo-contractive mapping, define the mapping T_r as follows: $x \in E$, $r \in (0, \infty)$*

$$T_r(x) = \left\{ z \in C : \langle j(y-z), Tz\rangle - \frac{1}{r}\langle j(y-z), (1+r)z - x\rangle \leq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is a non-expansive mapping;*
(iii) *$F(T_r) = F(T)$;*
(iv) *$F(T)$ is closed and convex.*

Proof Let $A := I - T$, we note that A is a continuous accretive mapping and that $\langle j(y-z), Tz\rangle - \frac{1}{r}\langle j(y-z), (1+r)z - x\rangle \leq 0$ is equivalent to $\langle j(y-z), Az\rangle + \frac{1}{r}\langle j(y-z), z - x\rangle \geq 0$. Thus, by a similar argument as in [24], the conclusions of (i)-(iv) hold. □

3 Main results

Let C be a nonempty, closed and convex subset of a smoothly, strictly convex and reflexive real Banach space E with dual E^*. Let $\{T_i : C \to C, i = 1, 2, \ldots, m\}$ be a finite family of continuous pseudo-contractive mappings. For the rest of this article, $T_{ir_n}x$ and W_n are defined as follows: for $x \in E$, $r_n \in (0, \infty)$,

$$T_{ir_n}(x) := \left\{ z \in C : \langle j(y-z), T_i z\rangle - \frac{1}{r_n}\langle j(y-z), (1+r_n)z - x\rangle \leq 0, \forall y \in C \right\}, \tag{3.1}$$

$$\begin{cases} U_{n,1} = \alpha_{n,1}T_{1r_n} + (1-\alpha_{n,1})I, \\ U_{n,2} = \alpha_{n,2}T_{2r_n}U_{n,1} + (1-\alpha_{n,2})I, \\ \cdots \\ W_n := U_{n,m} = \alpha_{n,m}T_{mr_n}U_{n,m-1} + (1-\alpha_{n,m})I. \end{cases} \tag{3.2}$$

We know from Lemma 2.7 and Takahashi [12] that T_{ir_n} and W_n are firmly non-expansive mappings and $F(W_n) = \bigcap_{i=1}^m F(T_{ir_n}) = \bigcap_{i=1}^m F(T_i)$. Denote $F := F(W_n)$.

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $\{T_i : C \to C, i = 1, 2, \ldots, m\}$ be a finite continuous pseudo-contractive mapping, for each bounded sequence x_n and for each Banach limit μ_n, $C_{\min}$ is defined as (2.2) satisfying $F \cap C_{\min} \neq \emptyset$. Let $f : C \to C$ be a contraction with a contraction* *coefficient* $\rho \in (0,1)$. *The mappings* T_{ir_n} *and* W_n *are defined as* (3.1) *and* (3.2), *respectively. Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$:

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n) W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \end{cases} \tag{3.3}$$

where $\lambda_n \in [0,1]$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences of nonnegative real numbers in* $[0,1]$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$;
(iv) $\lim_{n\to\infty}(\alpha_{n,i} - \alpha_{n-1,i}) = 0$; $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i : C \to C, i = 1, 2, \ldots, m\}$.

Proof First we prove that $\{x_n\}$ is bounded. Take $p \in F \cap C_{\min}$, because W_n is non-expansive, then we have that

$$\|y_n - p\| \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\|W_n x_n - W_n p\| \leq \|x_n - p\|. \tag{3.4}$$

For $n \geq 0$, because f is contractive, we have from (3.4) that

$$\begin{aligned} \|x_{n+1} - p\| &= \big\|\alpha_n\big(f(x_n) - p\big) + \beta_n(x_n - p) + \gamma_n(y_n - p)\big\| \\ &\leq \alpha_n \big\|f(x_n) - f(p)\big\| + \alpha_n \big\|f(p) - p\big\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \rho\alpha_n \|x_n - p\| + \alpha_n \big\|f(p) - p\big\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \big[1 - (1-\rho)\alpha_n\big]\|x_n - p\| + \alpha_n \big\|f(p) - p\big\| \\ &\leq \max\left\{\|x_0 - p\|, \frac{f(p) - p}{1-\rho}\right\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{W_n x_n\}$ and $\{y_n\}$, $\{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$. Let $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$. Hence we have that

$$\begin{aligned} z_{n+1} - z_n = {} & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(f(x_{n+1}) - f(x_n)\big) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right) f(x_n) \\ & + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(y_{n+1} - y_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) y_n. \end{aligned} \tag{3.5}$$

Because $y_n = \lambda_n x_n + (1-\lambda_n) W_n x_n$, so we have that

$$\begin{aligned} & \|y_{n+1} - y_n\| \\ & \quad \leq \lambda_{n+1}\|x_{n+1} - x_n\| + (1-\lambda_{n+1})\|W_{n+1} x_{n+1} - W_n x_n\| + |\lambda_{n+1} - \lambda_n| \|x_n - W_n x_n\| \\ & \quad \leq \|x_{n+1} - x_n\| + (1-\lambda_{n+1})\|W_{n+1} x_n - W_n x_n\| + |\lambda_{n+1} - \lambda_n| \|x_n - W_n x_n\| \\ & \quad \leq \|x_{n+1} - x_n\| + \|W_{n+1} x_n - W_n x_n\| + |\lambda_{n+1} - \lambda_n| \|x_n - W_n x_n\|. \end{aligned} \tag{3.6}$$

Because T_{ir_n} and $U_{n,m}$ are non-expansive mappings, we have from (3.2) that

$$\begin{aligned}
&\|W_{n+1}x_n - W_n x_n\| \\
&\quad = \|\alpha_{n+1,m}T_{mr_{n+1}}U_{n+1,m-1}x_n + (\alpha_{n,m} - \alpha_{n+1,m})x_n - \alpha_{n,m}T_{mr_n}U_{n,m-1}x_n\| \\
&\quad \le |\alpha_{n+1,m} - \alpha_{n,m}|\{\|x_n\| + \|T_{mr_n}U_{n,m-1}x_n\|\} \\
&\qquad + \alpha_{n+1,m}\|T_{mr_{n+1}}U_{n+1,m-1}x_n - T_{mr_n}U_{n,m-1}x_n\| \\
&\quad \le 2M|\alpha_{n+1,m} - \alpha_{n,m}| + \alpha_{n+1,m}\|T_{mr_{n+1}}U_{n+1,m-1}x_n - T_{mr_n}U_{n,m-1}x_n\| \\
&\quad \le 2M|\alpha_{n+1,m} - \alpha_{n,m}| + \|T_{mr_{n+1}}U_{n+1,m-1}x_n - T_{mr_n}U_{n,m-1}x_n\|,
\end{aligned} \tag{3.7}$$

where $M = \max\{\|x_n\|, \sup_{0\le i\le m-2}\|T_{m-i,r_n}U_{n,m-(i+1)}x_n\|\}$.

Let $u_n = T_{mr_n}v_n$, $u_{n+1} = T_{mr_{n+1}}v_{n+1}$, $v_n = U_{n,m-1}x_n$, $v_{n+1} = U_{n+1,m-1}x_n$, by the definition of mapping T_{ir_n}, we have that

$$\langle j(y-u_n), T_m u_n\rangle - \frac{1}{r_n}\langle j(y-u_n), (1+r_n)u_n - v_n\rangle \le 0, \quad \forall y \in C, \tag{3.8}$$

$$\langle j(y-u_{n+1}), T_m u_{n+1}\rangle - \frac{1}{r_{n+1}}\langle j(y-u_{n+1}), (1+r_{n+1})u_{n+1} - v_{n+1}\rangle \le 0, \quad \forall y \in C. \tag{3.9}$$

Let $y := u_{n+1}$ in (3.8), and let $y := u_n$ in (3.9), we have that

$$\langle j(u_{n+1}-u_n), T_m u_n\rangle - \frac{1}{r_n}\langle j(u_{n+1}-u_n), (1+r_n)u_n - v_n\rangle \le 0, \tag{3.10}$$

$$\langle j(u_n-u_{n+1}), T_m u_{n+1}\rangle - \frac{1}{r_{n+1}}\langle j(u_n-u_{n+1}), (1+r_{n+1})u_{n+1} - v_{n+1}\rangle \le 0. \tag{3.11}$$

Adding (3.10) and (3.11), and because T_m is pseudo-contractive, we have that

$$\left\langle j(u_{n+1}-u_n), \frac{u_n - v_n}{r_n} - \frac{u_{n+1} - v_{n+1}}{r_{n+1}}\right\rangle \ge 0.$$

Therefore we have

$$\left\langle j(u_{n+1}-u_n), u_n - v_n - \frac{r_n(u_{n+1} - v_{n+1})}{r_{n+1}} + u_{n+1} - u_{n+1}\right\rangle \ge 0.$$

Without loss of generality, let b be a real number such that $r_n > b > 0$, $\forall n \in N$, hence we have that

$$\|u_{n+1} - u_n\| \le \|v_{n+1} - v_n\| + \frac{1}{b}|r_{n+1} - r_n|M_1, \tag{3.12}$$

where $M_1 = \sup_{2\le i\le m}\{\|T_{i,r_n}U_{n,i-1}x_n\| + \|U_{n,i-1}x_n\|\}$.

Since $v_n = U_{n,m-1}x_n$, $v_{n+1} = U_{n+1,m-1}x_n$, so we have that

$$\begin{aligned}
&\|v_{n+1} - v_n\| \\
&\quad = \|\alpha_{n+1,m-1}T_{m-1r_{n+1}}U_{n+1,m-2}x_n + (\alpha_{n,m-1} - \alpha_{n+1,m-1})x_n - \alpha_{n,m-1}T_{m-1r_n}U_{n,m-2}x_n\| \\
&\quad \le |\alpha_{n+1,m-1} - \alpha_{n,m-1}|\{\|x_n\| + \|T_{m-1r_n}U_{n,m-2}x_n\|\}
\end{aligned}$$

$$\begin{aligned}&+\alpha_{n+1,m-1}\|T_{m-1r_{n+1}}U_{n+1,m-2}x_n-T_{m-1r_n}U_{n,m-2}x_n\|\\&\le 2M|\alpha_{n+1,m-1}-\alpha_{n,m-1}|+\alpha_{n+1,m-1}\|T_{m-1r_{n+1}}U_{n+1,m-2}x_n-T_{m-1r_n}U_{n,m-2}x_n\|.\end{aligned}\tag{3.13}$$

By the definition of T_{ir_n}, repeating steps from (3.8) to (3.12), we have that

$$\begin{aligned}&\|T_{m-1r_{n+1}}U_{n+1,m-2}x_n-T_{m-1r_n}U_{n,m-2}x_n\|\\&\quad\le\|U_{n+1,m-2}x_n-U_{n,m-2}x_n\|+\frac{1}{b}|r_{n+1}-r_n|M_1.\end{aligned}\tag{3.14}$$

Consequently, we have from (3.12) and (3.13), (3.14) that

$$\begin{aligned}&\|u_{n+1}-u_n\|\\&\quad\le 2M|\alpha_{n+1,m-1}-\alpha_{n,m-1}|+\alpha_{n+1,m-1}\left\{\|U_{n+1,m-2}x_n-U_{n,m-2}x_n\|+\frac{1}{b}|r_{n+1}-r_n|M_1\right\}\\&\qquad+\frac{1}{b}|r_{n+1}-r_n|M_1\\&\quad\le 2M|\alpha_{n+1,m-1}-\alpha_{n,m-1}|+\|U_{n+1,m-2}x_n-U_{n,m-2}x_n\|+\frac{2}{b}|r_{n+1}-r_n|M_1\\&\quad\le\cdots\\&\quad\le 2M\sum_{i=2}^{m-1}|\alpha_{n+1,i}-\alpha_{n,i}|+\frac{m-1}{b}|r_{n+1}-r_n|M_1+\|U_{n+1,1}x_n-U_{n,1}x_n\|.\end{aligned}\tag{3.15}$$

From (3.2) we have that

$$\begin{aligned}\|U_{n+1,1}x_n-U_{n,1}x_n\|&=\big\|(\alpha_{n,1}-\alpha_{n+1,1})x_n+\alpha_{n+1,1}T_{1r_{n+1}}x_n-\alpha_{n,1}T_{1r_n}x_n\big\|\\&\le 2M|\alpha_{n+1,1}-\alpha_{n,1}|+\|T_{1r_{n+1}}x_n-T_{1r_n}x_n\|.\end{aligned}\tag{3.16}$$

By the definition of T_{ir_n}, repeating steps from (3.8) to (3.12), we have that

$$\|T_{1r_{n+1}}x_n-T_{1r_n}x_n\|\le\frac{|r_{n+1}-r_n|}{b}M_2,\tag{3.17}$$

where $M_2=\max\{M_1,\sup\{\|T_{1r_n}x_n\|+\|x_n\|\}\}$. Substituting (3.17) into (3.16), (3.16) into (3.15), (3.15) into (3.7), we have that

$$\|W_{n+1}x_n-W_nx_n\|\le 2M\sum_{i=1}^{m}|\alpha_{n+1,i}-\alpha_{n,i}|+\frac{m}{b}|r_{n+1}-r_n|M_2.\tag{3.18}$$

Hence we have from (3.5)-(3.7) and (3.18) that

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{(\rho-1)\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(2M\sum_{i=1}^{N}|\alpha_{n+1,i}-\alpha_{n,i}|+\frac{m}{b}|r_{n+1}-r_n|M_2\right)\\&\qquad+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\left\{\|f(x_n)\|+\|y_n\|\right\}+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1}-\lambda_n|\|x_n-W_nx_n\|.\end{aligned}$$

Notice conditions (ii) and (iii), (iv), we have that

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\big)=0.$$

Hence we have from Lemma 2.4 that

$$\limsup_{n\to\infty}\|z_n-x_n\|=0. \tag{3.19}$$

Therefore we have that

$$\|x_{n+1}-x_n\|=|1-\beta_n|\|z_n-x_n\|\to 0. \tag{3.20}$$

Finally we show that $\{x_n\}$ converges strongly to $p\in F\cap C_{\min}$. Because $\mu_n\|x_n-p\|^2=\inf_{y\in C}\mu_n\|x_n-y\|^2$, we have from Lemma 2.1 that

$$\mu_n\langle f(p)-p,J(x_n-p)\rangle\le 0. \tag{3.21}$$

Due to the norm-weak* uniform continuity of the duality mapping J, it follows from (3.20) that

$$\lim_{n\to\infty}\big(\langle f(p)-p,J(x_{n+1}-p)\rangle-\langle f(p)-p,J(x_n-p)\rangle\big)=0.$$

Hence, the sequence $\{\langle f(p)-p,J(x_n-p)\rangle\}$ satisfies the conditions of Lemma 2.2. As a result, we must have

$$\limsup_{n\to\infty}\langle f(p)-p,J(x_{n+1}-p)\rangle\le 0. \tag{3.22}$$

On the other hand, since f is contractive with a contraction coefficient $\rho\in(0,1)$, we have from (3.3), (3.4) and Lemma 2.5 that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad=\big\|\alpha_n\big(f(x_n)-p\big)+\beta_n(x_n-p)+\gamma_n(y_n-p)\big\|^2\\
&\quad\le\big\|\beta_n(x_n-p)+\gamma_n(y_n-p)\big\|^2+2\alpha_n\langle f(x_n)-p,J(x_{n+1}-p)\rangle\\
&\quad\le(1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\langle f(x_n)-f(p),J(x_{n+1}-p)\rangle+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&\quad\le(1-\alpha_n)^2\|x_n-p\|^2+2\rho\alpha_n\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&\quad\le(1-\alpha_n)^2\|x_n-p\|^2+\rho\alpha_n\big[\|x_n-p\|^2+\|x_{n+1}-p\|^2\big]+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1}-p\|^2\le{}&\left[1-\frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}\right]\|x_n-p\|^2+\frac{2\alpha_n}{1-\rho\alpha_n}\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&+\frac{\alpha_n^2}{1-\rho\alpha_n}\|x_n-p\|^2.
\end{aligned} \tag{3.23}$$

Let $\theta_n = \frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}$, $\sigma_n = \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(p)-p, J(x_{n+1}-p)\rangle + \frac{\alpha_n^2}{1-\rho\alpha_n}\|x_n - p\|^2$. Since $\{x_n\}$ is bounded, according to Lemma 2.3 and formula (3.23), we have that $\lim_{n\to\infty}\|x_n - p\| = 0$, *i.e.*, the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_i : C \to C$, $i = 1, 2, \ldots, m$. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $\{T_i : C \to C, i = 1, 2, \ldots, m\}$ be a finite family of continuous pseudo-contractive mappings, for each bounded sequence x_n and for each Banach limit μ_n, $C_{\min}$ is defined as* (2.2) *satisfying $F \cap C_{\min} \neq \varnothing$, $f : C \to C$ is a contraction with a contraction coefficient $\rho \in (0,1)$. The mappings T_{ir_n} and W_n are defined as* (3.1) *and* (3.2), *respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n)W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \end{cases} \tag{3.24}$$

where $\lambda_n \in [0,1]$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of nonnegative real numbers in $[0,1]$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $\lim_{n\to\infty}\lambda_n = 1$;

(iv) $\lim_{n\to\infty}(\alpha_{n,i} - \alpha_{n-1,i}) = 0$; $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : C \to C, i = 1, 2, \ldots, m\}$.

Proof Take $p \in F \cap C_{\min}$, from (3.24) we can obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\ &\leq \alpha_n\|f(x_n) - x_n\| + \gamma_n(1-\lambda_n)\|W_n x_n - x_n\|. \end{aligned}$$

Notice the boundedness of the sequences $\{x_n\}$ and $\{W_n x_n\}$. According to conditions (ii) and (iii), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Similar to Theorem 3.1, we can obtain the result. □

If in Theorem 3.1 and Theorem 3.2 we let $f :\equiv u \in C$ be a constant mapping, we have the following corollary.

Corollary 3.3 *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $\{T_i : C \to C, i = 1, 2, \ldots, m\}$ be a finite family of continuous pseudo-contractive mappings, for each bounded sequence x_n and for each Banach limit μ_n, $C_{\min}$ is defined as* (2.2) *satisfying $F \cap C_{\min} \neq \varnothing$. The mappings T_{ir_n} and W_n are defined as* (3.1) *and* (3.2), *respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$*

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n)W_n x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n, \end{cases} \tag{3.25}$$

where $\lambda_n \in [0,1]$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are the sequences of nonnegative real numbers in $[0,1]$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$ *or* $\lim_{n\to\infty} \lambda_n = 1$;
(iv) $\lim_{n\to\infty}(\alpha_{n,i} - \alpha_{n-1,i}) = 0$; $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i : C \to C, i = 1, 2, \ldots, m\}$.

Theorem 3.4 *Let* C *be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space* E. *Let* $T : C \to C$ *be a continuous pseudo-contractive mapping for each bounded sequence* x_n *and for each Banach limit* μ_n, $C_{\min}$ *be defined as* (2.2) *satisfying* $F(T) \cap C_{\min} \neq \emptyset$, $f : C \to C$ *be a contraction with a contraction coefficient* $\rho \in (0, 1)$. *Mapping* T_{r_n} *is defined as follows*: $x \in C$, $r_n \in (0, \infty)$

$$T_{r_n}(x) = \left\{ z \in C : \langle j(y - z), Tz \rangle - \frac{1}{r_n} \langle j(y - z), (1 + r_n)z - x \rangle \leq 0, \forall y \in C \right\}.$$

Let $\{x_n\}$ *be a sequence generated by* $x_0 \in C$

$$\begin{cases} y_n = \lambda_n x_n + (1 - \lambda_n) T_{r_n} x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \end{cases} \tag{3.26}$$

where $\lambda_n \in [0, 1]$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are the sequences of nonnegative real numbers in* $[0, 1]$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$ *or* $\lim_{n\to\infty} \lambda_n = 1$;
(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence x_n *converges strongly to a fixed point of* T.

Proof Putting $\alpha_{n,i} = 1$ in (3.2), we have $W_n = T_{r,n}$; from Lemma 2.4 and Theorems 3.1 and 3.2, we can obtain the result. □

Competing interests
The author declares that they have no competing interests.

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