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Some eigenvalue results for perturbations of maximal monotone operators

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Abstract

We study the solvability of a nonlinear eigenvalue problem for maximal monotone operators under a normalization observation. The investigation is based on degree theories for appropriate classes of operators, and a regularization method by the duality operator is used. Let X be a real reflexive Banach space with its dual X^* and Ω be a bounded open set in X. Suppose that $T : D(T) \subset X \to X^*$ is a maximal monotone operator and $C : (0, \infty) \times \overline{\Omega} \to X^*$ is a bounded demicontinuous operator satisfying condition (S_+). Applying the Browder degree theory, we solve a nonlinear eigenvalue problem of the form $Tx + C(\lambda, x) = 0$. In the case where $Tx + \lambda Cx = 0$, an eigenvalue result for generalized pseudomonotone densely defined perturbations is obtained by the Kartsatos-Skrypnik degree theory.

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1 Introduction and preliminaries

Eigenvalue theory is closely related to the problem of solving nonlinear equations which was initiated by Krasnosel'skii [1] for compact operators. Regarding maximal monotone operators, it has been extensively investigated by many researchers in various aspects, with applications to evolution equations and differential equations; see, *e.g.*, [2–5]. The study was mostly based on degree theories for appropriate classes of operators and the usual method of regularization by means of the duality operator; see [6–10].

Let X be a real reflexive Banach space with its dual X^* and Ω be a bounded open set in X. Suppose that $T : D(T) \subset X \to X^*$ is a maximal monotone operator. We consider the nonlinear eigenvalue problem

$$Tx + \lambda Cx = 0, \tag{E}$$

where $C : D(C) \subset X \to X^*$ is an operator. When the operator C or the resolvents of the operator T are compact, this problem was studied, for instance, by Guan-Kartsatos [11], Kartsatos [12], and Li-Huang [13], where the method is to use the Leray-Schauder degree theory. More generally, an implicit eigenvalue problem of the form

$$Tx + C(\lambda, x) = 0 \tag{IE}$$

was investigated in [3, 4], where $C : (0, \infty) \times \overline{\Omega} \to X^*$ is a bounded operator to be specified later. Kartsatos and Skrypnik [4] observed the above eigenvalue problem provided that the following property ($\mathcal{P}$) is fulfilled: For $\varepsilon > 0$, there exists a $\lambda > 0$ such that the equation

$$Tx + C(\lambda, x) + \varepsilon Jx = 0$$

has no solution in $D(T) \cap \Omega$, where J denotes the duality operator. This property has a close relation to the use of topological degree in eigenvalue theory by the regularization method. It is shown in [3] that two conditions about the weak closure of a certain set consisting of normalized vectors and the asymptotic behavior of the operator C at infinity of λ, called *normalized conditions*, are main ingredients in solving an eigenvalue problem. In this connection, we are now interested in finding eigenvectors under normalized conditions more concrete than property ($\mathcal{P}$).

The purpose of this paper is to establish the existence of solutions for the above eigenvalue problems under normalized conditions, motivated by the works of Kartsatos and Skrypnik [3, 4]. We first study implicit eigenvalue problem (IE), where C is assumed to be a bounded demicontinuous operator satisfying condition (S_+). For this, a key tool is the Browder degree given in [6]. Next, we consider two types of the operator C for eigenvalue problem (E). For the one, we apply the Browder degree theory for nonlinear operators of the form $T + f$ with T maximal monotone and f bounded with condition (S_+) introduced in [7]. In the other case, where C is a generalized pseudomonotone, quasibounded, and densely defined operator, we solve this problem by using the Kartsatos-Skrypnik degree theory for densely defined $(\tilde{S}_+)$-perturbations of maximal monotone operators developed in [8].

This paper is organized as follows. In Section 2, we study the solvability of implicit eigenvalue problem (IE) with normalized conditions based on the Browder degree theory. Section 3 contains an eigenvalue result for problem (E) as a special case of (IE). In Section 4, we deal with eigenvalue problem (E) for densely defined perturbations of maximal monotone operators under normalized conditions.

Let X be a real Banach space with dual space X^*, Ω be a nonempty subset of X, and Y be another real Banach space. Let $\overline{\Omega}$, $\operatorname{int}\Omega$, and $\partial\Omega$ denote the closure, the interior, and the boundary of Ω in X, respectively. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence. An operator $F : \Omega \to Y$ is said to be *bounded* if F maps bounded subsets of Ω into bounded subsets of Y. F is said to be *demicontinuous* if for every $x_0 \in \Omega$ and for every sequence $\{x_n\}$ in Ω with $x_n \to x_0$, we have $Fx_n \rightharpoonup Fx_0$.

Let $T : D(T) \subset X \to X^*$ be an operator. Then T is said to be *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \text{for every } x, y \in D(T).$$

T is said to be *maximal monotone* if it is monotone and it follows from $(x, x^*) \in X \times X^*$ and

$$\langle x^* - Ty, x - y\rangle \geq 0 \quad \text{for every } y \in D(T)$$

that $x \in D(T)$ and $Tx = x^*$.

T is said to be *generalized pseudomonotone* if for every sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0$, $Tx_n \rightharpoonup h^*$ and

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x_0\rangle \leq 0,$$

we have $x_0 \in D(T)$, $Tx_0 = h^*$, and $\lim_{n\to\infty} \langle Tx_n, x_n\rangle = \langle Tx_0, x_0\rangle$.

We say that T satisfies condition $(\tilde{S}_+)$ if for every sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0$, $Tx_n \rightharpoonup h^*$ and

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$, $x_0 \in D(T)$, and $Tx_0 = h^*$.

We say that T satisfies condition (S_+) if for every sequence $\{x_n\}$ in $D(T)$ with $x_n \rightharpoonup x_0$ and

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$.

We say that T satisfies condition (S_0) on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and $Tx_n \rightharpoonup h^*$ and

$$\lim_{n\to\infty} \langle Tx_n, x_n \rangle = \langle h^*, x_0 \rangle,$$

we have $x_n \to x_0$.

We say that T satisfies condition (S_q) on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and $Tx_n \to h^*$, we have $x_n \to x_0$.

We say that T satisfies condition $(T_\infty^{(0)})$ on a set $M \subset D(T)$ if for every sequence $\{x_n\}$ in M with $\|Tx_n\| \to \infty$, we have $\|Tx_n\|^{-1} Tx_n \rightharpoonup 0$.

It is obvious from the definitions that (S_+) implies (S_0) and (S_0) implies (S_q). Note that if T satisfies condition $(\tilde{S}_+)$ and X is reflexive, then T is generalized pseudomonotone. For the above definitions, we refer to, *e.g.*, [3, 4, 8], and [5].

Let $C : [0,\infty) \times M \to X^*$ be an operator, where M is a subset of X. Then $C(t,x)$ is said to be *continuous in t uniformly* with respect to $x \in M$ if for every $t_0 \in [0,\infty)$ and for every sequence $\{t_n\}$ in $[0,\infty)$ with $t_n \to t_0$, we have $C(t_n, x) \to C(t_0, x)$ uniformly with respect to $x \in M$.

We say that C satisfies condition (S_+) if for every $\lambda \in (0,\infty)$ and for every sequence $\{x_n\}$ in M with $x_n \rightharpoonup x_0$ and

$$\limsup_{n\to\infty} \langle C(\lambda, x_n), x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$.

Throughout this paper, X will always be an infinite dimensional real reflexive Banach space which has been renormed so that X and X^* are locally uniformly convex.

An operator $J_\psi : X \to X^*$ is said to be a *duality operator* if

$$\langle J_\psi x, x \rangle = \psi(\|x\|)\|x\| \quad \text{and} \quad \|J_\psi x\| = \psi(\|x\|) \quad \text{for } x \in X,$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, $\psi(0) = 0$, and $\psi(t) \to \infty$ as $t \to \infty$ called a *gauge function*. If ψ is the identity map I, then $J := J_I$ is called a *normalized duality operator*. It is known in [14] that J_ψ is continuous, bounded, surjective, strictly monotone, maximal monotone and satisfies condition (S_+).

The following demiclosedness property of maximal monotone operators will be frequently used; see [5].

Lemma 1.1 *Let $T : D(T) \subset X \to X^*$ be a maximal monotone operator. Then for every sequence $\{x_n\}$ in $D(T)$, $x_n \to x$ in X and $Tx_n \rightharpoonup x^*$ in X^* imply that $x \in D(T)$ and $Tx = x^*$.*

2 Implicit eigenvalue problem

In this section, we establish the existence of a solution for a nonlinear implicit eigenvalue problem under normalized conditions by using the Browder degree theory for class (S_+).

Recall that a mapping $H : [0,1] \times \overline{\Omega} \to X^*$ is of class (S_+) if the following condition holds:

For any sequence $\{u_j\}$ in $\overline{\Omega}$ with $u_j \rightharpoonup u_0$ and any sequence $\{t_j\}$ in $[0,1]$ with $t_j \to t_0$ for which

$$\limsup_{j\to\infty} \langle H(t_j,u_j), u_j - u_0 \rangle \le 0,$$

we have $u_j \to u_0$ and $H(t_j,u_j) \rightharpoonup H(t_0,u_0)$; see [2, 6].

As mentioned in the introduction, Kartsatos and Skrypnik [4] gave the following result provided that property $(\mathcal{P})$ is fulfilled. In a more concrete situation, we adopt the normalization method considered in [3].

Theorem 2.1 *Let Ω be a bounded open set in X with $0 \in \Omega$. Let $T : D(T) \subset X \to X^*$ be a maximal monotone operator such that the closure of Ω is included in the interior of $D(T)$ with $T(0) = 0$ and T satisfies condition $(T_\infty^{(0)})$ on $\overline{\Omega}$. Assume that $C : [0,\infty) \times \overline{\Omega} \to X^*$ is demicontinuous, bounded and satisfies condition (S_+) such that $C(0,x) = 0$ for all $x \in \overline{\Omega}$ and $C(t,x)$ is continuous in t uniformly with respect to $x \in \overline{\Omega}$. Further assume that*

(c1) *There exists a positive number $\mathcal{N}$ such that the weak sequential closure of the set*

$$G = \left\{ \frac{C(\lambda,x)}{\|C(\lambda,x)\|} : \lambda \ge \mathcal{N}, x \in \overline{\Omega}, \|Jx + Tx\| \le 2M(\lambda) \right\}$$

does not contain the origin 0, *where*

$$M(\lambda) = \sup\{ \| C(\lambda,x) \| : x \in \overline{\Omega} \}.$$

(c2) $\lim_{\lambda\to\infty} m(\lambda) = \infty$, *where* $m(\lambda) = \inf\{\|C(\lambda,x)\| : x \in \overline{\Omega}\}$.

Then the following statements hold:

(a) *For each $\varepsilon > 0$, there exists a point $(\lambda_\varepsilon, x_\varepsilon)$ in $(0,\infty) \times \partial\Omega$ such that*

$$Tx_\varepsilon + C(\lambda_\varepsilon, x_\varepsilon) + \varepsilon J x_\varepsilon = 0.$$

(b) *If $0 \notin T(\partial\Omega)$ and T satisfies condition (S_q) on $\partial\Omega$, then the implicit eigenvalue problem*

$$Tx + C(\lambda,x) = 0$$

has a solution (λ_0, x_0) in $(0,\infty) \times \partial\Omega$.

Proof (a) For our aim, we use the Browder degree d_B given in [6]. Let ε be any positive number. We first prove that there is a number $\Lambda \in (0,\infty)$ such that

$$d_B\big(T + C(\Lambda,\cdot) + \varepsilon J, \Omega, 0\big) = 0. \tag{2.1}$$

Assume the contrary. For a sequence $\{\Lambda_n\}$ in $(0,\infty)$ with $\Lambda_n \to \infty$, the following occurs:

For each $n \in \mathbb{N}$, either there exists a point $x_n \in \Omega$ such that $Tx_n + C(\Lambda_n, x_n) + \varepsilon Jx_n = 0$, in view of $d_B(T + C(\Lambda_n, \cdot) + \varepsilon J, \Omega, 0) \neq 0$, or there exists a point $x_n \in \partial\Omega$ such that $Tx_n + C(\Lambda_n, x_n) + \varepsilon Jx_n = 0$. Thus, we get a sequence $\{x_n\}$ in $\overline{\Omega}$ such that

$$Tx_n + C(\Lambda_n, x_n) + \varepsilon Jx_n = 0. \tag{2.2}$$

This implies

$$\|Jx_n + Tx_n\| \leq \big\|(1-\varepsilon)Jx_n\big\| + \big\|C(\Lambda_n, x_n)\big\| \leq 2M(\Lambda_n)$$

for sufficiently large n. Since the sequence $\{\|C(\Lambda_n, x_n)\|^{-1}C(\Lambda_n, x_n)\}$ is bounded in the reflexive Banach space X^*, we may suppose that $\|C(\Lambda_n, x_n)\|^{-1}C(\Lambda_n, x_n)$ converges weakly to some $h_0 \in X^*$. Using (c1), it is clear that $h_0 \neq 0$. It follows from (2.2) that

$$\frac{Tx_n}{\|Tx_n\|} \rightharpoonup -h_0. \tag{2.3}$$

However, $\|C(\Lambda_n, x_n)\| \to \infty$ by (c2) implies $\|Tx_n\| \to \infty$, which is a contradiction to condition $(T_\infty^{(0)})$. Hence assertion (2.1) holds.

Next, we consider a mapping $H : [0,1] \times \overline{\Omega} \to X^*$ given by

$$H(t,x) = Tx + C(t\Lambda, x) + \varepsilon Jx \quad \text{for } (t,x) \in [0,1] \times \overline{\Omega}.$$

Then H is of class (S_+). To prove this, let $\{u_j\}$ be any sequence in $\overline{\Omega}$ with $u_j \rightharpoonup u_0$ and $\{t_j\}$ be any sequence in $[0,1]$ with $t_j \to t_0$ such that

$$\limsup_{j\to\infty} \big\langle H(t_j, u_j), u_j - u_0\big\rangle \leq 0. \tag{2.4}$$

Since the operators T and J are monotone, it follows from

$$\big\langle H(t_j, u_j), u_j - u_0\big\rangle = \langle Tu_j, u_j - u_0\rangle + \big\langle C(t_j\Lambda, u_j), u_j - u_0\big\rangle + \varepsilon\langle Ju_j, u_j - u_0\rangle \tag{2.5}$$

that

$$\big\langle H(t_j, u_j), u_j - u_0\big\rangle \geq \langle Tu_0, u_j - u_0\rangle + \big\langle C(t_j\Lambda, u_j), u_j - u_0\big\rangle + \varepsilon\langle Ju_0, u_j - u_0\rangle. \tag{2.6}$$

By (2.4) and (2.6), we have

$$\limsup_{j\to\infty} \big\langle C(t_j\Lambda, u_j), u_j - u_0\big\rangle \leq 0. \tag{2.7}$$

There are two cases to consider. If $t_0 = 0$, then $C(t_j\Lambda, u_j) \to 0$ and so

$$\lim_{j\to\infty} \big\langle C(t_j\Lambda, u_j), u_j - u_0\big\rangle = 0. \tag{2.8}$$

Since (2.5) implies

$$\big\langle H(t_j,u_j),u_j-u_0\big\rangle \geq \langle Tu_0,u_j-u_0\rangle + \big\langle C(t_j\Lambda,u_j),u_j-u_0\big\rangle + \varepsilon\langle Ju_j,u_j-u_0\rangle,$$

it follows from (2.4) and (2.8) that

$$\limsup_{j\to\infty}\langle Ju_j,u_j-u_0\rangle \leq 0.$$

Since J satisfies condition (S_+), we obtain

$$u_j \to u_0,$$

which implies

$$Tu_j \rightharpoonup Tu_0, \qquad C(t_j\Lambda,u_j) \rightharpoonup C(0,u_0), \quad \text{and} \quad Ju_j \to Ju_0$$

on observing that T is demicontinuous on $\overline{\Omega}$. This means that $H(t_j,u_j) \rightharpoonup H(0,u_0)$. If $t_0 > 0$, we have

$$\begin{aligned}
&\limsup_{j\to\infty}\big\langle C(t_0\Lambda,u_j),u_j-u_0\big\rangle \\
&\quad\leq \limsup_{j\to\infty}\big\langle C(t_j\Lambda,u_j),u_j-u_0\big\rangle + \limsup_{j\to\infty}\big[-\big\langle C(t_j\Lambda,u_j)-C(t_0\Lambda,u_j),u_j-u_0\big\rangle\big]
\end{aligned}$$

and hence by (2.7),

$$\limsup_{j\to\infty}\big\langle C(t_0\Lambda,u_j),u_j-u_0\big\rangle \leq \limsup_{j\to\infty}\big[\big\|C(t_j\Lambda,u_j)-C(t_0\Lambda,u_j)\big\|\,\|u_j-u_0\|\big] = 0.$$

Since C satisfies condition (S_+), we get $u_j \to u_0$ from which $Tu_j \rightharpoonup Tu_0$, $C(t_j\Lambda,u_j) \rightharpoonup C(t_0\Lambda,u_0)$, and $Ju_j \to Ju_0$. Consequently, $H(t_j,u_j) \rightharpoonup H(t_0,u_0)$. We have just shown that the mapping H is of class (S_+).

We are now ready to apply the degree theory of Browder [6, 7]. Then we have

$$d_B\big(H(0,\cdot),\Omega,0\big) = d_B(T+\varepsilon J,\Omega,0) = 1.$$

The last equality is based on Theorem 3 in [6] because the operator $T+\varepsilon J$ is strictly monotone and demicontinuous on $\overline{\Omega}$ and satisfies condition (S_+). On the other hand, it is shown in (2.1) that

$$d_B\big(H(1,\cdot),\Omega,0\big) = d_B\big(T+C(\Lambda,\cdot)+\varepsilon J,\Omega,0\big) = 0.$$

Hence, in view of Theorem 4 in [6], there exist $t_0 \in [0,1]$ and $x_0 \in \partial\Omega$ such that

$$Tx_0 + C(t_0\Lambda,x_0) + \varepsilon Jx_0 = 0.$$

It follows from the injectivity of $T+\varepsilon J$ that $t_0 > 0$. Consequently, if we let $\lambda_\varepsilon := t_0\Lambda$ and $x_\varepsilon := x_0$, then we have $\lambda_\varepsilon \in (0,\infty)$ and $Tx_\varepsilon + C(\lambda_\varepsilon,x_\varepsilon) + \varepsilon Jx_\varepsilon = 0$.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0,\infty)$ such that $\varepsilon_n \to 0$. According to statement (a), there exists a sequence $\{(\lambda_{\varepsilon_n}, x_{\varepsilon_n})\}$ in $(0,\infty) \times \partial\Omega$ such that

$$Tx_{\varepsilon_n} + C(\lambda_{\varepsilon_n}, x_{\varepsilon_n}) + \varepsilon_n J x_{\varepsilon_n} = 0.$$

If we set $x_n := x_{\varepsilon_n}$ and $\lambda_n := \lambda_{\varepsilon_n}$, it can be written in the form

$$Tx_n + C(\lambda_n, x_n) + \varepsilon_n J x_n = 0. \tag{2.9}$$

Without loss of generality, we may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \quad \text{and} \quad C(\lambda_n, x_n) \rightharpoonup c^*, \tag{2.10}$$

where $\lambda_0 \in [0,\infty]$, $x_0 \in X$, and $c^* \in X^*$. Note that λ_0 belongs to $(0,\infty)$. In fact, if $\lambda_0 = 0$, then $C(\lambda_n, x_n) \to 0$ implies $Tx_n \to 0$. Since T satisfies condition (S_q) on $\partial\Omega$, we obtain that $x_n \to x_0 \in \partial\Omega$ and therefore $Tx_0 = 0$, which contradicts the hypothesis that $0 \notin T(\partial\Omega)$. If $\lambda_0 = \infty$, then (c2) implies $\|C(\lambda_n, x_n)\| \to \infty$ and so $\|Tx_n\| \to \infty$. As in (2.3), a similar argument proves that $\|Tx_n\|^{-1} Tx_n$ converges weakly to some nonzero vector, which contradicts condition $(T_\infty^{(0)})$. Thus we have shown that $\lambda_0 \in (0,\infty)$.

For the next aim, we now show that

$$\limsup_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \le 0. \tag{2.11}$$

Assume that (2.11) is false. Then there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle > 0.$$

Hence we obtain from (2.9) that

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x_0 \rangle < 0.$$

Noticing by (2.9) and (2.10) that $Tx_n \rightharpoonup -c^*$, we get

$$\limsup_{n\to\infty} \langle Tx_n, x_n \rangle < \lim_{n\to\infty} \langle Tx_n, x_0 \rangle = \langle -c^*, x_0 \rangle. \tag{2.12}$$

For every $x \in D(T)$, we have by the monotonicity of T

$$\begin{aligned} \liminf_{n\to\infty} \langle Tx_n, x_n \rangle &\ge \liminf_{n\to\infty} \big[\langle Tx_n, x \rangle + \langle Tx, x_n - x \rangle \big] \\ &= \langle -c^*, x \rangle + \langle Tx, x_0 - x \rangle, \end{aligned}$$

which implies along with (2.12)

$$\langle -c^* - Tx, x_0 - x \rangle > 0. \tag{2.13}$$

By the maximal monotonicity of T, we have $x_0 \in D(T)$ and $Tx_0 = -c^*$. Letting $x = x_0$ in (2.13), we get a contradiction. Therefore, (2.11) is true.

Since $C(\lambda_n, x_n) - C(\lambda_0, x_n) \to 0$, it follows from (2.11) and

$$\langle C(\lambda_n, x_n), x_n - x_0 \rangle = \langle C(\lambda_n, x_n) - C(\lambda_0, x_n), x_n - x_0 \rangle + \langle C(\lambda_0, x_n), x_n - x_0 \rangle$$

that

$$\limsup_{n\to\infty} \langle C(\lambda_0, x_n), x_n - x_0 \rangle \leq 0.$$

Since C satisfies condition (S_+) and $\lambda_0 \in (0, \infty)$, we have $x_n \to x_0 \in \partial\Omega$, which implies $C(\lambda_n, x_n) \rightharpoonup C(\lambda_0, x_0)$. Hence we obtain from (2.9) that $Tx_n \rightharpoonup -C(\lambda_0, x_0)$. By Lemma 1.1, we conclude that $Tx_0 + C(\lambda_0, x_0) = 0$. This completes the proof. □

Remark 2.2 In the proof of Theorem 2.1, the demicontinuity of T on $\overline{\Omega}$ is needed to show that H is of class (S_+). This is guaranteed under an additional condition $\overline{\Omega} \subset \operatorname{int} D(T)$. Actually, local boundedness of T on $\operatorname{int} D(T)$ implies the demicontinuity of T; see, *e.g.*, [5].

3 Eigenvalue problem with condition (S_+)

In this section, we study a multiplicative eigenvalue problem as a special case of the implicit eigenvalue problem in the previous section. As a key tool, we employ the Browder degree for nonlinear operators of the form $T + f$ with T maximal monotone and f bounded with condition (S_+).

We give a variant of Corollary 1 in [4] under normalized conditions.

Theorem 3.1 *Let Ω be a bounded open set in X with $0 \in \Omega$. Let $T : D(T) \subset X \to X^*$ be a maximal monotone operator with $0 \in D(T)$ and $T(0) = 0$ such that T satisfies condition $(T_\infty^{(0)})$ on $D(T) \cap \overline{\Omega}$. Assume that $C : \overline{\Omega} \to X^*$ is a demicontinuous bounded operator which satisfies condition (S_+) and the two additional conditions*:

(c1) *There is a positive number $\mathcal{N}$ such that the weak sequential closure of the set*

$$G = \left\{ \frac{Cx}{\|Cx\|} : \lambda \geq \mathcal{N}, x \in \overline{\Omega}, \|Jx + Tx\| \leq 2M(\lambda) \right\}$$

does not contain zero vector, where

$$M(\lambda) = |\lambda| \sup\{\|Cx\| : x \in \overline{\Omega}\}.$$

(c2) $\inf\{\|Cx\| : x \in \overline{\Omega}\}$ *is not equal to* 0.

Then we have the following properties:

(a) *For each $\varepsilon > 0$, there exists $(\lambda_\varepsilon, x_\varepsilon) \in (0, \infty) \times (D(T) \cap \partial\Omega)$ such that*

$$Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon Jx_\varepsilon = 0.$$

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$ and T satisfies condition (S_q) on $D(T) \cap \partial\Omega$, then the eigenvalue problem*

$$Tx + \lambda Cx = 0$$

has a solution (λ_0, x_0) in $(0, \infty) \times (D(T) \cap \partial\Omega)$.

Proof (a) Fix $\varepsilon > 0$. Let d_B denote the Browder degree in the sense of [7]. We first claim that there exists a number Λ in $(0,\infty)$ such that

$$d_B(T + \Lambda C + \varepsilon J, \Omega, 0) = 0. \tag{3.1}$$

Assume the contrary. As in the proof of (2.1), a similar argument establishes that for a sequence $\{\Lambda_n\}$ in $(0,\infty)$ with $\Lambda_n \to \infty$, there is a sequence $\{x_n\}$ in $D(T) \cap \overline{\Omega}$ such that

$$Tx_n + \Lambda_n Cx_n + \varepsilon Jx_n = 0. \tag{3.2}$$

This implies

$$\|Jx_n + Tx_n\| \leq \big\|(1-\varepsilon)Jx_n\big\| + \|\Lambda_n Cx_n\| \leq 2M(\Lambda_n)$$

for sufficiently large n. By the boundedness of the sequence $\{\|Cx_n\|^{-1}Cx_n\}$ in X^*, we may suppose that $\|Cx_n\|^{-1}Cx_n \rightharpoonup h_0$ for some $h_0 \in X^*$. It follows from (c1) and (3.2) that $h_0 \neq 0$ and

$$\frac{Tx_n}{\|Tx_n\|} \rightharpoonup -h_0.$$

But (c2) implies $\|Tx_n\| \to \infty$, which contradicts condition $(T_\infty^{(0)})$. Hence assertion (3.1) holds.

Now we consider a mapping $H : [0,1] \times (D(T) \cap \overline{\Omega}) \to X^*$ given by

$$H(t,x) := Tx + \varepsilon Jx + t\Lambda Cx \quad \text{for } (t,x) \in [0,1] \times \big(D(T) \cap \overline{\Omega}\big).$$

Using the normalization property of the Browder degree d_B, *e.g.*, Theorem 12 in [7], we have

$$d_B\big(H(0,\cdot), \Omega, 0\big) = d_B(T + \varepsilon J, \Omega, 0) = 1.$$

Moreover, (3.1) means that

$$d_B\big(H(1,\cdot), \Omega, 0\big) = d_B(T + \varepsilon J + \Lambda C, \Omega, 0) = 0.$$

Note that $T + \varepsilon J$ is maximal monotone and satisfies condition (S_+). Hence, there are $t_0 \in [0,1]$ and $x_0 \in D(T) \cap \partial\Omega$ such that

$$Tx_0 + t_0 \Lambda Cx_0 + \varepsilon Jx_0 = 0.$$

By the injectivity of the strictly monotone operator $T + \varepsilon J$, we know that $t_0 > 0$. Consequently, if we let $\lambda_\varepsilon := t_0\Lambda$ and $x_\varepsilon := x_0$, then $\lambda_\varepsilon \in (0,\infty)$ and $Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon Jx_\varepsilon = 0$.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0,\infty)$ such that $\varepsilon_n \to 0$. By (a), there exists a sequence $\{(\lambda_n, x_n)\}$ in $(0,\infty) \times (D(T) \cap \partial\Omega)$ such that

$$Tx_n + \lambda_n Cx_n + \varepsilon_n Jx_n = 0. \tag{3.3}$$

We may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \quad \text{and} \quad Cx_n \rightharpoonup c^*, \tag{3.4}$$

where $\lambda_0 \in [0,\infty]$, $x_0 \in X$, and $c^* \in X^*$. Note that λ_0 belongs to $(0,\infty)$. Indeed, if $\lambda_0 = 0$, then it follows from $Tx_n \to 0$ and condition (S_q) that $x_n \to x_0 \in \partial\Omega$ and therefore $x_0 \in D(T)$ and $Tx_0 = 0$, which contradicts the hypothesis that $0 \notin T(D(T)\cap\partial\Omega)$. If $\lambda_0 = \infty$, then (c2) implies $\|\lambda_n Cx_n\| \to \infty$ and so $\|Tx_n\| \to \infty$. But we can show as above that $\|Tx_n\|^{-1}Tx_n$ converges weakly to some nonzero vector, which contradicts condition $(T_\infty^{(0)})$.

To prove that

$$\limsup_{n\to\infty} \langle \lambda_n Cx_n, x_n - x_0\rangle \le 0, \tag{3.5}$$

we assume to the contrary that there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle \lambda_n Cx_n, x_n - x_0\rangle > 0.$$

Hence we obtain from (3.3) that

$$\limsup_{n\to\infty} \langle Tx_n, x_n - x_0\rangle < 0.$$

Noticing by (3.3) and (3.4) that $Tx_n \rightharpoonup -\lambda_0 c^*$, we get

$$\limsup_{n\to\infty} \langle Tx_n, x_n\rangle < \lim_{n\to\infty} \langle Tx_n, x_0\rangle = \langle -\lambda_0 c^*, x_0\rangle. \tag{3.6}$$

For every $x \in D(T)$, we have by the monotonicity of T

$$\liminf_{n\to\infty} \langle Tx_n, x_n\rangle \ge \langle -\lambda_0 c^*, x\rangle + \langle Tx, x_0 - x\rangle,$$

which implies along with (3.6)

$$\langle -\lambda_0 c^* - Tx, x_0 - x\rangle > 0.$$

By the maximal monotonicity of T, we get a contradiction. Therefore, (3.5) holds.

It follows from (3.5) and

$$\langle \lambda_n Cx_n, x_n - x_0\rangle = (\lambda_n - \lambda_0)\langle Cx_n, x_n - x_0\rangle + \langle \lambda_0 Cx_n, x_n - x_0\rangle$$

that

$$\lambda_0 \limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0.$$

Since C satisfies condition (S_+) and is demicontinuous, we have $x_n \to x_0 \in \partial\Omega$, which implies $Cx_n \rightharpoonup Cx_0$. Hence we obtain from (3.3) that $Tx_n \rightharpoonup -\lambda_0 Cx_0$. By Lemma 1.1, we conclude that $x_0 \in D(T)\cap\partial\Omega$ and $Tx_0 + \lambda_0 Cx_0 = 0$, what we wanted to prove. □

Remark 3.2 We point out that in Theorem 3.1, the condition $\overline{\Omega} \subset \operatorname{int} D(T)$ is not necessary to be assumed.

4 Densely defined perturbations

This section is devoted to the eigenvalue problem for densely defined quasibounded perturbations of maximal monotone operators in reflexive Banach spaces. To do this, we apply the Kartsatos-Skrypnik degree for densely defined $(\tilde{S}_+)$-perturbations of maximal monotone operators developed in [8].

Recall that an operator $C : D(C) \subset X \to X^*$ is *quasibounded* if for every $S > 0$ there exists a constant $K(S) > 0$ such that for all $u \in D(C)$ with $\|u\| \le S$ and $\langle Cu, u\rangle \le 0$, we have $\|Cu\| \le K(S)$.

As in Section 3, we employ a normalization method to obtain an eigenvalue result for generalized pseudomonotone operators.

Theorem 4.1 *Let Ω be a bounded open set in X with $0 \in \Omega$ and L be a dense subspace of X. Let $T : D(T) \subset X \to X^*$ be a maximal monotone operator with $0 \in D(T)$ and $T(0) = 0$ which satisfies condition $(T_\infty^{(0)})$ on $D(T) \cap \overline{\Omega}$. Assume that $C : D(C) \subset X \to X^*$ is a generalized pseudomonotone quasibounded operator with $L \subset D(C)$. Furthermore, assume that*

(h1) *There exists a positive number $\mathcal{N}$ such that the weak sequential closure of the set*

$$G = \left\{ \frac{Cx}{\|Cx\|} : \lambda \ge \mathcal{N}, x \in D(C) \cap \overline{\Omega}, \|J_\psi x + Tx\| \le 2M(\lambda) \right\}$$

does not contain zero vector, where

$$M(\lambda) = |\lambda| \sup\bigl\{ \|Cx\| : x \in D(C) \cap \overline{\Omega} \bigr\}.$$

(h2) $\inf\{\|Cx\| : x \in D(C) \cap \overline{\Omega}\}$ *is not equal to* 0.

(h3) *For every $F \in \mathcal{F}(L)$ and $v \in L$, the function $c(F, v) : F \to \mathbb{R}$, $c(F, v)(u) = \langle Cu, v\rangle$, is continuous on F, where $\mathcal{F}(L)$ denotes the set of all finite-dimensional subspaces of L.*

Then the following statements hold:

(a) *For each $\varepsilon > 0$, there exists a point $(\lambda_\varepsilon, x_\varepsilon)$ in $(0, \infty) \times (D(T + C) \cap \partial\Omega)$ such that*

$$Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon J_\psi x_\varepsilon = 0.$$

Here $D(T + C)$ denotes the intersection of $D(T)$ and $D(C)$.

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$ and T satisfies condition (S_0) on $D(T) \cap \partial\Omega$, then the eigenvalue problem*

$$Tx + \lambda Cx = 0$$

has a solution (λ_0, x_0) in $(0, \infty) \times (D(T + C) \cap \partial\Omega)$.

Proof (a) We will apply the Kartsatos-Skrypnik degree d_S given in [8]. Let ε be an arbitrary positive number. Since T satisfies condition $(T_\infty^{(0)})$ on $D(T) \cap \overline{\Omega}$ and J_ψ is bounded, we can prove as in Theorem 3.1 that under hypotheses (h1) and (h2), there is a positive number

Λ such that

$$d_S(T + \Lambda C + \varepsilon J_\psi, \Omega, 0) = 0. \tag{4.1}$$

For $t \in [0,1]$, we set $T^t := T$ and $C^t := t\Lambda C + \varepsilon J_\psi$, where $D(T^t)$ and $D(C^t)$ denote the domain of T^t and C^t, respectively. In this case, $D(T^t) = D(T)$ for $t \in [0,1]$, $D(C^0) = X$ for $t = 0$ and $D(C^t) = D(C)$ for $t \in (0,1]$. Notice that the operators $C^0 = \varepsilon J_\psi$ and $C^1 = \Lambda C + \varepsilon J_\psi$ satisfy condition $(\tilde{S}_+)$, based on the facts that C is generalized pseudomonotone and J_ψ is bounded and satisfies condition (S_+).

In the sense of Definition 4.2 in [8], we check the following conditions on two families $\{T^t\}$ and $\{C^t\}$. In fact, conditions on $\{T^t\}$ are obviously satisfied, with T^t independent of t, due to maximal monotonicity of T, $0 \in D(T)$, and $T(0) = 0$.

(c_1^t) Since J_ψ is monotone and bounded, it follows from the quasiboundedness of C that $\{C^t\}$ is uniformly quasibounded.

(c_2^t) Let $\{t_n\}$ be any sequence in $[0,1]$ and $\{u_n\}$ be any sequence in L such that $t_n \to t_0$, $u_n \rightharpoonup u_0$, $C^{t_n} u_n \rightharpoonup h^*$ and

$$\limsup_{n\to\infty} \langle C^{t_n} u_n, u_n - u_0 \rangle \le 0, \qquad \langle C^{t_n} u_n, u_n \rangle \le 0, \tag{4.2}$$

where $t_0 \in [0,1]$, $u_0 \in X$, and $h^* \in X^*$. If $t_0 = 0$, then the second inequality in(4.2) implies

$$\varepsilon \psi(\|u_n\|)\|u_n\| = \varepsilon \langle J_\psi u_n, u_n\rangle \le -t_n \Lambda \langle C u_n, u_n\rangle \to 0$$

and hence $u_n \to 0$, $u_0 = 0 \in D(C^0)$, and $C^0 u_0 = h^*$. Now let $t_0 > 0$. From the first inequality in (4.2) and the following inequality

$$\langle C^{t_n} u_n, u_n - u_0 \rangle \ge t_n \Lambda \langle C u_n, u_n - u_0\rangle + \varepsilon \langle J_\psi u_0, u_n - u_0\rangle,$$

we obtain that

$$\limsup_{n\to\infty} \langle C u_n, u_n - u_0\rangle \le 0. \tag{4.3}$$

In view of $C^{t_n} u_n \rightharpoonup h^*$, there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that $C u_n \rightharpoonup h_1^*$ and $J_\psi u_n \rightharpoonup h_2^*$ for some $h_1^*, h_2^* \in X^*$. Since C is generalized pseudomonotone, we obtain from (4.3) that $u_0 \in D(C)$, $C u_0 = h_1^*$, and $\langle C u_n, u_n\rangle \to \langle C u_0, u_0\rangle$. Thus,

$$\lim_{n\to\infty} \langle C u_n, u_n - u_0\rangle = \langle C u_0, u_0\rangle - \langle h_1^*, u_0\rangle = 0.$$

Hence it follows from the first inequality in (4.2) that

$$\varepsilon \limsup_{n\to\infty} \langle J_\psi u_n, u_n - u_0\rangle = \limsup_{n\to\infty} \langle t_n \Lambda C u_n + \varepsilon J_\psi u_n, u_n - u_0\rangle \le 0.$$

Since J_ψ satisfies condition (S_+), we have $u_n \to u_0$ and so $J_\psi u_n \to J_\psi u_0$. Consequently, $u_0 \in D(C^{t_0})$ and $C^{t_0} u_0 = t_0 \Lambda h_1^* + \varepsilon h_2^* = h^*$. Therefore, condition (c_2^t) is satisfied.

(c_3^t) For every $F \in \mathcal{F}(L)$ and $v \in L$, the function $\tilde{c}(F,v) : [0,1] \times F \to \mathbb{R}$, $\tilde{c}(F,v)(t,u) = \langle C^t u, v\rangle$, is continuous on $[0,1] \times F$ because $c(F,v)$ is continuous on F and J_ψ is continuous on X.

We can now consider a mapping $H : [0,1] \times (D(T+C) \cap \overline{\Omega}) \to X^*$ given by

$$H(t,x) := Tx + t\Lambda Cx + \varepsilon J_\psi x.$$

By Theorem 4.4 in [8] and (4.1), we have

$$d_S\big(H(0,\cdot),\Omega,0\big) = d_S(T + \varepsilon J_\psi, \Omega, 0) = 1$$

and

$$d_S\big(H(1,\cdot),\Omega,0\big) = d_S(T + \Lambda C + \varepsilon J_\psi, \Omega, 0) = 0.$$

According to Theorem 4.3 in [8], there exist $t_0 \in [0,1]$ and $x_0 \in D(T+C) \cap \partial\Omega$ such that

$$Tx_0 + t_0\Lambda Cx_0 + \varepsilon J_\psi x_0 = 0.$$

The injectivity of $T + \varepsilon J_\psi$ implies that $t_0 > 0$. If we let $\lambda_\varepsilon := t_0\Lambda$ and $x_\varepsilon := x_0$, then the conclusion follows.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0,\infty)$ such that $\varepsilon_n \to 0$. According to (a), there are sequences $\{\lambda_n\}$ in $(0,\infty)$ and $\{x_n\}$ in $D(T+C) \cap \partial\Omega$ such that

$$Tx_n + \lambda_n Cx_n + \varepsilon_n J_\psi x_n = 0. \tag{4.4}$$

Then it follows from the monotonicity of T with $T(0) = 0$ that

$$\langle Cx_n, x_n\rangle \le -\frac{\varepsilon_n}{\lambda_n}\psi\big(\|x_n\|\big)\|x_n\| \le 0.$$

Hence the quasiboundedness of C implies that $\{Cx_n\}$ is bounded. Without loss of generality, we may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \quad \text{and} \quad Cx_n \rightharpoonup c^*, \tag{4.5}$$

where $\lambda_0 \in [0,\infty]$, $x_0 \in X$, and $c^* \in X^*$. Since T satisfies conditions (S_q) and $(T_\infty^{(0)})$ and since $0 \notin T(D(T) \cap \partial\Omega)$, it is easily verified that λ_0 belongs to $(0,\infty)$.

As in the proof of Theorem 3.1, we can show that

$$\limsup_{n\to\infty} \langle \lambda_n Cx_n, x_n - x_0\rangle \le 0. \tag{4.6}$$

Then it follows from (4.6) and $\lambda_n \to \lambda_0$ that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0.$$

Since C is generalized pseudomonotone, we have by (4.5)

$$x_0 \in D(C), \qquad Cx_0 = c^*, \quad \text{and} \quad \lim_{n\to\infty} \langle Cx_n, x_n \rangle = \langle Cx_0, x_0 \rangle.$$

Hence we obtain from (4.4) that $Tx_n \rightharpoonup -\lambda_0 Cx_0$ and

$$\lim_{n\to\infty} \langle Tx_n, x_n \rangle = \lim_{n\to\infty} -\lambda_n \langle Cx_n, x_n \rangle = \langle -\lambda_0 Cx_0, x_0 \rangle.$$

Since T satisfies condition (S_0) on $D(T) \cap \partial\Omega$, we have $x_n \to x_0$. By the maximal monotonicity of T, we conclude that $x_0 \in D(T + C) \cap \partial\Omega$ and $Tx_0 + \lambda_0 Cx_0 = 0$. This completes the proof. □

As a consequence of Theorem 4.1, we get another eigenvalue result for densely defined operators satisfying condition $(\tilde{S}_+)$ in comparison with Theorem 4 in [4].

Corollary 4.2 *Let T, Ω, and L be as in Theorem* 4.1. *Assume that $C : D(C) \subset X \to X^*$ is a quasibounded operator with $L \subset D(C)$ and satisfies condition $(\tilde{S}_+)$. Furthermore, assume that conditions* (h1), (h2), *and* (h3) *in Theorem* 4.1 *are satisfied. Then*:

(a) *For each $\varepsilon > 0$, there exists $(\lambda_\varepsilon, x_\varepsilon) \in (0,\infty) \times (D(T + C) \cap \partial\Omega)$ such that* $Tx_\varepsilon + \lambda_\varepsilon Cx_\varepsilon + \varepsilon J_\psi x_\varepsilon = 0$.

(b) *If $0 \notin T(D(T) \cap \partial\Omega)$ and T satisfies condition (S_q) on $D(T) \cap \partial\Omega$, then there exists* $(\lambda_0, x_0) \in (0,\infty) \times (D(T + C) \cap \partial\Omega)$ *such that* $Tx_0 + \lambda_0 Cx_0 = 0$.

Proof Statement (a) follows from part (a) of Theorem 4.1 by noting that if C satisfies condition $(\tilde{S}_+)$, then C is generalized pseudomonotone.

(b) Let $\{\varepsilon_n\}$ be a sequence in $(0,\infty)$ such that $\varepsilon_n \to 0$. In view of (a), we can choose a sequence $\{(\lambda_n, x_n)\}$ in $(0,\infty) \times (D(T + C) \cap \partial\Omega)$ such that

$$Tx_n + \lambda_n Cx_n + \varepsilon_n J_\psi x_n = 0.$$

We may suppose that

$$\lambda_n \to \lambda_0, \qquad x_n \rightharpoonup x_0, \quad \text{and} \quad Cx_n \rightharpoonup c^*,$$

where $\lambda_0 \in [0,\infty]$, $x_0 \in X$, and $c^* \in X^*$. Obviously, $\lambda_0 \in (0,\infty)$. As before, the same argument shows that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.$$

Since C satisfies condition $(\tilde{S}_+)$, we have

$$x_n \to x_0, \qquad x_0 \in D(C), \quad \text{and} \quad Cx_0 = c^*.$$

Combining this with $Tx_n \rightharpoonup -\lambda_0 Cx_0$, we obtain that $x_0 \in D(T + C) \cap \partial\Omega$ and $Tx_0 + \lambda_0 Cx_0 = 0$ as required. □

Remark 4.3 When C satisfies condition $(\tilde{S}_+)$, condition (S_0) on T appearing in Theorem 4.1 may be replaced by weaker condition (S_q); see the proof of Theorem 4.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BJ participated in the sequence alignment and coordination. KI conceived of the study and drafted the manuscript. All authors read and approved the final manuscript.

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