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Common fixed point results for three pairs of self-maps satisfying new contractive condition

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Abstract

In this paper, we introduce a new twice power type contractive condition for six self-mappings in generalized metric spaces. Using the weakly commuting and weakly compatible conditions of self-mapping pairs, the existence and uniqueness of common fixed point in complete generalized metric spaces is discussed, and a new common fixed point theorem is obtained. We also provide illustrative examples in support of our new results. Our results generalize some well-known comparable results in the literature due to Ye and Gu.

MSC: 47H10; 54H25; 54E50

Keywords: G-metric space; weakly commuting mapping pairs; weakly compatible mapping pairs; common fixed point

1 Introduction and preliminaries

In 1976, Jungck [1] proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. This theorem has many applications in mathematics. The notion of weakly commuting maps is introduced by Sessa [2]. Jungck [3] generalized the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. The concept of weakly compatible maps is defined by Jungck [4].

In 2006, Mustafa and Sims [5] introduced a new notion of generalized metric space called G-metric space. Based on the notion of generalized metric spaces, Mustafa *et al.* [6, 7] obtained some fixed point results for mappings satisfying different contractive conditions. Aydi [8] obtained a fixed point result for a self-mapping satisfying (ψ, φ)-weakly contractive conditions. Shatanawi [9] proved some fixed point results for self-maps in a complete G-metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim_{n \to \infty} \phi^n(t) = 0$ for all $t \geq 0$. Chugh *et al.* [10] obtained some fixed point results for maps satisfying property P.

In 2009, Abbas and Rhoades [11] initiated the study of a common fixed point theory in generalized metric spaces. Kaewcharoen [12] obtained some common fixed point results for contractive mappings satisfying Φ-maps. Abbas *et al.* [13] obtained some periodic point results. Aydi *et al.* [14] obtained some common fixed point results for generalized weakly G-contraction mapping. Ye and Gu [15] obtained some common fixed point theorems of three maps for a class of twice power type contraction condition. In [16], Gu and

Ye introduce the concept of φ-weakly commuting self-mapping pairs in G-metric space, and used this concept, they establish a new common fixed point theorem of Altman integral type mappings. Aydi [17] obtained a common fixed point theorem of integral type contraction in generalized metric spaces. Tahat *et al.* [18] obtained some common fixed point theorems for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces. Manro *et al.* [19] obtained some common fixed point theorems for expansion mappings in G-metric spaces. Abbas *et al.* [20] and Manro *et al.* [21] gives some common fixed point theorems for R-weakly commuting maps in G-metric spaces. In [22, 23], the authors proved some common fixed point theorems of weakly compatible mappings in G-metric spaces. In [24–30], the authors proved some common fixed point results of three (or four, or six) mappings in G-metric spaces.

Recently, Abbas *et al.* [31] and Mustafa *et al.* [32] obtained some common fixed point results for a pair of mappings satisfying the $(E.A)$ property under certain generalized strict contractive conditions. Long *et al.* [33] obtained some common fixed points results of two pairs of mappings when only one pair satisfies the $(E.A)$ property. Gu and Yin [34] obtained some common fixed points results of three pairs of mappings for which only two pairs need to satisfy the common $(E.A)$ property in the framework of a generalized metric space. Very recently, Gu and Shatanawi [35] used the concept of the common $(E.A)$ property, proved some common fixed point theorems for three pairs of weakly compatible self-maps satisfying a generalized weakly G-contraction condition in generalized metric spaces.

Very recently, Jleli and Samet [36] and Samet *et al.* [37] noticed that some fixed point theorems in the context of a generalized metric space can be concluded by some existing results in the setting of a (quasi-)metric space. In fact, if the contraction condition of the fixed point theorem on a generalized metric space can be reduced to two variables instead of three variables, then one can construct an equivalent fixed point theorem in the setting of a usual metric space. More precisely, in [36, 37], the authors noticed that $d(x, y) = G(x, y, y)$ forms a quasi-metric. Therefore, if one can transform the contraction condition of existence results in a generalized metric space in such terms, $G(x, y, y)$, then the related fixed point results become the well-known fixed point results in the context of a quasi-metric space.

The purpose of this paper is to use the concept of weakly commuting mappings and weakly compatible mappings to discuss some new common fixed point problem for a class of twice power type contraction maps in G-metric spaces. The results presented in this paper extend and improve some well-known corresponding results in the literature due to Ye and Gu [15].

The following definitions and results will be needed in the sequel.

Definition 1.1 [5] Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;

(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

It is well known that the function $G(x, y, z)$ on G-metric space X is jointly continuous in all three of its variables, and $G(x, y, z) = 0$ if and only if $x = y = z$ (see [5]).

Definition 1.2 [5] Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n) if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, and we say that the sequence (x_n) is G-convergent to x or (x_n) G-convergent to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if, for any $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq k$.

Proposition 1.1 [5] *Let (X, G) be a G-metric space, then the following are equivalent*:
1. *(x_n) is G-convergent to x.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$.*
3. *$G(x_n, x, x) \to 0$ as $n \to +\infty$.*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.3 [5] Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if, for every $\epsilon > 0$, there is $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq k$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.2 [5] *Let (X, G) be a G-metric space. Then the following are equivalent*:
1. *The sequence (x_n) is G-Cauchy.*
2. *For every $\epsilon > 0$, there is $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq k$.*

Definition 1.4 [5] Let (X, G) and (X', G') be G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if, for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ imply $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if only if it is G-continuous at $a \in X$.

Definition 1.5 [5] A G-metric space (X, G) is G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Definition 1.6 [19] Two self-mappings f and g of a G-metric space (X, G) are said to be weakly commuting if $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$ for all x in X.

Definition 1.7 [19] Let f and g be two self-mappings from a G-metric space (X, G) into itself. Then the mappings f and g are said to be weakly compatible if $G(fgx, gfx, gfx) = 0$ whenever $G(fx, gx, gx) = 0$.

Proposition 1.3 [5] *Let (X, G) be a G-metric space. Then, for all x, y, z, a in X, it follows that $G(x, y, y) \leq 2G(y, x, x)$.*

2 Main results

Theorem 2.1 *Let (X, G) be a complete G-metric space, and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

(i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$G^2(Sx, Ty, Rz) \le k \max \left\{ \begin{array}{l} G(Ax, Sx, Sx)G(By, Ty, Ty), \\ G(By, Ty, Ty)G(Cz, Rz, Rz), \\ G(Cz, Rz, Rz)G(Ax, Sx, Sx) \end{array} \right\} \tag{2.1}$$

or

$$G^2(Sx, Ty, Rz) \le k \max \left\{ \begin{array}{l} G(Ax, Ax, Sx)G(By, By, Ty), \\ G(By, By, Ty)G(Cz, Cz, Rz), \\ G(Cz, Cz, Rz)G(Ax, Ax, Sx) \end{array} \right\}, \tag{2.2}$$

where $k \in [0, 1)$. Then one of the pairs (S, A), (T, B), and (R, C) has a coincidence point in X. Moreover, if one of the following conditions is satisfied:

(a) *either S or A is G-continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible*;

(b) *either T or B is G-continuous, the pair (T, B) is weakly commuting, the pairs (S, A) and (R, C) are weakly compatible*;

(c) *either F or C is G-continuous, the pair (R, C) is weakly commuting, the pairs (S, A) and (T, B) are weakly compatible*.

Then the mappings S, T, R, A, B, and C have a unique common fixed point in X.

Proof First, we suppose that the condition (2.1) holds.

Let x_0 in X be an arbitrary point, since $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$, there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{3n} = Sx_{3n} = Bx_{3n+1}, \qquad y_{3n+1} = Tx_{3n+1} = Cx_{3n+2}, \qquad y_{3n+2} = Rx_{3n+2} = Ax_{3n+3}$$

for $n = 0, 1, 2, \dots$. If $y_{3n+2} = y_{3n+3}$, then $Sp = Ap$ where $p = x_{3n+3}$. If $y_{3n} = y_{3n+1}$, then $Tp = Bp$ where $p = x_{3n+1}$. If $y_{3n+1} = y_{3n+2}$, then $Rp = Cp$ where $p = x_{3n+2}$. Without loss of generality, we can assume that $y_n \ne y_{n+1}$, for all $n = 0, 1, 2, \dots$.

Now we prove that $\{y_n\}$ is a G-Cauchy sequence in X.

Actually, using condition (2.1) and (G_3) we have

$$G^2(y_{3n-1}, y_{3n}, y_{3n+1}) = G^2(Sx_{3n}, Tx_{3n+1}, Rx_{3n-1})$$

$$\le k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Sx_{3n})G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1})G(Cx_{3n-1}, Rx_{3n-1}, Rx_{3n-1}), \\ G(Cx_{3n-1}, Rx_{3n-1}, Rx_{3n-1})G(Ax_{3n}, Sx_{3n}, Sx_{3n}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n})G(y_{3n}, y_{3n+1}, y_{3n+1}), \\ G(y_{3n}, y_{3n+1}, y_{3n+1})G(y_{3n-2}, y_{3n-1}, y_{3n-1}), \\ G(y_{3n-2}, y_{3n-1}, y_{3n-1})G(y_{3n-1}, y_{3n}, y_{3n}) \end{array} \right\}$$

$$\le k \max \left\{ \begin{array}{c} G(y_{3n-2}, y_{3n-1}, y_{3n})G(y_{3n-1}, y_{3n}, y_{3n+1}), \\ G(y_{3n-1}, y_{3n}, y_{3n+1})G(y_{3n-2}, y_{3n-1}, y_{3n}), \\ G(y_{3n-2}, y_{3n-1}, y_{3n})G(y_{3n-1}, y_{3n}, y_{3n+1}) \end{array} \right\}$$

$$= kG(y_{3n-2}, y_{3n-1}, y_{3n})G(y_{3n-1}, y_{3n}, y_{3n+1}).$$

This implies that

$$G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq kG(y_{3n-2}, y_{3n-1}, y_{3n}). \tag{2.3}$$

Again using condition (2.1) and (G_3) we have

$$G^2(y_{3n}, y_{3n+1}, y_{3n+2}) = G^2(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})$$

$$\leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Sx_{3n})G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1})G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(Ax_{3n}, Sx_{3n}, Sx_{3n}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n})G(y_{3n}, y_{3n+1}, y_{3n+1}), \\ G(y_{3n}, y_{3n+1}, y_{3n+1})G(y_{3n+1}, y_{3n+2}, y_{3n+2}), \\ G(y_{3n+1}, y_{3n+2}, y_{3n+2})G(y_{3n-1}, y_{3n}, y_{3n}) \end{array} \right\}$$

$$\leq k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1})G(y_{3n}, y_{3n+1}, y_{3n+2}), \\ G(y_{3n-1}, y_{3n}, y_{3n+1})G(y_{3n}, y_{3n+1}, y_{3n+2}), \\ G(y_{3n}, y_{3n+1}, y_{3n+2})G(y_{3n-1}, y_{3n}, y_{3n+1}) \end{array} \right\}$$

$$= kG(y_{3n-1}, y_{3n}, y_{3n+1})G(y_{3n}, y_{3n+1}, y_{3n+2}).$$

This gives

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq kG(y_{3n-1}, y_{3n}, y_{3n+1}). \tag{2.4}$$

Similarly, using condition (2.1) and (G_3) we have

$$G^2(y_{3n+1}, y_{3n+2}, y_{3n+3}) = G^2(Sx_{3n+3}, Tx_{3n+1}, Rx_{3n+2})$$

$$\leq k \max \left\{ \begin{array}{c} G(Ax_{3n+3}, Sx_{3n+3}, Sx_{3n+3})G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1})G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(Ax_{3n+3}, Sx_{3n+3}, Sx_{3n+3}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n+2}, y_{3n+3}, y_{3n+3})G(y_{3n}, y_{3n+1}, y_{3n+1}), \\ G(y_{3n}, y_{3n+1}, y_{3n+1})G(y_{3n+1}, y_{3n+2}, y_{3n+2}), \\ G(y_{3n+1}, y_{3n+2}, y_{3n+2})G(y_{3n+2}, y_{3n+3}, y_{3n+3}) \end{array} \right\}$$

$$\leq k \max \left\{ \begin{array}{c} G(y_{3n+1}, y_{3n+2}, y_{3n+3})G(y_{3n}, y_{3n+1}, y_{3n+2}), \\ G(y_{3n}, y_{3n+1}, y_{3n+2})G(y_{3n+1}, y_{3n+2}, y_{3n+3}), \\ G(y_{3n}, y_{3n+1}, y_{3n+2})G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \end{array} \right\}$$

$$= kG(y_{3n}, y_{3n+1}, y_{3n+2})G(y_{3n+1}, y_{3n+2}, y_{3n+3}).$$

This implies that

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq kG(y_{3n}, y_{3n+1}, y_{3n+2}). \tag{2.5}$$

Combining (2.3), (2.4), and (2.5) we have

$$G(y_n, y_{n+1}, y_{n+2}) \leq kG(y_{n-1}, y_n, y_{n+1}) \leq k^2 G(y_{n-2}, y_{n-1}, y_n) \leq \cdots \leq k^n G(y_0, y_1, y_2).$$

Therefore, for all $n, m \in \mathbb{N}$, $n < m$, by (G$_5$) and (G$_3$) we have

$$
\begin{aligned}
G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) + \cdots \\
&\quad + G(y_{m-1}, y_m, y_m) \\
&\leq G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_{m+1}) \\
&\leq \left(k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}\right) G(y_0, y_1, y_2) \\
&\leq \frac{k^n}{1-k} G(y_0, y_1, y_2) \to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence $\{y_n\}$ is a G-Cauchy sequence in X. Since X is a complete G-metric space, there exists a point $u \in X$ such that $y_n \to u$ $(n \to \infty)$.

Since the sequences $\{Sx_{3n}\} = \{Bx_{3n+1}\}$, $\{Tx_{3n+1}\} = \{Cx_{3n+2}\}$ and $\{Rx_{3n-1}\} = \{Ax_{3n}\}$ are all subsequences of $\{y_n\}$, they all converge to u. We have

$$
\begin{aligned}
y_{3n} = Sx_{3n} = Bx_{3n+1} \to u, &\qquad y_{3n+1} = Tx_{3n+1} = Cx_{3n+2} \to u, \\
y_{3n-1} = Rx_{3n-1} = Ax_{3n} \to u &\quad (n \to \infty).
\end{aligned}
\tag{2.6}
$$

Now we prove that u is a common fixed point of S, T, R, A, B, and C under condition (a).

First, we suppose that A is continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible.

Step 1. We prove that $u = Su = Au$.

By (2.6) and the weakly commuting of the mapping pair (S, A) we have

$$
G(SAx_{3n}, ASx_{3n}, ASx_{3n}) \leq G(Sx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \quad (n \to \infty).
\tag{2.7}
$$

Since A is continuous, $A^2 x_{3n} \to Au$ $(n \to \infty)$, $ASx_{3n} \to Au$ $(n \to \infty)$. By (2.7) we know $SAx_{3n} \to Au$ $(n \to \infty)$.

From condition (2.1) we know

$$
\begin{aligned}
&G^2(SAx_{3n}, Tx_{3n+1}, Rx_{3n+2}) \\
&\leq k \max \left\{ \begin{array}{l}
G(A^2 x_{3n}, SAx_{3n}, SAx_{3n}) G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\
G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}) G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\
G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) G(A^2 x_{3n}, SAx_{3n}, SAx_{3n})
\end{array} \right\}.
\end{aligned}
$$

Letting $n \to \infty$ we have

$$
G^2(Au, u, u) \leq k \max \left\{ \begin{array}{c}
G(Au, Au, Au) G(u, u, u), \\
G(u, u, u) G(u, u, u), \\
G(u, u, u) G(Au, Au, Au)
\end{array} \right\} = 0.
$$

This implies that $G(Au, u, u) = 0$, and so $Au = u$.

Again by use of condition (2.1) we have

$$
G^2(Su, Tx_{3n+1}, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c}
G(Au, Su, Su) G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\
G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}) G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\
G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) G(Au, Su, Su)
\end{array} \right\}.
$$

Letting $n \to \infty$ we have

$$G^2(Su, u, u) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Su)G(u, u, u), \\ G(u, u, u)G(u, u, u), \\ G(u, u, u)G(Au, Su, Su) \end{array} \right\} = 0.$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$.

So we have $u = Au = Su$.

Step 2. We prove that $u = Tu = Bu$.

Since $S(X) \subset B(X)$ and $u = Su \in S(X)$, there is a point $v \in X$ such that $u = Su = Bv$. Again, by use of condition (ii), we have

$$G^2(Su, Tv, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Su)G(Bv, Tv, Tv), \\ G(Bv, Tv, Tv)G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(Au, Su, Su) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Au = Su$ we have

$$G^2(u, Tv, u) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(u, Tv, Tv), \\ G(u, Tv, Tv)G(u, u, u), \\ G(u, u, u)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, Tv, u) = 0$, and so $Tv = u$.

Since the pair (T, B) is weakly compatible, we have

$$Tu = TBv = BTv = Bu.$$

Again, by use of condition (2.1), we have

$$G^2(Su, Tu, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Su)G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(Au, Su, Su) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Au = Su$ and $Tu = Bu$ we have

$$G^2(u, Tu, u) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(Tu, Tu, Tu), \\ G(Tu, Tu, Tu)G(u, u, u), \\ G(u, u, u)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, Tu, u) = 0$, and so $Tu = u$.

So we have $u = Tu = Bu$.

Step 3. We prove that $u = Ru = Cu$.

Since $T(X) \subset C(X)$ and $u = Tu \in T(X)$, there is a point $w \in X$ such that $u = Tu = Cw$. Again, by use of condition (2.1), we have

$$G^2(Su, Tu, Rw) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Su)G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cw, Rw, Rw), \\ G(Cw, Rw, Rw)G(Au, Su, Su) \end{array} \right\}.$$

Using $u = Au = Su$ and $u = Tu = Bu = Cw$, we obtain

$$G^2(u, u, Rw) \leq k \max \left\{ \begin{array}{l} G(u, u, u)G(u, u, u), \\ G(u, u, u)G(u, Rw, Rw), \\ G(u, Rw, Rw)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, u, Rw) = 0$, and so $Rw = u = Cw$.

Since the pair (R, C) is weakly compatible, we have

$$Ru = RCw = CRw = Cu.$$

Again by use of condition (2.1), $Su = Au$ and $Ru = Cu$ we have

$$G^2(u, u, Ru) = G^2(Su, Tu, Ru) \leq k \max \left\{ \begin{array}{l} G(Au, Su, Su)G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cu, Ru, Ru), \\ G(Cu, Ru, Ru)G(Au, Su, Su) \end{array} \right\} = 0.$$

So we have $G(u, u, Ru) = 0$, and so $u = Ru = Cu$.

Therefore u is the common fixed point of S, T, R, A, B and C when A is continuous and the pair (S, A) is weakly commuting, the pairs (T, B) and (F, C) are weakly compatible.

Next, we suppose that S is continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible.

Step 1. We prove that $u = Su$.

By (2.6) and the weakly commuting of the mapping pair (S, A) we have

$$G(SAx_{3n}, ASx_{3n}, ASx_{3n}) \leq G(Sx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \quad (n \to \infty). \tag{2.8}$$

Since S is continuous, $S^2x_{3n} \to Su$ $(n \to \infty)$, $SAx_{3n} \to Su$ $(n \to \infty)$. By (2.8) we know $ASx_{3n} \to Su$ $(n \to \infty)$.

From condition (2.1) we have

$$G^2\left(S^2x_{3n}, Tx_{3n+1}, Rx_{3n+2}\right)$$
$$\leq k \max \left\{ \begin{array}{l} G(ASx_{3n}, S^2x_{3n}, S^2x_{3n})G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Tx_{3n+1})G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(ASx_{3n}, S^2x_{3n}, S^2x_{3n}) \end{array} \right\}.$$

Letting $n \to \infty$ we have

$$G^2(Su, u, u) \leq k \max \left\{ \begin{array}{l} G(Su, Su, Su)G(u, u, u), \\ G(u, u, u)G(u, u, u), \\ G(u, u, u)G(Su, Su, Su) \end{array} \right\} = 0.$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$.

Step 2. We prove that $u = Tu = Bu$.

Since $S(X) \subset B(X)$ and $u = Su \in S(X)$, there is a point $z \in X$ such that $u = Su = Bz$. Again by use of condition (2.1), we have

$$G^2\left(S^2 x_{3n}, Tz, Rx_{3n+2}\right) \leq k \max \left\{ \begin{array}{c} G(ASx_{3n}, S^2 x_{3n}, S^2 x_{3n})G(Bz, Tz, Tz), \\ G(Bz, Tz, Tz)G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(ASx_{3n}, S^2 x_{3n}, S^2 x_{3n}) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Su$ we have

$$G^2(u, Tz, u) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(u, Tz, Tz), \\ G(u, Tz, Tz)G(u, u, u), \\ G(u, u, u)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, Tz, u) = 0$, and so $Tz = u = Bz$.

Since the pair (T, B) is weakly compatible, we have

$$Tu = TBz = BTz = Bu.$$

Again, by use of condition (2.1), we have

$$G^2(Sx_{3n}, Tu, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Sx_{3n})G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Rx_{3n+2})G(Ax_{3n}, Sx_{3n}, Sx_{3n}) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Su$ and $Tu = Bu$ we have

$$G^2(u, Tu, u) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(Tu, Tu, Tu), \\ G(Tu, Tu, Tu)G(u, u, u), \\ G(u, u, u)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, Tu, u) = 0$, and so $Tu = u = Bu$.

So we have $u = Tu = Bu$.

Step 3. We prove that $u = Ru = Cu$.

Since $T(X) \subset C(X)$ and $u = Tu \in T(X)$, there is a point $t \in X$ such that $u = Tu = Ct$. Again by use of condition (2.1), we have

$$G^2(Sx_{3n}, Tu, Rt) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Sx_{3n})G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Ct, Rt, Rt), \\ G(Ct, Rt, Rt)G(Ax_{3n}, Sx_{3n}, Sx_{3n}) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Tu = Bu$, we obtain

$$G^2(u, u, Rt) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(u, u, u), \\ G(u, u, u)G(u, Rt, Rt), \\ G(u, Rt, Rt)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, u, Rt) = 0$, and so $Rt = u = Ct$.

Since the pair (R, C) is weakly compatible, we have

$$Ru = RCt = CRt = Cu.$$

Again, by use of condition (2.1), we have

$$G^2(Sx_{3n}, Tu, Ru) \leq k \max \left\{ \begin{array}{l} G(Ax_{3n}, Sx_{3n}, Sx_{3n})G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cu, Ru, Ru), \\ G(Cu, Ru, Ru)G(Ax_{3n}, Sx_{3n}, Sx_{3n}) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Tu = Bu$ we have

$$G^2(u, u, Ru) \leq k \max \left\{ \begin{array}{l} G(u, u, u)G(u, u, u), \\ G(u, u, u)G(Ru, Ru, Ru), \\ G(Ru, Ru, Ru)G(u, u, u) \end{array} \right\} = 0.$$

This implies that $G(u, u, Ru) = 0$, and so $Ru = u = Cu$.

Step 4. We prove that $u = Au$.

Since $R(X) \subset A(X)$ and $u = Ru \in R(X)$, there is a point $p \in X$ such that $u = Ru = Ap$. Again by use of condition (2.1), we have

$$G^2(Sp, Tu, Ru) \leq k \max \left\{ \begin{array}{l} G(Ap, Sp, Sp)G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cu, Ru, Ru), \\ G(Cu, Ru, Ru)G(Ap, Sp, Sp) \end{array} \right\}.$$

Using $u = Tu = Bu$ and $u = Ru = Cu$, we obtain

$$G^2(Sp, u, u) \leq k \max \left\{ \begin{array}{l} G(u, Sp, Sp)G(u, u, u), \\ G(u, u, u)G(u, u, u), \\ G(u, u, u)G(u, Sp, Sp) \end{array} \right\} = 0.$$

This implies that $G(Sp, u, u) = 0$, and $Sp = u = Ap$.

Since the pair (S, A) is weakly compatible, we have

$$Su = SAp = ASp = Au = u.$$

Therefore u is the common fixed point of S, T, R, A, B, and C when S is continuous and the pair (S, A) is weakly commuting, the pairs (T, B) and (F, C) are weakly compatible.

Similarly we can prove that u is the unique common fixed point of the maps S, T, R, A, B, and C under the conditions of (b) and (c).

Next we prove the uniqueness of a common fixed point u.

Let u and v be two common fixed points of S, T, R, A, B, and C, by use of condition (2.1), we have

$$G^2(u, u, v) = G^2(Su, Tu, Rv) \leq k \max \left\{ \begin{array}{l} G(Au, Su, Su)G(Bu, Tu, Tu), \\ G(Bu, Tu, Tu)G(Cv, Rv, Rv), \\ G(Cv, Rv, Rv)G(Au, Su, Su)) \end{array} \right\} = 0.$$

This shows that $G(u, u, v) = 0$, and so $u = v$. Thus the common fixed point is unique.

If condition (2.2) holds, then the argument is similar to that above, so we omit it. □

Theorem 2.2 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

(i) *$S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$*;

(ii) *the pairs (S, A), (T, B), and (R, C) are commuting mappings*;

(iii) *$\forall x, y, z \in X$,*

$$G^2\left(S^p x, T^q y, R^r z\right) \leq k \max \left\{ \begin{array}{l} G(Ax, S^p x, S^p x)G(By, T^q y, T^q y), \\ G(By, T^q y, T^q y)G(Cz, R^r z, R^r z), \\ G(Cz, R^r z, R^r z)G(Ax, S^p x, S^p x) \end{array} \right\} \tag{2.9}$$

or

$$G^2\left(S^p x, T^q y, R^r z\right) \leq k \max \left\{ \begin{array}{l} G(Ax, Ax, S^p x)G(By, By, T^q y), \\ G(By, By, T^q y)G(Cz, Cz, R^r z), \\ G(Cz, Cz, R^r z)G(Ax, Ax, S^p x) \end{array} \right\}, \tag{2.10}$$

where $k \in [0, 1)$, $p, q, r \in \mathbb{N}$, then S, T, R, A, B, and C have a unique common fixed point in X.

Proof Suppose the condition (2.9) holds. Since $S^p X \subset S^{p-1} X \subset \cdots \subset SX$, $SX \subset BX$, so that $S^p X \subset BX$. Similar, we can show that $T^q X \subset CX$ and $R^r X \subset AX$. From Theorem 2.1, we see that S^p, T^q, R^r, A, B, and C have a unique common fixed point u.

Since $Su = S(S^p u) = S^{p+1} u = S^p(Su)$, so that

$$G^2\left(S^p Su, T^q u, R^r u\right) \leq k \max \left\{ \begin{array}{l} G(ASu, S^p Su, S^p Su)G(Bu, T^q u, T^q u), \\ G(Bu, T^q u, T^q u)G(Cu, R^r u, R^r u), \\ G(Cu, R^r u, R^r u)G(ASu, S^p Su, S^p Su) \end{array} \right\}.$$

Note that $u = Au = Bu = Cu = S^p u = T^q u = R^r u$ and $S^p Su = Su$, we obtain

$$G^2(Su, u, u) = G^2\left(S^p Su, T^q u, R^r u\right) \leq k \max \left\{ \begin{array}{l} G(ASu, Su, Su)G(u, u, u), \\ G(u, u, u)G(u, u, u), \\ G(u, u, u)G(ASu, Su, Su) \end{array} \right\} = 0.$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$.

By the same argument, we can prove $Tu = u$ and $Ru = u$. Thus we have $u = Su = Tu = Fu = Au = Bu = Cu$, so that S, T, R, A, B, and C have a common fixed point u in X. Let v be any other common fixed point of S, T, R, A, B, and C, then use of condition (2.9), we have

$$G^2(u, u, v) = G^2\left(S^p u, T^q u, R^r v\right) \leq k \max \left\{ \begin{array}{l} G(Au, S^p u, S^p u)G(Bu, T^q u, T^q u), \\ G(Bu, T^q u, T^q u)G(Cv, R^r v, R^r v), \\ G(Cv, R^r v, R^r v)G(Au, S^p u, S^p u) \end{array} \right\} = 0.$$

This implies that $G(u, u, v) = 0$, and so $u = v$. Thus the common fixed point is unique.

If condition (2.10) holds, then the argument is similar to that above, so we omit it. □

Remark 2.1 Theorems 2.1 and 2.2 improve and extend the corresponding results in Ye and Gu [15, Theorem 2.1, Corollary 2.2] from three self-mappings to six self-mappings.

Corollary 2.1 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

(i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Sx, Sx)G(By, Ty, Ty) + bG(By, Ty, Ty)G(Cz, Rz, Rz)$$
$$+ cG(Cz, Rz, Rz)G(Ax, Sx, Sx) \tag{2.11}$$

or

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Ax, Sx)G(By, By, Ty) + bG(By, By, Ty)G(Cz, Cz, Rz)$$
$$+ cG(Cz, Cz, Rz)G(Ax, Ax, Sx), \tag{2.12}$$

where $0 \leq a + b + c < 1$. Then one of the pairs (S, A), (T, B), and (R, C) has a coincidence point in X. Moreover, assume one of the following conditions is satisfied:

(a) *either S or A is G-continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible*;

(b) *either T or B is G-continuous, the pair (T, B) is weakly commuting, the pairs (S, A) and (R, C) are weakly compatible*;

(c) *either F or C is G-continuous, the pair (R, C) is weakly commuting, the pairs (S, A) and (T, B) are weakly compatible.*

Then the mappings S, T, R, A, B, and C have a unique common fixed point in X.

Proof Suppose the condition (2.11) holds. For $x, y, z \in X$, let

$$M(x, y, z) = \max \left\{ \begin{array}{l} G(Ax, Sx, Sx)G(By, Ty, Ty), \\ G(By, Ty, Ty)G(Cz, Rz, Rz), \\ G(Cz, Rz, Rz)G(Ax, Sx, Sx) \end{array} \right\}.$$

Then

$$aG(Ax, Sx, Sx)G(By, Ty, Ty) + bG(By, Ty, Ty)G(Cz, Rz, Rz)$$
$$+ cG(Cz, Rz, Rz)G(Ax, Sx, Sx)$$
$$\leq (a + b + c)M(x, y, z).$$

So, if

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Sx, Sx)G(By, Ty, Ty) + bG(By, Ty, Ty)G(Cz, Rz, Rz)$$
$$+ cG(Cz, Rz, Rz)G(Ax, Sx, Sx),$$

then $G(Sx, Ty, Rz) \leq (a + b + c)M(x, y, z)$. Taking $k = a + b + c$ in Theorem 2.1, the conclusion of Corollary 2.1 can be obtained from Theorem 2.1 immediately.

If condition (2.12) holds, then the argument is similar to that above, so we omit it. This completes the proof of Corollary 2.1. □

Remark 2.2 If $A = B = C = I$ (I is the identity mapping, here and below), Corollary 2.1 is reduced to Theorem 2.1 of Ye and Gu [15].

Corollary 2.2 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:
  (i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;
 (ii) *the pairs (S, A), (T, B), and (R, C) are commuting mappings*;
(iii) $\forall x, y, z \in X$,

$$G^2\big(S^p x, T^q y, R^r z\big)$$
$$\leq aG\big(Ax, S^p x, S^p x\big)G\big(By, T^q y, T^q y\big) + bG\big(By, T^q y, T^q y\big)G\big(Cz, R^r z, R^r z\big)$$
$$+ cG\big(Cz, R^r z, R^r z\big)G\big(Ax, S^p x, S^p x\big) \tag{2.13}$$

or

$$G^2\big(S^p x, T^q y, R^r z\big)$$
$$\leq aG\big(Ax, Ax, S^p x\big)G\big(By, By, T^q y\big) + bG\big(By, By, T^q y\big)G\big(Cz, Cz, R^r z\big)$$
$$+ cG\big(Cz, Cz, R^r z\big)G\big(Ax, Ax, S^p x\big), \tag{2.14}$$

where $0 \leq a + b + c + d < 1$, $p, q, r \in \mathbb{N}$, then S, T, R, A, B, and C have a unique common fixed point in X.

Proof The proof follows from Theorem 2.2, and from an argument similar to that used in Corollary 2.1. □

Remark 2.3 If $A = B = C = I$, Corollary 2.2 is reduced to Corollary 2.2 of Ye and Gu [15].

In Theorem 2.1, if we take $A = B = C = I$, then we have the following corollary.

Corollary 2.3 *Let (X, G) be a complete G-metric space and let S, T, and R be three mappings of X into itself satisfying the following conditions*:

$$G^2(Sx, Ty, Rz) \leq k \max \left\{ \begin{array}{l} G(x, Sx, Sx)G(y, Ty, Ty), \\ G(y, Ty, Ty)G(z, Rz, Rz), \\ G(z, Rz, Rz)G(x, Sx, Sx) \end{array} \right\} \tag{2.15}$$

or

$$G^2(Sx, Ty, Rz) \leq k \max \left\{ \begin{array}{l} G(x, x, Sx)G(y, y, Ty), \\ G(y, y, Ty)G(z, z, Rz), \\ G(z, z, Rz)G(x, x, Sx) \end{array} \right\} \tag{2.16}$$

for all $x, y, z \in X$, where $k \in [0, 1)$.
  Then the mappings S, T, and R have a unique common fixed point in X.

Remark 2.4 In Theorems 2.1, 2.2, Corollaries 2.1, 2.2 and 2.3, we have taken: (1) $S = T = R$; (2) $A = B = C$; (3) $A = B = C = I$; (4) $T = R$ and $B = C$; (5) $T = R$, $B = C = I$, several new result can be obtain.

Theorem 2.3 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

(i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$G^2(Sx, Ty, Rz) \leq k \max \begin{cases} G(Ax, Sx, Ty)G(By, Ty, Rz), \\ G(By, Ty, Rz)G(Cz, Rz, Sx), \\ G(Cz, Rz, Sx)G(Ax, Sx, Ty) \end{cases} \tag{2.17}$$

or

$$G^2(Sx, Ty, Rz) \leq k \max \begin{cases} G(Ax, Ax, Ty)G(By, By, Rz), \\ G(By, By, Rz)G(Cz, Cz, Sx), \\ G(Cz, Cz, Sx)G(Ax, Ax, Ty) \end{cases}, \tag{2.18}$$

where $k \in [0, \frac{1}{2})$. Then one of the pairs (S, A), (T, B), and (R, C) has a coincidence point in X. Moreover, assume one of the following conditions is satisfied:

(a) *either S or A is G-continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible*;

(b) *either T or B is G-continuous, the pair (T, B) is weakly commuting, the pairs (S, A) and (R, C) are weakly compatible*;

(c) *either F or C is G-continuous, the pair (R, C) is weakly commuting, the pairs (S, A) and (T, B) are weakly compatible.*

Then the mappings S, T, R, A, B, and C have a unique common fixed point in X.

Proof First, we suppose that the condition (2.17) holds.

Let x_0 in X be an arbitrary point, since $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$ there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$y_{3n} = Sx_{3n} = Bx_{3n+1}, \qquad y_{3n+1} = Tx_{3n+1} = Cx_{3n+2}, \qquad y_{3n+2} = Rx_{3n+2} = Ax_{3n+3}$$

for $n = 0, 1, 2, \ldots$.

If $y_{3n+2} = y_{3n+3}$, then $Sp = Ap$ where $p = x_{3n+3}$. If $y_{3n} = y_{3n+1}$, then $Tp = Bp$ where $p = x_{3n+1}$. If $y_{3n+1} = y_{3n+2}$, then $Rp = Cp$ where $p = x_{3n+2}$. Without loss of generality, we can assume that $y_n \neq y_{n+1}$, for all $n = 0, 1, 2, \ldots$.

Now we prove that $\{y_n\}$ is a G-Cauchy sequence in X.

In fact, using condition (2.17) we have

$$G^2(y_{3n-1}, y_{3n}, y_{3n+1}) = G^2(Sx_{3n}, Tx_{3n+1}, Rx_{3n-1})$$

$$\leq k \max \begin{cases} G(Ax_{3n}, Sx_{3n}, Tx_{3n+1})G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n-1}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n-1})G(Cx_{3n-1}, Rx_{3n-1}, Sx_{3n}), \\ G(Cx_{3n-1}, Rx_{3n-1}, Sx_{3n})G(Ax_{3n}, Sx_{3n}, Tx_{3n+1}) \end{cases}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}) G(y_{3n}, y_{3n+1}, y_{3n-1}), \\ G(y_{3n}, y_{3n+1}, y_{3n-1}) G(y_{3n-2}, y_{3n-1}, y_{3n}), \\ G(y_{3n-2}, y_{3n-1}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G^2(y_{3n-1}, y_{3n}, y_{3n+1}), \\ G(y_{3n-2}, y_{3n-1}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1}) \end{array} \right\}. \tag{2.19}$$

If

$$\max\{G^2(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-2}, y_{3n-1}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1})\} = G^2(y_{3n-1}, y_{3n}, y_{3n+1}),$$

then by the inequality (2.19) we obtain

$$G^2(y_{3n-1}, y_{3n}, y_{3n+1}) \le k G^2(y_{3n-1}, y_{3n}, y_{3n+1}),$$

which is a contradiction since $0 \le k < \frac{1}{2}$, and hence

$$\max\{G^2(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-2}, y_{3n-1}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1})\}$$
$$= G(y_{3n-2}, y_{3n-1}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1}).$$

Therefore, the inequality (2.19) implies that

$$G(y_{3n-1}, y_{3n}, y_{3n+1}) \le k G(y_{3n-2}, y_{3n-1}, y_{3n}). \tag{2.20}$$

Again using the condition (2.17) we have

$$G^2(y_{3n}, y_{3n+1}, y_{3n+2}) = G^2(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})$$

$$\le k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Tx_{3n+1}) G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) G(Cx_{3n+2}, Rx_{3n+2}, Sx_{3n}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Sx_{3n}) G(Ax_{3n}, Sx_{3n}, Tx_{3n+1}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}) G(y_{3n}, y_{3n+1}, y_{3n+2}), \\ G(y_{3n}, y_{3n+1}, y_{3n+2}) G(y_{3n+1}, y_{3n+2}, y_{3n}), \\ G(y_{3n+1}, y_{3n+2}, y_{3n}) G(y_{3n-1}, y_{3n}, y_{3n+1}) \end{array} \right\}$$

$$= k \max \left\{ \begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}) G(y_{3n}, y_{3n+1}, y_{3n+2}), \\ G^2(y_{3n}, y_{3n+1}, y_{3n+2}) \end{array} \right\}. \tag{2.21}$$

If

$$\max\{G(y_{3n-1}, y_{3n}, y_{3n+1}) G(y_{3n}, y_{3n+1}, y_{3n+2}), G^2(y_{3n}, y_{3n+1}, y_{3n+2})\} = G^2(y_{3n}, y_{3n+1}, y_{3n+2}),$$

then the inequality (2.21) implies that

$$G^2(y_{3n}, y_{3n+1}, y_{3n+2}) \le k G^2(y_{3n}, y_{3n+1}, y_{3n+2}),$$

this is a contradiction, and so

$$\max\big\{G(y_{3n-1},y_{3n},y_{3n+1})G(y_{3n},y_{3n+1},y_{3n+2}),G^2(y_{3n},y_{3n+1},y_{3n+2})\big\}$$
$$= G(y_{3n-1},y_{3n},y_{3n+1})G(y_{3n},y_{3n+1},y_{3n+2}).$$

Therefore, the inequality (2.21) implies that

$$G(y_{3n},y_{3n+1},y_{3n+2}) \le kG(y_{3n-1},y_{3n},y_{3n+1}). \tag{2.22}$$

Similarly, using condition (2.17), we have

$$G^2(y_{3n+1},y_{3n+2},y_{3n+3})$$
$$= G^2(Sx_{3n+3},Tx_{3n+1},Rx_{3n+2})$$
$$\le k\max\left\{\begin{array}{l} G(Ax_{3n+3},Sx_{3n+3},Tx_{3n+1})G(Bx_{3n+1},Tx_{3n+1},Rx_{3n+2}), \\ G(Bx_{3n+1},Tx_{3n+1},Rx_{3n+2})G(Cx_{3n+2},Rx_{3n+2},Sx_{3n+3}), \\ G(Cx_{3n+2},Rx_{3n+2},Sx_{3n+3})G(Ax_{3n+3},Sx_{3n+3},Tx_{3n+1}) \end{array}\right\}$$
$$= k\max\left\{\begin{array}{l} G(y_{3n+2},y_{3n+3},y_{3n+1})G(y_{3n},y_{3n+1},y_{3n+2}), \\ G(y_{3n},y_{3n+1},y_{3n+2})G(y_{3n+1},y_{3n+2},y_{3n+3}), \\ G(y_{3n+1},y_{3n+2},y_{3n+3})G(y_{3n+2},y_{3n+3},y_{3n+1}) \end{array}\right\}$$
$$= k\max\left\{\begin{array}{c} G(y_{3n},y_{3n+1},y_{3n+2})G(y_{3n+1},y_{3n+2},y_{3n+3}), \\ G^2(y_{3n+1},y_{3n+2},y_{3n+3}) \end{array}\right\}. \tag{2.23}$$

If

$$\max\big\{G(y_{3n},y_{3n+1},y_{3n+2})G(y_{3n+1},y_{3n+2},y_{3n+3}),G^2(y_{3n+1},y_{3n+2},y_{3n+3})\big\}$$
$$= G^2(y_{3n+1},y_{3n+2},y_{3n+3}),$$

then from the inequality (2.23) we get

$$G^2(y_{3n+1},y_{3n+2},y_{3n+3}) \le kG^2(y_{3n+1},y_{3n+2},y_{3n+3}),$$

which is a contradiction, hence we have

$$\max\big\{G(y_{3n},y_{3n+1},y_{3n+2})G(y_{3n+1},y_{3n+2},y_{3n+3}),G^2(y_{3n+1},y_{3n+2},y_{3n+3})\big\}$$
$$= G(y_{3n},y_{3n+1},y_{3n+2})G(y_{3n+1},y_{3n+2},y_{3n+3}).$$

Therefore, the above inequality (2.23) becomes

$$G^2(y_{3n+1},y_{3n+2},y_{3n+3}) \le kG(y_{3n},y_{3n+1},y_{3n+2}). \tag{2.24}$$

By combining (2.20), (2.22), and (2.24), $\forall n \in \mathbb{N}$, we have

$$G(y_n,y_{n+1},y_{n+2}) \le kG(y_{n-1},y_n,y_{n+1}) \le \cdots \le k^n G(y_0,y_1,y_2). \tag{2.25}$$

Therefore, for all $m, n \in \mathbb{N}$, $m > n$, by (G$_3$), (G$_5$), and (2.25) we have

$$G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{m-1}, y_m, y_m)$$

$$\leq G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_{m+1})$$

$$\leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) G(y_0, y_1, y_2)$$

$$< \frac{k^n}{1-k} G(y_0, y_1, y_2) \to 0 \quad (n \to \infty).$$

This implies that $G(y_n, y_m, y_m) \to 0$, as $n, m \to \infty$. Thus $\{y_n\}$ is a G-Cauchy sequence in X. Due to the G-completeness of X, there exists $u \in X$, such that $\{y_n\}$ is G-convergent to u.

Since the sequences $\{Sx_{3n}\} = \{Bx_{3n+1}\}$, $\{Tx_{3n+1}\} = \{Cx_{3n+2}\}$ and $\{Rx_{3n-1}\} = \{Ax_{3n}\}$ are all subsequences of $\{y_n\}$, they all converge to u. We have

$$y_{3n} = Sx_{3n} = Bx_{3n+1} \to u, \qquad y_{3n+1} = Tx_{3n+1} = Cx_{3n+2} \to u,$$

$$y_{3n-1} = Rx_{3n-1} = Ax_{3n} \to u \quad (n \to \infty) \tag{2.26}$$

Now we prove that u is a common fixed point of S, T, R, A, B, and C under the condition (a).

First, we suppose that A is continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible.

Step 1. We prove that $u = Su = Au$.

By (2.24) and the weakly commuting of the mapping pair (S, A) we have

$$G(SAx_{3n}, ASx_{3n}, ASx_{3n}) \leq G(Sx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \quad (n \to \infty). \tag{2.27}$$

Since A is continuous, $A^2 x_{3n} \to Au$ $(n \to \infty)$, $ASx_{3n} \to Au$ $(n \to \infty)$. By (2.27) we know $SAx_{3n} \to Au$ $(n \to \infty)$.

From condition (2.17) we know

$$G^2(SAx_{3n}, Tx_{3n+1}, Rx_{3n+2})$$

$$\leq k \max \left\{ \begin{array}{l} G(A^2 x_{3n}, SAx_{3n}, Tx_{3n+1}) G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) G(Cx_{3n+2}, Rx_{3n+2}, SAx_{3n}), \\ G(Cx_{3n+2}, Rx_{3n+2}, SAx_{3n}) G(A^2 x_{3n}, SAx_{3n}, Tx_{3n+1}) \end{array} \right\}.$$

Letting $n \to \infty$ we have

$$G^2(Au, u, u) \leq k \max \left\{ \begin{array}{l} G(Au, Au, u) G(u, u, u), \\ G(u, u, u) G(u, u, Au), \\ G(u, u, Au) G(Au, Au, u) \end{array} \right\} = kG(u, u, Au) G(Au, Au, u). \tag{2.28}$$

If $G(Au, u, u) \neq 0$, then from (2.28) and Proposition 1.3, we obtain

$$G(Au, u, u) \leq kG(Au, Au, u) \leq 2kG(Au, u, u),$$

which is a contradiction since $0 \leq k < \frac{1}{2}$. So $G(Au, u, u) = 0$, this is $Au = u$.

Again, by use of condition (2.17) we have

$$G^2(Su, Tx_{3n+1}, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Tx_{3n+1})G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2})G(Cx_{3n+2}, Rx_{3n+2}, Su), \\ G(Cx_{3n+2}, Rx_{3n+2}, Su)G(Au, Su, Tx_{3n+1}) \end{array} \right\}.$$

Letting $n \to \infty$ and using $Au = u$ we have

$$G^2(Su, u, u) \leq k \max \left\{ \begin{array}{c} G(u, Su, u)G(u, u, u), \\ G(u, u, u)G(u, u, Su), \\ G(u, u, Su)G(u, Su, u) \end{array} \right\} = kG^2(Su, u, u).$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$. Therefore we have $u = Au = Su$.

Step 2. We prove that $u = Tu = Bu$.

Since $S(X) \subset B(X)$ and $u = Su \in S(X)$, there is a point $v \in X$ such that $u = Su = Bv$. Again by use of condition (2.15), we have

$$G^2(Su, Tv, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Tv)G(Bv, Tv, Rx_{3n+2}), \\ G(Bv, Tv, Rx_{3n+2})G(Cx_{3n+2}, Rx_{3n+2}, Su), \\ G(Cx_{3n+2}, Rx_{3n+2}, Su)G(Au, Su, Tv) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Au = Su = Bv$ we have

$$G^2(u, Tv, u) \leq k \max \left\{ \begin{array}{c} G(u, u, Tv)G(u, Tv, u), \\ G(u, Tv, u)G(u, u, u), \\ G(u, u, u)G(u, u, Tv) \end{array} \right\} = kG^2(u, Tv, u).$$

This implies that $G(u, Tv, u) = 0$, and so $Tv = u$.

Since the pair (T, B) is weakly compatible, we have

$$Tu = TBv = BTv = Bu.$$

Again by use of condition (2.17), we have

$$G^2(Su, Tu, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Tu)G(Bu, Tu, Rx_{3n+2}), \\ G(Bu, Tu, Rx_{3n+1})G(Cx_{3n+2}, Rx_{3n+2}, Su), \\ G(Cx_{3n+2}, Rx_{3n+2}, Su)G(Au, Su, Tu) \end{array} \right\}.$$

Letting $n \to \infty$, using $u = Au = Su$, $Tu = Bu$, and Proposition 1.3, we have

$$\begin{aligned} G^2(u, Tu, u) &\leq k \max \left\{ \begin{array}{c} G(u, u, Tu)G(Tu, Tu, u), \\ G(Tu, Tu, u)G(u, u, u), \\ G(u, u, u)G(u, u, Tu) \end{array} \right\} \\ &= kG(u, u, Tu)G(Tu, Tu, u) \\ &\leq 2kG^2(u, Tu, u). \end{aligned}$$

This implies that $G(u, Tu, u) = 0$, and so $Tu = u$.

So we have $u = Tu = Bu$.

Step 3. We prove that $u = Ru = Cu$.

Since $T(X) \subset C(X)$ and $u = Tu \in T(X)$, there is a point $w \in X$ such that $u = Tu = Cw$. Again by use of condition (2.17), we have

$$G^2(Su, Tu, Rw) \leq k \max \left\{ \begin{array}{c} G(Au, Su, Tu)G(Bu, Tu, Rw), \\ G(Bu, Tu, Rw)G(Cw, Rw, Su), \\ G(Cw, Rw, Su)G(Au, Su, Tu) \end{array} \right\}.$$

Using $u = Au = Su$ and $u = Tu = Bu = Cw$, we obtain

$$G^2(u, u, Rw) \leq k \max \left\{ \begin{array}{c} G(u, u, u)G(u, u, Rw), \\ G(u, u, Rw)G(u, Rw, u), \\ G(u, Rw, u)G(u, u, u) \end{array} \right\} = kG^2(u, u, Rw).$$

This implies that $G(u, u, Rw) = 0$, and so $Rw = u = Cw$.

Since the pair (R, C) is weakly compatible, we have

$$Ru = RCw = CRw = Cu.$$

Again, by use of condition (2.17), $u = Su = Au = Tu = Bu$, $Ru = Cu$, and Proposition 1.3, we have

$$\begin{aligned}
G^2(u, u, Ru) &= G^2(Su, Tu, Ru) \\
&\leq k \max \left\{ \begin{array}{c} G(Au, Su, Tu)G(Bu, Tu, Ru), \\ G(Bu, Tu, Ru)G(Cu, Ru, Su), \\ G(Cu, Ru, Su)G(Au, Su, Tu) \end{array} \right\} \\
&= k \max \left\{ \begin{array}{c} G(u, u, u)G(u, u, Ru), \\ G(u, u, Ru)G(Ru, Ru, u), \\ G(Ru, Ru, u)G(u, u, u) \end{array} \right\} \\
&= kG(u, u, Ru)G(Ru, Ru, u) \\
&\leq 2kG^2(u, u, Ru).
\end{aligned}$$

This implies that $G(u, u, Ru) = 0$, and so $Ru = u = Cu$.

Therefore u is the common fixed point of S, T, R, A, B and C when A is continuous and the pair (S, A) is weakly commuting, the pairs (T, B) and (F, C) are weakly compatible.

Next, we suppose that S is continuous, the pair (S, A) is weakly commuting, the pairs (T, B) and (R, C) are weakly compatible.

Step 1. We prove that $u = Su$.

By (2.24) and the weakly commuting of the mapping pair (S, A) we have

$$G(SAx_{3n}, ASx_{3n}, ASx_{3n}) \leq G(Sx_{3n}, Ax_{3n}, Ax_{3n}) \rightarrow 0 \quad (n \rightarrow \infty). \tag{2.29}$$

Since S is continuous, $S^2 x_{3n} \rightarrow Su$ $(n \rightarrow \infty)$, $SAx_{3n} \rightarrow Su$ $(n \rightarrow \infty)$. By (2.29) we know $ASx_{3n} \rightarrow Su$ $(n \rightarrow \infty)$.

From condition (2.17) we have

$$G^2\left(S^2x_{3n}, Tx_{3n+1}, Rx_{3n+2}\right)$$

$$\leq k \max \left\{ \begin{array}{c} G(ASx_{3n}, S^2x_{3n}, Tx_{3n+1})G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Rx_{3n+2})G(Cx_{3n+2}, Rx_{3n+2}, S^2x_{3n}), \\ G(Cx_{3n+2}, Rx_{3n+2}, S^2x_{3n})G(ASx_{3n}, S^2x_{3n}, Tx_{3n+1}) \end{array} \right\}.$$

Letting $n \to \infty$, and using Proposition 1.3, we have

$$G^2(Su, u, u) \leq k \max \left\{ \begin{array}{c} G(Su, Su, u)G(u, u, u), \\ G(u, u, u)G(u, u, Su), \\ G(u, u, Su)G(Su, Su, u) \end{array} \right\}$$

$$= kG(u, u, Su)G(Su, Su, u)$$

$$\leq 2kG^2(Su, u, u).$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$.

Step 2. We prove that $u = Tu = Bu$.

Since $S(X) \subset B(X)$ and $u = Su \in S(X)$, there is a point $z \in X$ such that $u = Su = Bz$. Again by use of condition (2.17), we have

$$G^2\left(S^2x_{3n}, Tz, Rx_{3n+2}\right)$$

$$\leq k \max \left\{ \begin{array}{c} G(ASx_{3n}, S^2x_{3n}, Tz)G(Bz, Tz, Rx_{3n+2}), \\ G(Bz, Tz, Rx_{3n+2})G(Cx_{3n+2}, Rx_{3n+2}, S^2x_{3n}), \\ G(Cx_{3n+2}, Rx_{3n+2}, S^2x_{3n})G(ASx_{3n}, S^2x_{3n}, Tz) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Su = Bz$ we have

$$G^2(u, Tz, u) \leq k \max \left\{ \begin{array}{c} G(u, u, Tz)G(u, Tz, u), \\ G(u, Tz, u)G(u, u, u), \\ G(u, u, u)G(u, u, Tz) \end{array} \right\} = kG^2(u, Tz, u).$$

This implies that $G(u, Tz, u) = 0$, and so $Tz = u = Bz$.

Since the pair (T, B) is weakly compatible, we have

$$Tu = TBz = BTz = Bu.$$

Again, by use of condition (2.17), we have

$$G^2(Sx_{3n}, Tu, Rx_{3n+2}) \leq k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Tu)G(Bu, Tu, Rx_{3n+2}), \\ G(Bu, Tu, Rx_{3n+2})G(Cx_{3n+2}, Rx_{3n+2}, Sx_{3n}), \\ G(Cx_{3n+2}, Rx_{3n+2}, Sx_{3n})G(Ax_{3n}, Sx_{3n}, Tu) \end{array} \right\}.$$

Letting $n \to \infty$, using $u = Su$, $Tu = Bu$, and Proposition 1.3, we have

$$G^2(u, Tu, u) \le k \max \left\{ \begin{array}{c} G(u, u, Tu)G(Tu, Tu, u), \\ G(Tu, Tu, u)G(u, u, u), \\ G(u, u, u)G(u, u, Tu) \end{array} \right\}$$

$$= kG(u, u, Tu)G(Tu, Tu, u)$$

$$\le 2kG^2(u, Tu, u).$$

This implies that $G(u, Tu, u) = 0$, and so $Tu = u = Bu$.

So we have $u = Tu = Bu$.

Step 3. We prove that $u = Ru = Cu$.

Since $T(X) \subset C(X)$ and $u = Tu \in T(X)$, there is a point $t \in X$ such that $u = Tu = Ct$. Again, by use of condition (2.17), we have

$$G^2(Sx_{3n}, Tu, Rt) \le k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Tu)G(Bu, Tu, Rt), \\ G(Bu, Tu, Rt)G(Ct, Rt, Sx_{3n}), \\ G(Ct, Rt, Sx_{3n})G(Ax_{3n}, Sx_{3n}, Tu) \end{array} \right\}.$$

Letting $n \to \infty$ and using $u = Tu = Bu = Ct$, we obtain

$$G^2(u, u, Rt) \le k \max \left\{ \begin{array}{c} G(u, u, u)G(u, u, Rt), \\ G(u, u, Rt)G(u, Rt, u), \\ G(u, Rt, u)G(u, u, u) \end{array} \right\} = kG^2(u, u, Rt).$$

This implies that $G(u, u, Rt) = 0$, and so $Rt = u = Ct$.

Since the pair (R, C) is weakly compatible, we have

$$Ru = RCt = CRt = Cu.$$

Again, by use of condition (2.17), we have

$$G^2(Sx_{3n}, Tu, Ru) \le k \max \left\{ \begin{array}{c} G(Ax_{3n}, Sx_{3n}, Tu)G(Bu, Tu, Ru), \\ G(Bu, Tu, Ru)G(Cu, Ru, Sx_{3n}), \\ G(Cu, Ru, Sx_{3n})G(Ax_{3n}, Sx_{3n}, Tu) \end{array} \right\}.$$

Letting $n \to \infty$, using $u = Tu = Bu$, $Ru = Cu$, and Proposition 1.3, we have

$$G^2(u, u, Ru) \le k \max \left\{ \begin{array}{c} G(u, u, u)G(u, u, Ru), \\ G(u, u, Ru)G(Ru, Ru, u), \\ G(Ru, Ru, u)G(u, u, u) \end{array} \right\}$$

$$= kG(u, u, Ru)G(Ru, Ru, u)$$

$$\le 2kG^2(u, u, Ru).$$

This implies that $G(u, u, Ru) = 0$, and so $Ru = u = Cu$.

Step 4. We prove that $u = Au$.

Since $R(X) \subset A(X)$ and $u = Ru \in R(X)$, there is a point $p \in X$ such that $u = Ru = Ap$. Again, by use of condition (2.17), we have

$$G^2(Sp, Tu, Ru) \le k \max \left\{ \begin{array}{l} G(Ap, Sp, Tu)G(Bu, Tu, Ru), \\ G(Bu, Tu, Ru)G(Cu, Ru, Sp), \\ G(Cu, Ru, Sp)G(Ap, Sp, Tu) \end{array} \right\}.$$

Using $u = Tu = Bu$ and $u = Ru = Cu = Ap$, we obtain

$$G^2(Sp, u, u) \le k \max \left\{ \begin{array}{l} G(u, Sp, uu)G(u, u, u), \\ G(u, u, u)G(u, u, Sp), \\ G(u, u, Sp)G(u, Sp, u) \end{array} \right\} = kG^2(Sp, u, u).$$

This implies that $G(Sp, u, u) = 0$, and $Sp = u = Ap$.

Since the pair (S, A) is weakly compatible, we have

$$u = Su = SAp = ASp = Au.$$

Therefore u is the common fixed point of S, T, R, A, B and C when S is continuous and the pair (S, A) is weakly commuting, the pairs (T, B) and (F, C) are weakly compatible.

Similarly we can prove that u is the unique common fixed point of the maps S, T, R, A, B, and C under the conditions of (b) and (c).

Next we prove the uniqueness of a common fixed point u.

Let u and v are two common fixed point of S, T, R, A, B, and C, by (2.17) and Proposition 1.3 we have

$$G^2(u, u, v) = G^2(Su, Tu, Rv)$$

$$\le k \max \left\{ \begin{array}{l} G(Au, Su, Tu)G(Bu, Tu, Rv), \\ G(Bu, Tu, Rv)G(Cv, Rv, Su), \\ G(Cv, Rv, Su)G(Au, Su, Tu)) \end{array} \right\}$$

$$= kG(u, u, v)G(v, v, u)$$

$$\le 2kG^2(u, u, v).$$

This shows that $G(u, u, v) = 0$, and so $u = v$. Thus the common fixed point is unique.

If condition (2.18) holds, then the argument is similar to that above, so we omit it. □

Now we introduce an example to support Theorem 2.3.

Example 2.1 Let $X = [0, 1]$, and (X, G) be a G-metric space defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all x, y, z in X. Let f, g, h, A, B, and C be self-mappings defined by

$$Sx = \frac{18}{19}, \qquad Tx = \begin{cases} \frac{20}{21}, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1], \end{cases} \qquad Rx = \begin{cases} \frac{19}{20}, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1], \end{cases}$$

$$Ax = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1), \\ \frac{19}{20}, & x = 1, \end{cases} \qquad Bx = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1], \end{cases} \qquad Cx = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1), \\ \frac{20}{21}, & x = 1. \end{cases}$$

Note that S is G-continuous in X, and T, R, A, B, and C are not G-continuous in X.

Clearly we get $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$.

By the definition of the mappings of S and A, for all $x \in [0,1]$, we have

$$G(SAx, ASx, ASx) = G\left(\frac{18}{19}, \frac{18}{19}, \frac{18}{19}\right) = 0 \leq G(Sx, Ax, Ax),$$

so we can see the pair (S, A) is weakly commuting.

By the definition of the mappings of T and B, only for $x \in (\frac{1}{2}, 1]$, $Tx = Bx$, at this time $TBx = T(\frac{18}{19}) = \frac{18}{19} = B(\frac{18}{19}) = BTx$, so $TBx = BTx$, so we can see that the pair (T, B) is weakly compatible. Similarly we can prove the pair (R, C) is also weakly compatible.

Now we prove the mappings S, T, R, A, B, and C satisfy condition (2.17) of Theorem 2.3 with $k = \frac{2}{21} \in [0, \frac{1}{2})$. Let

$$N(x, y, z) = \max \begin{cases} G(Ax, Sx, Ty)G(By, Ty, Rz), \\ G(By, Ty, Rz)G(Cz, Rz, Sx), \\ G(Cz, Rz, Sx)G(Ax, Sx, Ty) \end{cases}.$$

Case 1. If $x, y, z \in [0, \frac{1}{2}]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{20}{21}, \frac{19}{20}\right) = \left(\frac{4}{399}\right)^2,$$

$$G(Ax, Sx, Ty) = G\left(1, \frac{18}{19}, \frac{20}{21}\right) = \frac{2}{19}, \qquad G(By, Ty, Rz) = G\left(1, \frac{20}{21}, \frac{19}{20}\right) = \frac{1}{10}.$$

Thus we have

$$\begin{aligned} G^2(Sx, Ty, Rz) &= \left(\frac{4}{399}\right)^2 < \frac{2}{21} \cdot \frac{2}{19} \cdot \frac{1}{10} \\ &= \frac{2}{21} G(Ax, Sx, Ty)G(By, Ty, Rz) \\ &\leq \frac{2}{21} N(x, y, z). \end{aligned}$$

Case 2. If $x, y \in [0, \frac{1}{2}]$, $z \in (\frac{1}{2}, 1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{20}{21}, \frac{18}{19}\right) = \left(\frac{4}{399}\right)^2,$$

$$G(Ax, Sx, Ty) = G\left(1, \frac{18}{19}, \frac{20}{21}\right) = \frac{2}{19}, \qquad G(By, Ty, Rz) = G\left(1, \frac{20}{21}, \frac{18}{19}\right) = \frac{2}{19}.$$

Therefore we get

$$\begin{aligned} G^2(Sx, Ty, Rz) &= \left(\frac{4}{399}\right)^2 < \frac{2}{21} \cdot \frac{2}{19} \cdot \frac{2}{19} \\ &= \frac{2}{21} G(Ax, Sx, Ty)G(By, Ty, Rz) \\ &\leq \frac{2}{21} N(x, y, z). \end{aligned}$$

Case 3. If $x, z \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{18}{19}, \frac{19}{20}\right) = \left(\frac{1}{190}\right)^2,$$

$$G(Ax, Sx, Ty) = G\left(1, \frac{18}{19}, \frac{18}{19}\right) = \frac{2}{19}, \qquad G(By, Ty, Rz) = G\left(\frac{18}{19}, \frac{18}{19}, \frac{19}{20}\right) = \frac{1}{190}.$$

Hence we have

$$\begin{aligned}
G^2(Sx, Ty, Rz) &= \left(\frac{1}{190}\right)^2 < \frac{2}{21} \cdot \frac{2}{19} \cdot \frac{1}{190} \\
&= \frac{2}{21} G(Ax, Sx, Ty) G(By, Ty, Rz) \\
&\leq \frac{2}{21} N(x, y, z).
\end{aligned}$$

Case 4. If $y, z \in [0, \frac{1}{2}]$, $x \in (\frac{1}{2}, 1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{20}{21}, \frac{19}{20}\right) = \left(\frac{4}{399}\right)^2,$$

$$G(By, Ty, Rz) = G\left(1, \frac{20}{21}, \frac{18}{19}\right) = \frac{2}{19},$$

$$G(Cz, Rz, Sx) = G\left(1, \frac{19}{20}, \frac{18}{19}\right) = \frac{2}{19}.$$

So we get

$$\begin{aligned}
G^2(Sx, Ty, Rz) &= \left(\frac{4}{399}\right)^2 < \frac{2}{21} \cdot \frac{2}{19} \cdot \frac{2}{19} \\
&= \frac{2}{21} G(By, Ty, Rz) G(Cz, Rz, Sx) \\
&\leq \frac{2}{21} N(x, y, z).
\end{aligned}$$

Case 5. If $x \in [0, \frac{1}{2}]$, $y, z \in (\frac{1}{2}, 1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{18}{19}, \frac{18}{19}\right) = 0 \leq \frac{2}{21} N(x, y, z).$$

Case 6. If $y \in [0, \frac{1}{2}]$, $x, z \in (\frac{1}{2}, 1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{20}{21}, \frac{18}{19}\right) = \left(\frac{4}{399}\right)^2,$$

$$G(Ax, Sx, Ty) = \begin{cases} G(\frac{18}{19}, \frac{18}{19}, \frac{20}{21}), & x \in (\frac{1}{2}, 1), \\ G(\frac{19}{20}, \frac{18}{19}, \frac{20}{21}), & x = 1, \end{cases} = \begin{cases} \frac{4}{399}, & x \in (\frac{1}{2}, 1), \\ \frac{4}{399}, & x = 1, \end{cases} = \frac{4}{399},$$

$$G(By, Ty, Rz) = G\left(1, \frac{20}{21}, \frac{18}{19}\right) = \frac{2}{19}.$$

Thus we have

$$
\begin{aligned}
G^2(Sx, Ty, Rz) &= \left(\frac{4}{399}\right)^2 = \frac{2}{21} \cdot \frac{4}{399} \cdot \frac{2}{19} \\
&= \frac{2}{21} G(Ax, Sx, Ty) G(By, Ty, Rz) \\
&\leq \frac{2}{21} N(x, y, z).
\end{aligned}
$$

Case 7. If $z \in [0, \frac{1}{2}]$, $x, y \in (\frac{1}{2}, 1]$, then

$$
G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{18}{19}, \frac{20}{21}\right) = \left(\frac{1}{190}\right)^2,
$$

$$
G(By, Ty, Rz) = G\left(\frac{18}{19}, \frac{18}{19}, \frac{19}{20}\right) = \frac{1}{190}, \qquad G(Cz, Rz, Sx) = G\left(1, \frac{19}{20}, \frac{18}{19}\right) = \frac{2}{19}.
$$

Hence we have

$$
\begin{aligned}
G^2(Sx, Ty, Rz) &= \left(\frac{1}{190}\right)^2 < \frac{2}{21} \cdot \frac{1}{190} \cdot \frac{2}{19} \\
&= \frac{2}{21} G(By, Ty, Rz) G(Cz, Rz, Sx) \\
&\leq \frac{2}{21} N(x, y, z).
\end{aligned}
$$

Case 8. If $x, y, z \in (\frac{1}{2}, 1]$, then

$$
G^2(Sx, Ty, Rz) = G^2\left(\frac{18}{19}, \frac{18}{19}, \frac{18}{19}\right) = 0 \leq \frac{2}{21} N(x, y, z).
$$

Then in all the above cases, the mappings S, T, R, A, B, and C are satisfying the condition (2.17) of Theorem 2.3 with $k = \frac{2}{21}$. Thus all the conditions of Theorem 2.3 are satisfied. Moreover, $\frac{18}{19}$ is the unique common fixed point for all of the mappings S, T, R, A, B, and C.

Theorem 2.4 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

  (i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;

 (ii) *the pairs (S, A), (T, B), and (R, C) are commuting mappings*;

(iii) $\forall x, y, z \in X$,

$$
G^2\left(S^p x, T^q y, R^r z\right) \leq k \max \left\{
\begin{array}{l}
G(Ax, S^p x, T^q y) G(By, T^q y, R^r z), \\
G(By, T^q y, R^r z) G(Cz, R^r z, S^p x), \\
G(Cz, R^r z, S^p x) G(Ax, S^p x, T^q y)
\end{array}
\right\}
\tag{2.30}
$$

   or

$$
G^2\left(S^p x, T^q y, R^r z\right) \leq k \max \left\{
\begin{array}{l}
G(Ax, Ax, T^q y) G(By, By, R^r z), \\
G(By, By, R^r z) G(Cz, Cz, S^p x), \\
G(Cz, Cz, S^p x) G(Ax, Ax, T^q y)
\end{array}
\right\},
\tag{2.31}
$$

where $k \in [0, \frac{1}{2})$, $p, q, r \in \mathbb{N}$, then S, T, R, A, B, and C have a unique common fixed point in X.

Proof Suppose the condition (2.30) holds. Since $S^p X \subset S^{p-1} X \subset \cdots \subset SX$, $SX \subset BX$, so that $S^p X \subset BX$. Similar, we can show that $T^q X \subset CX$ and $R^r X \subset AX$. From Theorem 2.3, we see that S^p, T^q, R^r, A, B, and C have a unique common fixed point u.

Since $Su = S(S^p u) = S^{p+1} u = S^p(Su)$,

$$G^2\left(S^p Su, T^q u, R^r u\right) \le k \max \left\{ \begin{array}{l} G(ASu, S^p Su, T^q u) G(Bu, T^q u, R^r u), \\ G(Bu, T^q u, R^r u) G(Cu, R^r u, S^p Su), \\ G(Cu, R^r u, S^p Su) G(ASu, S^p Su, T^q u) \end{array} \right\}.$$

Note that $u = Au = Bu = Cu = S^p u = T^q u = R^r u$, $S^p Su = Su$, and $AS = SA$, using Proposition 1.3, we obtain

$$
\begin{aligned}
G^2(Su, u, u) &= G^2\left(S^p Su, T^q u, R^r u\right) \\
&\le k \max \left\{ \begin{array}{l} G(Su, Su, u) G(u, u, u), \\ G(u, u, u) G(u, u, Su), \\ G(u, u, Su) G(Su, Su, u) \end{array} \right\} \\
&= k G(u, u, Su) G(Su, Su, u) \\
&\le 2k G^2(Su, u, u).
\end{aligned}
$$

This implies that $G(Su, u, u) = 0$, and so $Su = u$.

By the same argument, we can prove $Tu = u$ and $Ru = u$. Thus we have $u = Su = Tu = Ru = Au = Bu = Cu$, so that S, T, R, A, B, and C have a common fixed point u in X. Let v be any other common fixed point of S, T, R, A, B, and C, then by use of condition (2.28) and Proposition 1.3, we have

$$
\begin{aligned}
G^2(u, u, v) &= G^2\left(S^p u, T^q u, R^r v\right) \\
&\le k \max \left\{ \begin{array}{l} G(Au, S^p u, T^q u) G(Bu, T^q u, R^r v), \\ G(Bu, T^q u, R^r v) G(Cv, R^r v, S^p u), \\ G(Cv, R^r v, S^p u) G(Au, S^p u, T^q u) \end{array} \right\} \\
&= k G(u, u, v) G(v, v, u) \\
&\le G^2(u, u, v).
\end{aligned}
$$

This implies that $G(u, u, v) = 0$, and so $u = v$. Thus the common fixed point is unique.

If condition (2.31) holds, then the argument is similar to that above, so we omit it. □

Remark 2.5 Theorems 2.3 and 2.4 improve and extend the corresponding results in Ye and Gu [15, Theorem 2.5, Corollary 2.7] from three self-mappings to six self-mappings.

Corollary 2.4 *Let (X, G) be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions:*
   (i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Sx, Ty)G(By, Ty, Rz) + bG(By, Ty, Rz)G(Cz, Rz, Sx)$$
$$+ cG(Cz, Rz, Sx)G(Ax, Sx, Ty) \tag{2.32}$$

or

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Ax, Ty)G(By, By, Rz) + bG(By, By, Rz)G(Cz, Cz, Sx)$$
$$+ cG(Cz, Cz, Sx)G(Ax, Ax, Ty), \tag{2.33}$$

where $0 \leq a + b + c + d < \frac{1}{2}$. *Then one of the pairs* (S, A), (T, B), *and* (R, C) *has a coincidence point in* X. *Moreover, assume one of the following conditions is satisfied*:

(a) *either S or A is G-continuous, the pair* (S, A) *is weakly commuting, the pairs* (T, B) *and* (R, C) *are weakly compatible*;
(b) *either T or B is G-continuous, the pair* (T, B) *is weakly commuting, the pairs* (S, A) *and* (R, C) *are weakly compatible*;
(c) *either F or C is G-continuous, the pair* (R, C) *is weakly commuting, the pairs* (S, A) *and* (T, B) *are weakly compatible*.

Then the mappings S, T, R, A, B, and C have a unique common fixed point in X.

Proof Suppose the condition (2.32) holds. For $x, y, z \in X$, let

$$M(x, y, z) = \max \left\{ \begin{array}{l} G(Ax, Sx, Ty)G(By, Ty, Rz), \\ G(By, Ty, Rz)G(Cz, Rz, Sx), \\ G(Cz, Rz, Sx)G(Ax, Sx, Ty) \end{array} \right\}.$$

Then

$$aG(Ax, Sx, Ty)G(By, Ty, Rz) + bG(By, Ty, Rz)G(Cz, Rz, Sx)$$
$$+ cG(Cz, Rz, Sx)G(Ax, Sx, Ty)$$
$$\leq (a + b + c)M(x, y, z).$$

So, if

$$G^2(Sx, Ty, Rz) \leq aG(Ax, Sx, Ty)G(By, Ty, Rz) + bG(By, Ty, Rz)G(Cz, Rz, Sx)$$
$$+ cG(Cz, Rz, Rz)G(Ax, Sx, Sx),$$

then $G^2(Sx, Ty, Rz) \leq (a + b + c)M(x, y, z)$. Taking $k = a + b + c$ in Theorem 2.3, the conclusion of Corollary 2.4 can be obtained from Theorem 2.3 immediately.

If condition (2.33) holds, then the argument is similar to that above, so we omit it. This completes the proof of Corollary 2.4. □

Remark 2.6 Corollary 2.4 improve and extend the corresponding results in Ye and Gu [15, Theorem 2.5] from three self-mappings to six self-mappings.

Corollary 2.5 *Let* (X, G) *be a complete G-metric space and let S, T, R, A, B, and C be six mappings of X into itself satisfying the following conditions*:

  (i) $S(X) \subset B(X)$, $T(X) \subset C(X)$, $R(X) \subset A(X)$;
 (ii) *the pairs* (S, A), (T, B), *and* (R, C) *are commuting mappings*;
(iii) $\forall x, y, z \in X$,

$$
\begin{aligned}
G^2\big(S^p x, T^q y, R^r z\big) \leq {} & aG\big(Ax, S^p x, T^q y\big)G\big(By, T^q y, R^r z\big) \\
& + bG\big(By, T^q y, R^r z\big)G\big(Cz, R^r z, S^p x\big) \\
& + cG\big(Cz, R^r z, S^p x\big)G\big(Ax, S^p x, T^q y\big)
\end{aligned}
\tag{2.34}
$$

or

$$
\begin{aligned}
G^2\big(S^p x, T^q y, R^r z\big) \leq {} & aG\big(Ax, Ax, T^q y\big)G\big(By, By, R^r z\big) \\
& + bG\big(By, By, R^r z\big)G\big(Cz, Cz, S^p x\big) \\
& + cG\big(Cz, Cz, S^p x\big)G\big(Ax, Ax, T^q y\big),
\end{aligned}
\tag{2.35}
$$

where $0 \leq a + b + c + d < \frac{1}{2}$, $p, q, r \in \mathbb{N}$, *then S, T, R, A, B, and C have a unique common fixed point in X.*

Proof The proof follows from Theorem 2.4, and from an argument similar to that used in Corollary 2.4. □

Remark 2.7 Corollary 2.5 improve and extend the corresponding results in Ye and Gu [15, Corollary 2.7] from three self-mappings to six self-mappings.

In Theorem 2.3, if we take $A = B = C = I$, then we have the following corollary.

Corollary 2.6 *Let* (X, G) *be a complete G-metric space and let S, T, and R be three mappings of X into itself satisfying the following conditions*:

$$
G^2(Sx, Ty, Rz) \leq k \max \left\{ \begin{array}{l} G(x, Sx, Ty)G(y, Ty, Rz), \\ G(y, Ty, Rz)G(z, Rz, Sx), \\ G(z, Rz, Sx)G(x, Sx, Ty) \end{array} \right\}
\tag{2.36}
$$

or

$$
G^2(Sx, Ty, Rz) \leq k \max \left\{ \begin{array}{l} G(x, x, Ty)G(y, y, Rz), \\ G(y, y, Rz)G(z, z, Sx), \\ G(z, z, Sx)G(x, x, Ty) \end{array} \right\}
\tag{2.37}
$$

for all $x, y, z \in X$, *where* $k \in [0, \frac{1}{2})$.
Then the mappings S, T, and R have a unique common fixed point in X.

Remark 2.8 In Theorems 2.3, 2.4, Corollaries 2.4, 2.5 and 2.6, we have taken: (1) $S = T = R$; (2) $A = B = C$; (3) $A = B = C = I$; (4) $T = R$ and $B = C$; (5) $T = R$, $B = C = I$, several new result can be obtained.

Now we introduce an example to support Corollary 2.6.

Example 2.2 Let $X = [0,1]$, and (X, G) be a G-metric space defined by $G(x,y,z) = |x-y| + |y-z| + |z-x|$, for all x, y, z in X. Let T, S, and R be three self-mappings defined by

$$Sx = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{11}{12}, & x \in (\frac{1}{2}, 1], \end{cases} \qquad Tx = \begin{cases} \frac{12}{13}, & x \in [0, \frac{1}{2}], \\ \frac{11}{12}, & x \in (\frac{1}{2}, 1], \end{cases} \qquad Rx = \frac{11}{12}, \quad x \in [0,1].$$

Next we prove the mappings S, T, and R satisfy condition (2.36) of Corollary 2.6 with $k = \frac{1}{3}$.

Case 1. If $x, y \in [0, \frac{1}{2}]$, $z \in [0,1]$, then

$$G^2(Sx, Ty, Rz) = G^2\left(1, \frac{12}{13}, \frac{11}{12}\right) = \frac{1}{36},$$

$$G(x, Sx, Ty) = G\left(x, 1, \frac{12}{13}\right) = |x-1| + \left|x - \frac{12}{13}\right| + \frac{1}{13} \geq \frac{1}{2} + \frac{11}{26} + \frac{1}{13} = 1,$$

$$G(y, Ty, Rz) = G\left(y, \frac{12}{13}, \frac{11}{12}\right) = \left|y - \frac{12}{13}\right| + \left|y - \frac{11}{12}\right| + \frac{1}{156} \geq \frac{11}{26} + \frac{5}{12} + \frac{1}{156} = \frac{11}{13}.$$

Thus, we have

$$G^2(Sx, Ty, Rz) = \frac{1}{36} < \frac{1}{3} \cdot 1 \cdot \frac{11}{13} \leq \frac{1}{3} G(x, Sx, Ty)G(y, Ty, Rz)$$

$$\leq \frac{1}{3} \max \left\{ \begin{array}{c} G(x, Sx, Ty)G(y, Ty, Rz), \\ G(y, Ty, Rz)G(z, Rz, Sx), \\ G(z, Rz, Sx)G(x, Sx, Ty) \end{array} \right\}.$$

Case 2. If $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$, $z \in [0,1]$, then we get

$$G^2(Sx, Ty, Rz) = G^2\left(1, \frac{11}{12}, \frac{11}{12}\right) = \frac{1}{36},$$

$$G(x, Sx, Ty) = G\left(x, 1, \frac{11}{12}\right) = |x-1| + \left|x - \frac{11}{12}\right| + \frac{1}{12} \geq \frac{1}{2} + \frac{5}{12} + \frac{1}{12} = 1,$$

$$G(z, Rz, Sx) = G\left(z, \frac{11}{12}, 1\right) = \left|z - \frac{11}{12}\right| + |z-1| + \frac{1}{12} \geq 0 + 0 + \frac{1}{12} = \frac{1}{12}.$$

Thus, we have

$$G^2(Sx, Ty, Rz) = \frac{1}{36} = \frac{1}{3} \cdot 1 \cdot \frac{1}{12} \leq \frac{1}{3} G(x, Sx, Ty)G(z, Rz, Sx)$$

$$\leq \frac{1}{3} \max \left\{ \begin{array}{c} G(x, Sx, Ty)G(y, Ty, Rz), \\ G(y, Ty, Rz)G(z, Rz, Sx), \\ G(z, Rz, Sx)G(x, Sx, Ty) \end{array} \right\}.$$

Case 3. If $x \in (\frac{1}{2}, 1]$, $y \in [0, \frac{1}{2}]$, $z \in [0,1]$, then we have

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{11}{12}, \frac{12}{13}, \frac{11}{12}\right) = \frac{1}{6084},$$

$$G(x, Sx, Ty) = G\left(x, \frac{11}{12}, \frac{12}{13}\right) = \left|x - \frac{11}{12}\right| + \left|x - \frac{12}{13}\right| + \frac{1}{156} \geq 0 + 0 + \frac{1}{156} = \frac{1}{156},$$

$$G(y, Ty, Rz) = G\left(y, \frac{12}{13}, \frac{11}{12}\right) = \left|y - \frac{12}{13}\right| + \left|y - \frac{11}{12}\right| + \frac{1}{156} \geq \frac{11}{26} + \frac{5}{12} + \frac{1}{156} = \frac{11}{13}.$$

Thus, we have

$$G^2(Sx, Ty, Rz) = \frac{1}{6084} \leq \frac{1}{3} \cdot \frac{1}{156} \cdot \frac{11}{13} \leq \frac{1}{3} G(x, Sx, Ty) G(y, Ty, Rz)$$

$$\leq \frac{1}{3} \max \left\{ \begin{array}{l} G(x, Sx, Ty) G(y, Ty, Rz), \\ G(y, Ty, Rz) G(z, Rz, Sx), \\ G(z, Rz, Sx) G(x, Sx, Ty) \end{array} \right\}.$$

Case 4. If $x, y \in (\frac{1}{2}, 1]$, $z \in [0, 1]$, then we have

$$G^2(Sx, Ty, Rz) = G^2\left(\frac{11}{12}, \frac{11}{12}, \frac{11}{12}\right) = 0.$$

Thus, we have

$$G^2(Sx, Ty, Rz) \leq \frac{1}{3} \max \left\{ \begin{array}{l} G(x, Sx, Ty) G(y, Ty, Rz), \\ G(y, Ty, Rz) G(z, Rz, Sx), \\ G(z, Rz, Sx) G(x, Sx, Ty) \end{array} \right\}.$$

Then in all of the above cases, the mappings S, T, and R satisfy the contractive condition (2.36) of Corollary 2.6 with $k = \frac{1}{3}$. Thus all the conditions of Corollary 2.6 are satisfied. Moreover, $\frac{11}{12}$ is the unique common fixed point for all of the three mappings S, T, and R.

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