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Generalized retarded nonlinear integral inequalities involving iterated integrals and an application

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Abstract
In this work, some new generalized retarded nonlinear integral inequalities, which include nonlinear composite functions of unknown functions between iterated integrals, are discussed. By adopting novel analysis techniques, the upper bounds of the embedded unknown functions are estimated explicitly. The derived results can be applied in the study of differential-integral equations and some practical problems in engineering.

MSC: 26D15; 26D20; 34A12

Keywords: integral inequality; iterated integrals; analysis technique; estimation

1 Introduction

Integral inequality that provides an explicit bound to the unknown function furnishes a handy tool to investigate qualitative properties of solutions of differential and integral equations. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall-Bellman inequality [1, 2], which can be stated as follows: If u and f are nonnegative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)\,ds, \quad t \in [a, b],$$

for some constant $c \ge 0$, then

$$u(t) \le c\exp\left(\int_a^t f(s)\,ds\right), \quad t \in [a, b]. \tag{1.1}$$

It has become one of the very few classical and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (1.1) have been established, such as [3–23].

Among these references, Bainov *et al.* [7, p.107] considered the following interesting Gronwall-type inequality

$$u(t) \le a(t) + \sum_{i=1}^n \int_\alpha^t f_1(t, t_1)\left(\int_\alpha^{t_1} f_2(t_1, t_2)\cdots \times \left(\int_\alpha^{t_{i-1}} f_i(t_{i-1}, t_i)u(t_i)\,dt_i\right)\cdots\right)dt_1, \tag{1.2}$$

in which the unknown function only exists in the innermost layer of iterated integrals. In 2005 Kim [8] considered analogous Gronwall-type integral inequalities involving iterated integrals by replacing the unknown function u in the right-hand side of (1.2) with u^p for some constant p. In 2007, Agarwal *et al.* [10] investigated some nonlinear retarded inequalities with iterated integrals to extend Kim's results in [8],

$$\begin{aligned}\varphi\big(u(t)\big) &\leq a(t) + \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) f_1(t_1) u(t_1) g\big(u(t_1)\big)\, dt_1 \\ &\quad + \sum_{i=2}^{n} \int_{\phi(\alpha)}^{\phi(t)} p_1(t_1) \biggl(\int_{\phi(\alpha)}^{\phi(t_1)} p_2(t_2) \biggl(\cdots \biggl(\int_{\phi(\alpha)}^{\phi(t_{i-2})} p_{i-1}(t_{i-1}) \\ &\quad \times \biggl(\int_{\phi(\alpha)}^{\phi(t_{i-1})} p_i(t_i) f_i(t_i) u(t_i) g\big(u(t_i)\big)\, dt_i \biggr) dt_{i-1} \biggr) \cdots \biggr) dt_2 \biggr) dt_1,\end{aligned}$$

which include the composite functions of unknown function only in the innermost layer of iterated integrals.

In 2011, Abdeldaim *et al.* [11] studied some new integral inequalities of Gronwall-Bellman-Pachpatte-type such as

$$u(t) \leq u_0 + \int_0^t f(s) u(s) \biggl[u(s) + \int_0^s h(\tau) \biggl[u(\tau) + \int_0^\tau g(\xi) u(\xi)\, d\xi \biggr] d\tau \biggr] ds,$$

and

$$u(t) \leq u_0 + \int_0^t \bigl[f(s) u(s) + q(s) \bigr] ds + \int_0^t f(s) u(s) \biggl[u(s) + \int_0^s g(\tau) u(\tau)\, d\tau \biggr] ds,$$

which include the composite functions of unknown functions in every layer of iterated integrals, but the iterated integrals are double integrals.

In this paper, we extend certain results that were proved in [7–11] to obtain new generalizations of formerly famous Gronwall-Bellman-Pachpatte-type inequalities. There are not only composite functions of unknown functions in iterated integrals on the right hand side of our inequalities, but also the composite functions of unknown function exist in every layer of the iterated integrals. In this work, we give the upper bounds of the embedded unknown functions by adopting novel analysis techniques in three different scenarios and illustrate an application of our results, which verifies that our results are handy tools to study the qualitative properties of nonlinear differential equations and integral equations.

2 Main result

In this section, we state and prove some new integral inequalities of Gronwall-Bellman-Pachpatte-type, which can be used in the analysis of various problems in the theory of nonlinear ordinary differential and integral equations.

First, we give five assumptions for functions that will appear in our main results.

1. $u(t)$ and $a(t)$ are nonnegative and continuous functions on $[t_0, +\infty)$. In addition, $a(t)$ is nondecreasing;
2. $f_i(t,s)$, $i = 1, 2, 3$ are nonnegative and continuous functions for $t_0 \leq s \leq t \leq +\infty$, and nondecreasing in t for fixed $s \in [t_0, +\infty)$;
3. $w(u)$ is a nondecreasing and continuous function on $[0, +\infty)$ with $w(u) > 0$ for $u > 0$;

4. $\varphi(u)$ is an increasing continuous function with $\varphi(u) > 0$ for $u > 0$ and $\varphi(\infty) = \infty$;
5. $\alpha(t)$ is a continuous, differentiable and nondecreasing function on $[t_0, +\infty)$ with $\alpha(t) < t$, $\alpha(t_0) = t_0$.

In order to clearly present our main idea, we first consider a class of simple integral inequalities, namely, the composite function of unknown function $\varphi(u)$ is involved in the innermost layer of iterated integrals only.

Theorem 1 *Assume that Assumptions* 1-5 *and the following inequality hold*

$$\begin{aligned}\varphi\bigl(u(t)\bigr) \le{}& a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w\bigl(u(s)\bigr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)w\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)w\bigl(u(\xi)\bigr)\,d\xi\biggr)\,d\tau\biggr)\,ds \end{aligned} \tag{2.1}$$

for $t \in [t_0, \infty)$. *Then we have*

$$u(t) \le \varphi^{-1}\bigl(W^{-1}\bigl(U_1(t)\bigr)\bigr), \quad \forall t \in [t_0, T_1), \tag{2.2}$$

where

$$\begin{aligned}U_1(t) :={}& W\bigl(a(t)\bigr) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\biggr)\,d\tau\biggr)\,ds,\end{aligned} \tag{2.3}$$

$$W(u) := \int_{u_0}^{u} \frac{ds}{w(\varphi^{-1}(s))}, \quad u > u_0, \tag{2.4}$$

and φ^{-1}, W^{-1} *are the inverse functions of* φ, W, *respectively, and*

$$T_1 := \max\bigl\{t \in [t_0, +\infty) | U_1(t) \in \operatorname{Dom}\bigl(W^{-1}\bigr)\bigr\}.$$

Proof Choose $T \in [t_0, T_1)$ arbitrarily. For $\forall t \in [t_0, T]$, we obtain that

$$\begin{aligned}\varphi\bigl(u(t)\bigr) \le{}& a(T) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w\bigl(u(s)\bigr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)w\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)w\bigl(u(\xi)\bigr)\,d\xi\biggr)\,d\tau\biggr)\,ds,\end{aligned} \tag{2.5}$$

from (2.1), by Assumption 2 that $f_i(t,s)$ are nondecreasing in t. Let $z_1(t)$ be the right-hand side of (2.5), which is a positive and nondecreasing function on $[t_0, T]$ with $z_1(t_0) = a(T)$. Then (2.5) can be written as

$$u(t) \le \varphi^{-1}\bigl(z_1(t)\bigr), \quad \forall t \in [t_0, T], \tag{2.6}$$

since the inverse function φ^{-1} of φ exists by Assumption 4. From (2.5) and (2.6), we can obtain that

$$\begin{aligned}\frac{dz_1(t)}{dt} &\leq \alpha'(t)f_1\big(T,\alpha(t)\big)w\big(\varphi^{-1}\big(z_1(t)\big)\big) \\ &\quad+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)w\big(\varphi^{-1}\big(z_1(\tau)\big)\big)\,d\tau \\ &\quad+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w\big(\varphi^{-1}\big(z_1(\xi)\big)\big)\,d\xi\right)d\tau, \qquad (2.7)\end{aligned}$$

for all $t \in [t_0, T]$. Applying the monotonicity of w, φ and z_1, (2.7) can be written as

$$\begin{aligned}\frac{dz_1(t)}{w(\varphi^{-1}(z_1(t)))} &\leq \left(\alpha'(t)f_1\big(T,\alpha(t)\big)+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\,d\tau\right. \\ &\quad\left.+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)dt, \qquad (2.8)\end{aligned}$$

for all $t \in [t_0, T]$. Integrating both sides of the above inequality from t_0 to t, we can obtain that

$$\begin{aligned}W\big(z_1(t)\big) &\leq W\big(z_1(t_0)\big)+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\,ds \\ &\quad+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds, \quad \forall t \in [t_0,T], \qquad (2.9)\end{aligned}$$

where W is defined as (2.4). In consequence, we get that

$$\begin{aligned}u(t) &\leq \varphi^{-1}\left(W^{-1}\left(W\big(a(T)\big)\right.\right. \\ &\quad+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\,ds+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad\left.\left.+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds\right)\right), \quad \forall t \in [t_0,T], \qquad (2.10)\end{aligned}$$

by (2.6) and (2.9). Let $t = T$ on both hand sides of (2.10), then we have that

$$\begin{aligned}u(T) &\leq \varphi^{-1}\left(W^{-1}\left(W\big(a(T)\big)+\int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds\right.\right. \\ &\quad+\int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad\left.\left.+\int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds\right)\right). \qquad (2.11)\end{aligned}$$

Thus, we obtain that

$$u(t) \leq \varphi^{-1}\bigl(W^{-1}\bigl(U_1(t)\bigr)\bigr)$$

from (2.11), where $U_1(t)$ is defined as (2.3), since T is chosen arbitrarily. □

Next, consider a more general scenario: the composite function of unknown function exists not only in the innermost layer of iterated integrals, but also in the outermost layer of iterated integrals.

Theorem 2 *Assume that Assumptions* 1-5 *and the following inequality hold*

$$\begin{aligned}\varphi\bigl(u(t)\bigr) \leq{}& a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w\bigl(u(s)\bigr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w\bigl(u(s)\bigr)\biggl(\int_{t_0}^{s} f_2(s,\tau)w\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w\bigl(u(s)\bigr)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)w\bigl(u(\xi)\bigr)\,d\xi\biggr)\,d\tau\biggr)\,ds. \qquad (2.12)\end{aligned}$$

Then the integral inequality (2.12) *implies that*

$$u(t) \leq \varphi^{-1}\bigl\{W^{-1}\bigl[J^{-1}\bigl(U_2(t)\bigr)\bigr]\bigr\}, \quad \forall t \in [t_0, T_2], \qquad (2.13)$$

where W is defined in (2.4) *and*

$$J(u) := \int_{u_0}^{u} \frac{ds}{w(\varphi^{-1}(W^{-1}(s)))}, \quad u > u_0, \qquad (2.14)$$

$$\begin{aligned}U_2(t) :={}& J\biggl(W\bigl(a(t)\bigr) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\,ds\biggr) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\biggr)\,d\tau\biggr)\,ds, \qquad (2.15)\end{aligned}$$

and φ^{-1}, W^{-1}, J^{-1} are the inverse functions of φ, W, J, respectively, and

$$T_2 := \max\bigl\{t \in [t_0, +\infty) | U_2(t) \in \mathrm{Dom}\bigl(J^{-1}\bigr), J^{-1}\bigl(U_2(t)\bigr) \in \mathrm{Dom}\bigl(W^{-1}\bigr)\bigr\}.$$

Proof Choose $T \in [t_0, T_2]$ arbitrarily. For $\forall t \in [t_0, T]$, from (2.12), we have that

$$\begin{aligned}\varphi\bigl(u(t)\bigr) \leq{}& a(T) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w\bigl(u(s)\bigr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w\bigl(u(s)\bigr)\biggl(\int_{t_0}^{s} f_2(s,\tau)w\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w\bigl(u(s)\bigr)\biggl(\int_{t_0}^{s} f_2(s,\tau)\biggl(\int_{t_0}^{\tau} f_3(\tau,\xi)w\bigl(u(\xi)\bigr)\,d\xi\biggr)\,d\tau\biggr)\,ds \qquad (2.16)\end{aligned}$$

by Assumption 2. Denote the right-hand side of (2.16) by $z_2(t)$, which can be proved that it is positive and nondecreasing on $[t_0, T]$ with $z_2(t_0) = a(T)$. Then (2.16) can be written as

$$u(t) \le \varphi^{-1}\bigl(z_2(t)\bigr), \quad \forall t \in [t_0, T], \tag{2.17}$$

by Assumption 4. From (2.16) and (2.17), we obtain that

$$\begin{aligned} \frac{dz_2(t)}{dt} &\le \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) w\bigl(\varphi^{-1}\bigl(z_2(t)\bigr)\bigr) + \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \\ &\quad \times w\bigl(\varphi^{-1}\bigl(z_2(t)\bigr)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) w\bigl(\varphi^{-1}\bigl(z_2(\tau)\bigr)\bigr)\, d\tau + \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \\ &\quad \times w\bigl(\varphi^{-1}\bigl(z_2(t)\bigr)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) \left(\int_{t_0}^{\tau} f_3(\tau, \xi) w\bigl(\varphi^{-1}\bigl(z_2(\xi)\bigr)\bigr)\, d\xi \right) d\tau, \end{aligned} \tag{2.18}$$

By the property of monotonicity of functions w, φ and z_2, we can obtain that

$$\begin{aligned} \frac{dz_2(t)}{w(\varphi^{-1}(z_2(t)))} \le{}& \Bigl(\alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \\ &+ \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) w\bigl(\varphi^{-1}\bigl(z_2(\tau)\bigr)\bigr)\, d\tau \\ &+ \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) \\ &\times \left(\int_{t_0}^{\tau} f_3(\tau, \xi) w\bigl(\varphi^{-1}\bigl(z_2(\xi)\bigr)\bigr)\, d\xi \right) d\tau \Bigr)\, dt, \end{aligned}$$

from (2.18). Integrating both sides of the above inequality from t_0 to t, we have that

$$\begin{aligned} W\bigl(z_2(t)\bigr) \le{}& W\bigl(z_2(t_0)\bigr) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s)\, ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) w\bigl(\varphi^{-1}\bigl(z_2(\tau)\bigr)\bigr)\, d\tau \right) ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) \left(\int_{t_0}^{\tau} f_3(\tau, \xi) w\bigl(\varphi^{-1}\bigl(z_2(\xi)\bigr)\bigr)\, d\xi \right) d\tau \right) ds \\ \le{}& W\bigl(z_2(t_0)\bigr) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s)\, ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) w\bigl(\varphi^{-1}\bigl(z_2(\tau)\bigr)\bigr)\, d\tau \right) ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) \left(\int_{t_0}^{\tau} f_3(\tau, \xi) w\bigl(\varphi^{-1}\bigl(z_2(\xi)\bigr)\bigr)\, d\xi \right) d\tau \right) ds, \end{aligned} \tag{2.19}$$

for all $t \in [t_0, T]$, where W is given in (2.4). Let $v_1(t)$ denote the right-hand side of (2.19), which can be proved to be a positive and nondecreasing function on $[t_0, T]$ with $v_1(t_0) = W(a(T)) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s)\, ds$. Then (2.19) is equivalent to

$$z_2(t) \le W^{-1}\bigl(v_1(t)\bigr), \quad \forall t \in [t_0, T]. \tag{2.20}$$

Differentiating v_1, we get that

$$\begin{aligned}\frac{dv_1(t)}{dt} &\le \alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)w\big(\varphi^{-1}\big(W^{-1}\big(v_1(\tau)\big)\big)\big)\,d\tau \\ &\quad + \alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big) \\ &\quad \times \left(\int_{t_0}^{\tau} f_3(\tau,\xi)w\big(\varphi^{-1}\big(W^{-1}\big(v_1(\xi)\big)\big)\big)\,d\xi\right)d\tau,\end{aligned} \tag{2.21}$$

using (2.20), for all $t \in [t_0, T]$. By (2.21) and the monotonicity of w, φ^{-1}, W^{-1} and v_1, we further obtain that

$$\begin{aligned}\frac{dv_1(t)}{w(\varphi^{-1}(W^{-1}(v_1(t))))} &\le \Bigg(\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\,d\tau \\ &\quad + \alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\Bigg)\,dt,\end{aligned}$$

for all $t \in [t_0, T]$. Integrating both sides of the above inequality from t_0 to t, we obtain

$$\begin{aligned}J\big(v_1(t)\big) &\le J\big(v_1(t_0)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds,\end{aligned} \tag{2.22}$$

for all $t \in [t_0, T]$, where J is defined by (2.14). Hence, inequalities (2.17), (2.20) and (2.22) yield that

$$\begin{aligned}u(t) &\le \varphi^{-1}\big\{z_2(t)\big\} \le \varphi^{-1}\big\{W^{-1}\big[v_1(t)\big]\big\} \\ &= \varphi^{-1}\Bigg\{W^{-1}\Bigg[J^{-1}\Bigg(J\Bigg(W\big(a(T)\big) \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds\Bigg) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds\Bigg)\Bigg]\Bigg\}, \quad \forall t \in [t_0, T].\end{aligned} \tag{2.23}$$

Let $t = T$ on both hand sides of (2.23), we have that

$$\begin{aligned}u(T) &\le \varphi^{-1}\Bigg\{W^{-1}\Bigg[J^{-1}\Bigg(J\Bigg(W\big(a(T)\big) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds\Bigg) \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds\Bigg)\Bigg]\Bigg\}.\end{aligned} \tag{2.24}$$

Due to the randomness of T, (2.13) is achieved immediately from (2.24). □

Obviously, the most general scenario is that the composite function of unknown function is involved in every layer of iterated integrals. For this kind of integral inequalities, we have the following result.

Theorem 3 *Assume that Assumptions* 1-5 *and the following inequality hold*

$$\begin{aligned}\varphi\big(u(t)\big) &\le a(t) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w_1\big(u(s)\big)\,ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w_1\big(u(s)\big)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(u(\tau)\big)\,d\tau\right)ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)w_1\big(u(s)\big)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(u(\tau)\big)\right. \\ &\quad \left.\times\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\big(u(\xi)\big)\,d\xi\right)d\tau\right)ds. \end{aligned} \tag{2.25}$$

Then we have that

$$u(t) \le \varphi^{-1}\big\{\Phi_1^{-1}\big[\Phi_2^{-1}\big(\Phi_3^{-1}\big(U_3(t)\big)\big)\big]\big\}, \quad \forall t \in [t_0, T_3], \tag{2.26}$$

where

$$\Phi_1(u) := \int_{u_0}^{u} \frac{ds}{w_1(\varphi^{-1}(s))}, \quad u > u_0, \tag{2.27}$$

$$\Phi_2(u) := \int_{u_0}^{u} \frac{ds}{w_2(\varphi^{-1}(\Phi_1^{-1}(s)))}, \quad u > u_0, \tag{2.28}$$

$$\Phi_3(u) := \int_{u_0}^{u} \frac{ds}{w_3(\varphi^{-1}(\Phi_1^{-1}(\Phi_2^{-1}(s))))}, \quad u > u_0, \tag{2.29}$$

$$\begin{aligned}U_3(t) &:= \Phi_3\left\{\Phi_2\left[\Phi_1\big(a(t)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\,ds\right]\right. \\ &\quad \left.+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right)ds\right\} \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)\,d\xi\right)d\tau\right)ds, \end{aligned} \tag{2.30}$$

and φ^{-1}, Φ_i^{-1}, $i = 1,2,3$ *are the inverse functions of* φ, Φ_i, $i = 1,2,3$, *respectively, and*

$$\begin{aligned}T_3 &:= \max\big\{t \in [t_0, +\infty) \,|\, U_3(t) \in \mathrm{Dom}\big(\Phi_3^{-1}\big), \\ &\quad \Phi_3^{-1}\big(U_3(t)\big) \in \mathrm{Dom}\big(\Phi_2^{-1}\big), \Phi_2^{-1}\big(\Phi_3^{-1}\big(U_3(t)\big)\big) \in \mathrm{Dom}\big(\Phi_1^{-1}\big)\big\}.\end{aligned}$$

Proof Choose $T \in [t_0, T_3]$ arbitrarily. For $\forall t \in [t_0, T]$, we obtain that

$$\begin{aligned}\varphi\big(u(t)\big) &\le a(T) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w_1\big(u(s)\big)\,ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w_1\big(u(s)\big)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(u(\tau)\big)\,d\tau\right)ds\end{aligned}$$

$$+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)w_1\big(u(s)\big)$$
$$\times\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(u(\tau)\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\big(u(\xi)\big)\,d\xi\right)d\tau\right)ds. \tag{2.31}$$

from (2.25) and the monotonicity of $f_i(t,s)$, $i = 1,2,3$ on t. Let $z_3(t)$ be the right-hand side of (2.31), which is a positive and nondecreasing function on $[t_0,T]$ with $z_3(t_0) = a(T)$. Then (2.31) is equivalent to

$$u(t)\le\varphi^{-1}\big(z_3(t)\big),\quad \forall n\in[t_0,T]. \tag{2.32}$$

Differentiating z_3, we can obtain that

$$\begin{aligned}\frac{dz_3(t)}{dt}\le{}&\alpha'(t)f_1\big(T,\alpha(t)\big)w_1\big(\varphi^{-1}\big(z_3(t)\big)\big)+\alpha'(t)f_1\big(T,\alpha(t)\big)w_1\big(\varphi^{-1}\big(z_3(t)\big)\big)\\&\times\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)w_2\big(\varphi^{-1}\big(z_3(\tau)\big)\big)\,d\tau\\&+\alpha'(t)f_1\big(T,\alpha(t)\big)w_1\big(\varphi^{-1}\big(z_3(t)\big)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\\&\times w_2\big(\varphi^{-1}\big(z_3(\tau)\big)\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\big(\varphi^{-1}\big(z_3(\xi)\big)\big)\,d\xi\right)d\tau,\end{aligned} \tag{2.33}$$

from (2.32) and the monotonicity of w_1, φ and z_3, for all $t\in[t_0,T]$. Thus, we have

$$\begin{aligned}\frac{dz_3(t)}{w_1(\varphi^{-1}(z_3(t)))}\le{}&\left(\alpha'(t)f_1\big(T,\alpha(t)\big)+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\right.\\&\times w_2\big(\varphi^{-1}\big(z_3(\tau)\big)\big)\,d\tau+\alpha'(t)f_1\big(T,\alpha(t)\big)\int_{t_0}^{\alpha(t)} f_2\big(\alpha(t),\tau\big)\\&\left.\times w_2\big(\varphi^{-1}\big(z_3(\tau)\big)\big)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\big(\varphi^{-1}\big(z_3(\xi)\big)\big)\,d\xi\right)d\tau\right)dt,\end{aligned}$$

by (2.33) for all $t\in[t_0,T]$. Integrating both sides of the above inequality from t_0 to t, we obtain

$$\begin{aligned}\Phi_1\big(z_3(t)\big)\le{}&\Phi_1\big(z_3(t_0)\big)+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\,ds\\&+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(\varphi^{-1}\big(z_2(\tau)\big)\big)\,d\tau\right)ds\\&+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(\varphi^{-1}\big(z_2(\tau)\big)\big)\right.\\&\left.\times\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\big(\varphi^{-1}\big(z_2(\xi)\big)\big)\,d\xi\right)d\tau\right)ds\\\le{}&\Phi_1\big(z_3(t_0)\big)+\int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds\\&+\int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\big(\varphi^{-1}\big(z_2(\tau)\big)\big)\,d\tau\right)ds\end{aligned}$$

$$+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)w_2\left(\varphi^{-1}\left(z_2(\tau)\right)\right) \times \left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\left(\varphi^{-1}\left(z_2(\xi)\right)\right) d\xi\right) d\tau\right) ds, \tag{2.34}$$

where Φ_1 is defined by (2.27). Let $v_2(t)$ denote the right-hand side of (2.34), which is a positive and nondecreasing function on $[t_0, T]$ with $v_2(t_0) = \Phi_1(a(T)) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds$. Then (2.34) is equivalent to

$$z_3(t) \le \Phi_1^{-1}\left(v_2(t)\right), \quad \forall t \in [t_0, T]. \tag{2.35}$$

Differentiating v_2, we obtain

$$\begin{aligned}\frac{dv_2(t)}{dt} &\le \alpha'(t)f_1\left(T,\alpha(t)\right)\int_{t_0}^{\alpha(t)} f_2\left(\alpha(t),\tau\right)w_2\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\tau)\right)\right)\right) d\tau \\ &\quad + \alpha'(t)f_1\left(T,\alpha(t)\right)\int_{t_0}^{\alpha(t)} f_2\left(\alpha(t),\tau\right)w_2\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\tau)\right)\right)\right) \\ &\quad \times \left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\xi)\right)\right)\right) d\xi\right) d\tau, \quad \forall t \in [t_0, T]\end{aligned} \tag{2.36}$$

by (2.35). Applying (2.36) and the monotonicity of w_2, φ^{-1}, Φ_1^{-1} and v_2, we can get that

$$\begin{aligned}\frac{dv_2(t)}{w_2(\varphi^{-1}(\Phi_1^{-1}(v_2(t))))} &\le \Bigg(\alpha'(t)f_1\left(T,\alpha(t)\right)\int_{t_0}^{\alpha(t)} f_2\left(\alpha(t),\tau\right) d\tau \\ &\quad + \alpha'(t)f_1\left(T,\alpha(t)\right)\int_{t_0}^{\alpha(t)} f_2\left(\alpha(t),\tau\right) \\ &\quad \times \left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\xi)\right)\right)\right) d\xi\right) d\tau\Bigg) dt\end{aligned}$$

for all $t \in [t_0, T]$. Integrating both sides of the above inequality from t_0 to t, we obtain

$$\begin{aligned}\Phi_2\left(v_2(t)\right) &\le \Phi_2\left(v_2(t_0)\right) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right) ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\xi)\right)\right)\right) d\xi\right) d\tau\right) ds \\ &\le \Phi_2\left(v_2(t_0)\right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right) ds + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau) \right. \\ &\quad \left. \times \left(\int_{t_0}^{\tau} f_3(\tau,\xi)w_3\left(\varphi^{-1}\left(\Phi_1^{-1}\left(v_2(\xi)\right)\right)\right) d\xi\right) d\tau\right) ds,\end{aligned} \tag{2.37}$$

for all $t \in [t_0, T]$, where Φ_2 is defined by (2.28). Now, let $v_3(t)$ be the right-hand side of (2.37), which is a positive and nondecreasing function on $[t_0, T]$ with

$$\begin{aligned}v_3(t_0) &= \Phi_2\left(v_2(t_0)\right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right) ds \\ &= \Phi_2\left(\Phi_1\left(a(T)\right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\,ds\right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T,s)\left(\int_{t_0}^{s} f_2(s,\tau)\,d\tau\right) ds.\end{aligned} \tag{2.38}$$

Then, (2.37) is equivalent to

$$v_2(t) \le \Phi_2^{-1}\bigl(v_3(t)\bigr), \quad \forall t \in [t_0, T]. \tag{2.39}$$

Differentiating v_3 and applying (2.39), we can obtain that

$$\begin{aligned} \frac{dv_3(t)}{dt} &\le \alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) \\ &\quad \times \left(\int_{t_0}^{\tau} f_3(\tau, \xi) w_3\bigl(\varphi^{-1}\bigl(\Phi_1^{-1}\bigl(\Phi_2^{-1}\bigl(v_3(\xi)\bigr)\bigr)\bigr)\bigr)\, d\xi \right) d\tau, \end{aligned} \tag{2.40}$$

for all $t \in [t_0, T]$. By (2.40) and the monotonicity of w_3, φ^{-1}, Φ_1^{-1}, Φ_2^{-1} and v_3, we get

$$\frac{dv_3(t)}{w_3(\varphi^{-1}(\Phi_1^{-1}(\Phi_2^{-1}(v_3(t)))))} \le \left(\alpha'(t) f_1\bigl(T, \alpha(t)\bigr) \int_{t_0}^{\alpha(t)} f_2\bigl(\alpha(t), \tau\bigr) \left(\int_{t_0}^{\tau} f_3(\tau, \xi)\, d\xi \right) d\tau \right) dt,$$

for all $t \in [t_0, T]$. Integrating both sides of the inequality above, from t_0 to t, we obtain

$$\Phi_3\bigl(v_3(t)\bigr) \le \Phi_3\bigl(v_3(t_0)\bigr) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) \left(\int_{t_0}^{\tau} f_3(\tau, \xi)\, d\xi \right) d\tau \right) ds, \tag{2.41}$$

for all $t \in [t_0, T]$, where Φ_3 is defined by (2.29). By combining (2.32), (2.35), (2.39) and (2.41), we can obtain that

$$\begin{aligned} u(t) &\le \varphi^{-1}\bigl\{z_3(t)\bigr\} \le \varphi^{-1}\bigl\{\Phi_1^{-1}\bigl[v_2(t)\bigr]\bigr\} \le \varphi^{-1}\bigl\{\Phi_1^{-1}\bigl[\Phi_2^{-1}\bigl(v_3(t)\bigr)\bigr]\bigr\} \\ &\le \varphi^{-1}\left\{ \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3\bigl(v_3(t_0)\bigr) + \sum_{s=t_0}^{n-1} f_1(T, s) \int_{t_0}^{s} f_2(s, \tau) \int_{t_0}^{\tau} f_3(\tau, \xi) \right)\right)\right]\right\} \\ &= \varphi^{-1}\left\{ \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3\left(\Phi_2\left(\Phi_1\bigl(a(T)\bigr) \right.\right.\right.\right.\right.\right. \\ &\quad \left. + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s)\, ds \right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau)\, d\tau \right) ds \Big) \\ &\quad \left.\left.\left.\left. + \int_{\alpha(t_0)}^{\alpha(t)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) \left(\int_{t_0}^{\tau} f_3(\tau, \xi)\, d\xi \right) d\tau \right) ds \right)\right)\right]\right\} \end{aligned} \tag{2.42}$$

for all $t \in [t_0, T]$. Let $t = T$ on both hand sides of (2.42), we have

$$\begin{aligned} u(T) &\le \varphi^{-1}\left\{ \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3\left(\Phi_2\left(\Phi_1\bigl(a(T)\bigr) \right.\right.\right.\right.\right.\right. \\ &\quad \left. + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s)\, ds \right) + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau)\, d\tau \right) ds \Big) \\ &\quad \left.\left.\left.\left. + \int_{\alpha(t_0)}^{\alpha(T)} f_1(T, s) \left(\int_{t_0}^{s} f_2(s, \tau) \left(\int_{t_0}^{\tau} f_3(\tau, \xi)\, d\xi \right) d\tau \right) ds \right)\right)\right]\right\}. \end{aligned} \tag{2.43}$$

Since T is chosen arbitrarily in (2.43), thus (2.26) is proved. □

As a generalization of Theorem 3, we can obtain the following corollary, which can be proved similarly as Theorem 3.

Corollary 1 *Assume that Assumptions* 1-5 *and the following inequality hold*

$$\begin{aligned} u(t) \le a(t) &+ \int_\alpha^t f_1(t,t_1) w_1\big(u(t_1)\big)\,dt_1 \\ &+ \sum_{i=2}^n \int_\alpha^t f_1(t,t_1) w_1\big(u(t_1)\big)\bigg(\int_\alpha^{t_1} f_2(t_1,t_2) \\ &\times w_2\big(u(t_2)\big)\bigg(\cdots\bigg(\int_\alpha^{t_{i-1}} f_i(t_{i-1},t_i) w_i\big(u(t_i)\big)\,dt_i\bigg)\cdots\bigg)\,dt_2\bigg)\,dt_1. \end{aligned} \tag{2.44}$$

Then we have that

$$u(t) \le \varphi^{-1}\big\{\Phi_1^{-1}\big[\Phi_2^{-1}\cdots\big(\Phi_{n-1}^{-1}\big(\Phi_n^{-1}\big(U_4(t)\big)\big)\big)\cdots\big]\big\}, \quad \forall t \in [t_0, T_4), \tag{2.45}$$

where

$$\begin{aligned} U_4(t) := \Phi_n\bigg\{\Phi_{n-1}\bigg[&\cdots\Phi_2\bigg(\Phi_1\big(a(t)\big) + \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,s)\bigg) + \cdots \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,t_1) \\ &\times \bigg(\int_{t_0}^{t_1} f_2(t_1,t_2)\cdots\bigg(\int_{t_0}^{t_{n-3}} f_{n-2}(t_{n-3},t_{n-2})\,dt_{n-2}\bigg)\,dt_{n-3}\cdots dt_2\bigg)\,dt_1\bigg] \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,t_1) \\ &\times \bigg(\int_{t_0}^{t_1} f_2(t_1,t_2)\cdots\bigg(\int_{t_0}^{t_{n-2}} f_{n-1}(t_{n-2},t_{n-1})\,dt_{n-1}\bigg)\,dt_{n-2}\cdots dt_2\bigg)\,dt_1\bigg\} \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t,t_1) \\ &\times \bigg(\int_{t_0}^{t_1} f_2(t_1,t_2)\cdots\bigg(\int_{t_0}^{t_{n-1}} f_n(t_{n-1},t_n)\,dt_n\bigg)\,dt_{n-1}\cdots dt_2\bigg)\,dt_1, \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} &\Phi_1(u) := \int_1^u \frac{ds}{w_1(\varphi^{-1}(s))}, \qquad \Phi_2(u) := \int_1^u \frac{ds}{w_2(\varphi^{-1}(\Phi_1^{-1}(s)))}, \qquad \ldots, \\ &\Phi_{n-1}(u) := \int_1^u \frac{ds}{w_{n-1}(\varphi^{-1}(\Phi_1^{-1}(\Phi_2^{-1}(\cdots(\Phi_{n-3}^{-1}(\Phi_{n-2}^{-1}(s)))\cdots))))}, \\ &\Phi_n(u) := \int_1^u \frac{ds}{w_{n-1}(\varphi^{-1}(\Phi_1^{-1}(\Phi_2^{-1}(\cdots(\Phi_{n-2}^{-1}(\Phi_{n-1}^{-1}(s)))\cdots))))}, \end{aligned} \tag{2.47}$$

and φ^{-1}, Φ_i^{-1}, $i = 1,2,\ldots,n$ *are the inverse functions of* φ, Φ_i, $i = 1,2,\ldots,n$, *respectively, and*

$$\begin{aligned} T_4 := \max\big\{&t \in [t_0,+\infty) \,|\, U_4(T_4) \in \operatorname{Dom}\big(\Phi_n^{-1}\big), \Phi_n^{-1}\big(U_4(T_4)\big) \in \operatorname{Dom}\big(\Phi_{n-1}^{-1}\big), \\ &\ldots, \Phi_3^{-1}\big(\Phi_4^{-1}\big(\cdots\big(\Phi_{n-1}^{-1}\big(\Phi_n^{-1}\big(U_4(T_4)\big)\big)\big)\cdots\big)\big) \in \operatorname{Dom}\big(\Phi_2^{-1}\big), \\ &\Phi_2^{-1}\big(\Phi_3^{-1}\big(\cdots\big(\Phi_{n-1}^{-1}\big(\Phi_n^{-1}\big(U_4(T_4)\big)\big)\big)\cdots\big)\big) \in \operatorname{Dom}\big(\Phi_1^{-1}\big)\big\}. \end{aligned}$$

3 Application

In this section, we apply our result in Theorem 3 to investigate the robust stability of a class of closed-loop control systems, which demonstrates that our results are handy tools to analyze the qualitative properties of solutions of some nonlinear ordinary differential and integral equations.

For a given control system

$$\frac{dx(t)}{dt} = A_0(t)x(t) + B_0(t)u(t), \tag{3.1}$$

there is no doubt that controller design plays a pivotal role. Choosing the full state feedback controller $u = -Fx$ with the appropriate gain F for (3.1), one can immediately obtain the following stable closed-loop system

$$\frac{dx(t)}{dt} = A(t)x(t), \tag{3.2}$$

where $A(t) \triangleq A_0(t) - B_0(t)F$. However, in practice, some undesirable system factors, including nonlinear uncertainties and input disturbance, will be involved. As such, before applying the designed controller to real processes, the stability of a closed-loop system against external perturbations must be verified, which is the so-called robust stability analysis.

Consider a perturbed system of (3.2)

$$\frac{dx(t)}{dt} = A(t)x(t) + f\bigl(t, x\bigl(\alpha(t)\bigr), \sigma\bigl(\alpha(t)\bigr)\bigr), \quad t \in [t_0, \infty), x(t_0) = x_0 \tag{3.3}$$

with

$$\sigma(t) = \theta(t) + \int_{t_0}^{t} k\bigl(t, s, x(s)\bigr)\, ds, \tag{3.4}$$

where $\alpha \in C^1([t_0, \infty), [t_0, \infty))$ is a nondecreasing function with $\alpha(t_0) = t_0$, $x, y, \theta, \sigma \in C^1(\mathbf{R}, \mathbf{R}^r)$, $A(t)$ is a $r \times r$ continuous nonsingular matrix, and the function $f \in C(\mathbf{R} \times \mathbf{R}^r \times \mathbf{R}^r, \mathbf{R}^r)$ and $k \in C(\mathbf{R} \times \mathbf{R} \times \mathbf{R}^r, \mathbf{R}^r)$ satisfy the following conditions

$$\bigl|f\bigl(t, x\bigl(\alpha(t)\bigr), \sigma\bigl(\alpha(t)\bigr)\bigr)\bigr| \le g_1(t)e^{-\beta t} w_1\bigl(\bigl|x\bigl(\alpha(t)\bigr)\bigr|e^{\beta\alpha(t)}\bigr)\bigl(1 + \bigl|\sigma\bigl(\alpha(t)\bigr)\bigr|\bigr), \tag{3.5}$$

$$\bigl|k\bigl(t, s, x(s)\bigr)\bigr| \le g_2(t, s) w_2\bigl(\bigl|x(s)\bigr|e^{\beta s}\bigr)\left(1 + \int_{t_0}^{s} g_3(s, \tau) w_3\bigl(\bigl|x(\tau)\bigr|e^{\beta\tau}\bigr)\, d\tau\right), \tag{3.6}$$

where $\beta > 0$ is a constant, $g_1 \in C([t_0, \infty), \mathbf{R}_+)$ and $g_i \in C([t_0, \infty) \times [t_0, \infty), \mathbf{R}_+)$, $i = 2, 3$, are nondecreasing in t for fixed $s \in [t_0, \infty)$, $w_i(u)$, $i = 1, 2, 3$, are positive and continuous functions defined on $[0, \infty)$. In general, the perturbation term $f(t, x(\alpha(t)), \sigma(\alpha(t)))$ could result from modeling errors, aging, uncertainties, disturbances, or some other reasons. Suppose that the nominal system (3.2) has a uniformly asymptotically stable equilibrium at the origin, we next exploit the stability of the perturbed system (3.3). The result is presented in the following proposition.

Proposition 1 *If there exists a constant $C > 0$ such that the fundamental solution matrix $X(t)$ of the linear system* (3.2) *satisfies*

$$\left|X(t)X^{-1}(s)\right| \leq C\exp\bigl(-\beta(t-s)\bigr), \quad 0 \leq s \leq t \leq \infty, \tag{3.7}$$

then we have that

$$\left|x_\sigma(t,t_0,x_0)\right| \leq \exp(-\beta t)\bigl\{\Phi_4^{-1}\bigl[\Phi_5^{-1}\bigl(\Phi_6^{-1}\bigl(U_4(t)\bigr)\bigr)\bigr]\bigr\}, \quad \forall t \in [t_0,+\infty), \tag{3.8}$$

where $x_\sigma(t,t_0,x_0)$ is a solution of the control system (3.3) *with* (3.4) *and*

$$f_1(s) := \frac{Cg_1(\alpha^{-1}(s))e^{\beta}(1+|\theta(s)|)}{\alpha'(\alpha^{-1}(s))}, \tag{3.9}$$

$$\Phi_4(u) := \int_1^u \frac{ds}{w_1(s)}, \quad u > 0, \tag{3.10}$$

$$\Phi_5(u) := \int_1^u \frac{ds}{w_2(\Phi_1^{-1}(s))}, \quad u > 0, \tag{3.11}$$

$$\Phi_6(u) := \int_1^u \frac{ds}{w_3(\Phi_1^{-1}(\Phi_2^{-1}(s)))}, \quad u > 0, \tag{3.12}$$

$$\begin{aligned} U_4(t) := \Phi_6\left\{\Phi_5\left[\Phi_4\bigl(|x_0|C\exp(\beta t_0)\bigr) + \int_{t_0}^{\alpha(t)} f_1(s)\,ds\right] + \int_{t_0}^{\alpha(t)} f_1(s)\left(\int_{t_0}^{s} g_2(s,\tau)\,d\tau\right)ds\right\} \\ + \int_{t_0}^{\alpha(t)} f_1(s)\left[\int_{t_0}^{s} g_2(s,\tau)\left(\int_{t_0}^{\tau} g_3(\tau,\xi)\,d\xi\right)d\tau\right]ds, \end{aligned} \tag{3.13}$$

and Φ_i^{-1}, $i = 4,5,6$ are the inverse functions of Φ_i, $i = 4,5,6$, respectively, and

$$\begin{aligned} T_4 := \max\bigl\{t \in [t_0,+\infty) | U_4(t) \in \operatorname{Dom}\bigl(\Phi_6^{-1}\bigr), \\ \Phi_6^{-1}\bigl(U_4(t)\bigr) \in \operatorname{Dom}\bigl(\Phi_5^{-1}\bigr), \Phi_5^{-1}\bigl(\Phi_6^{-1}\bigl(U_4(t)\bigr)\bigr) \in \operatorname{Dom}\bigl(\Phi_4^{-1}\bigr)\bigr\}. \end{aligned}$$

Further, if there exists a positive constant b such that

$$\bigl\{\Phi_4^{-1}\bigl[\Phi_5^{-1}\bigl(\Phi_6^{-1}\bigl(U_4(t)\bigr)\bigr)\bigr]\bigr\} \leq b,$$

any solution of the control system (3.3) *with* (3.4) *is exponentially asymptotically stable.*

Proof Firstly, we can obtain the solution of (3.3) with (3.4)

$$\begin{aligned} x_\sigma(t,t_0,x_0) = X(t)X^{-1}(t_0)x_0 \\ + \int_{t_0}^{t} X(s)X^{-1}(s+1)f\bigl(s,x_\sigma\bigl(\alpha(s),t_0,x_0\bigr),\sigma\bigl(\alpha(s)\bigr)\bigr)\,ds, \end{aligned} \tag{3.14}$$

by using the variation of constants formula. Then we have that

$$\begin{aligned} \left|x_\sigma(t,t_0,x_0)\right| \leq |x_0|C\exp\bigl(-\beta(t-t_0)\bigr) + \int_{t_0}^{t} C\exp\bigl(-\beta(t-s-1)\bigr) \\ \times g_1(s)e^{-\beta s}w_1\bigl(\bigl|x_\sigma\bigl(\alpha(s),t_0,x_0\bigr)\bigr|e^{\beta s}\bigr)\bigl(1+\bigl|\sigma\bigl(\alpha(s)\bigr)\bigr|\bigr), \end{aligned} \tag{3.15}$$

by conditions (3.5) and (3.7) from (3.14). Further, by using conditions (3.6), we can obtain that

$$\begin{aligned}
&\left|x_\sigma(t,t_0,x_0)\right| \\
&\quad \le |x_0| C\exp\bigl(-\beta(t-t_0)\bigr) + \int_{t_0}^{t} C\exp\bigl(-\beta(t-1)\bigr) g_1(s) \\
&\quad \times w_1\bigl(\left|x_\sigma\bigl(\alpha(s),t_0,x_0\bigr)\right| e^{\beta\alpha(s)}\bigr) \biggl\{ 1 + \left|\theta\bigl(\alpha(s)\bigr)\right| + \int_{t_0}^{\alpha(s)} g_2\bigl(\alpha(s),\tau\bigr) \\
&\quad \times w_2\bigl(\left|x_\sigma(\tau,t_0,x_0)\right| e^{\beta\tau}\bigr) \biggl(1 + \int_{t_0}^{\tau} g_3(\tau,\xi) w_3\bigl(\left|x_\sigma(\xi,t_0,x_0)\right| e^{\beta\xi}\bigr)\,d\xi \biggr)\,d\tau \biggr\}\,ds
\end{aligned}$$

from (3.4) and (3.15). Then we have that

$$\begin{aligned}
&\left|x_\sigma(t,t_0,x_0)\right| \\
&\quad \le |x_0| C\exp\bigl(-\beta(t-t_0)\bigr) + e^{-\beta t} \int_{t_0}^{\alpha(t)} C e^{\beta} g_1\bigl(\alpha^{-1}(\eta)\bigr) \\
&\quad \times w_1\bigl(\left|x_\sigma(\eta,t_0,x_0)\right| e^{\beta\eta}\bigr) \times \biggl\{ 1 + \left|\theta(\eta)\right| + \int_{t_0}^{\eta} g_2(\eta,\tau) w_2\bigl(\left|x_\sigma(\tau,t_0,x_0)\right| e^{\beta\tau}\bigr) \\
&\quad \times \biggl(1 + \int_{t_0}^{\tau} g_3(\tau,\xi) w_3\bigl(\left|x_\sigma(\xi,t_0,x_0)\right| e^{\beta\xi}\bigr)\,d\xi \biggr)\,d\tau \biggr\} \frac{d\eta}{\alpha'(\alpha^{-1}(\eta))},
\end{aligned} \tag{3.16}$$

where we use the change $\eta = \alpha(s)$. Let $u(t) = |x_\sigma(n,t_0,x_0)| \exp(\beta t)$, (3.16) can be rewritten as

$$\begin{aligned}
u(t) &\le |x_0| C\exp(\beta t_0) + \int_{t_0}^{\alpha(t)} C e^{\beta} g_1\bigl(\alpha^{-1}(s)\bigr) w_1\bigl(u(s)\bigr) \biggl\{ 1 + \left|\theta(s)\right| \\
&\quad + \int_{t_0}^{s} g_2(s,\tau) w_2\bigl(u(\tau)\bigr) \biggl(1 + \int_{t_0}^{\tau} g_3(\tau,\xi) w_3\bigl(u(\xi)\bigr)\,d\xi \biggr)\,d\tau \biggr\} \frac{ds}{\alpha'(\alpha^{-1}(s))}.
\end{aligned} \tag{3.17}$$

Letting $a(t) = |x_0| C\exp(\beta t_0)$, we have

$$\begin{aligned}
u(t) &\le a(t) + \int_{t_0}^{\alpha(t)} f_1(s) w_1\bigl(u(s)\bigr)\,ds + \int_{t_0}^{\alpha(t)} f_1(s) w_1\bigl(u(s)\bigr) \biggl(\int_{t_0}^{s} f_2(s,\tau) w_2\bigl(u(\tau)\bigr)\,d\tau \biggr)\,ds \\
&\quad + \int_{t_0}^{\alpha(t)} f_1(s) w_1\bigl(u(s)\bigr) \\
&\quad \times \biggl[\int_{t_0}^{s} f_2(s,\tau) w_2\bigl(u(\tau)\bigr) \biggl(\int_{t_0}^{\tau} f_3(\tau,\xi) w_3\bigl(u(\xi)\bigr)\,d\xi \biggr)\,d\tau \biggr]\,ds,
\end{aligned} \tag{3.18}$$

from (3.10) and (3.17). Applying the result of Theorem 3, the inequality

$$\left|x_\sigma(t,t_0,x_0)\right| \le \exp(-\beta t) \bigl\{ \Phi_4^{-1}\bigl[\Phi_5^{-1}\bigl(\Phi_6^{-1}\bigl(U_4(t)\bigr)\bigr)\bigr]\bigr\}$$

is proved.

If there exists a positive constant b such that

$$\bigl\{ \Phi_4^{-1}\bigl[\Phi_5^{-1}\bigl(\Phi_6^{-1}\bigl(U_4(t)\bigr)\bigr)\bigr]\bigr\} \le b, \quad \forall n \in \mathbf{N},$$

then we have that

$$\left|x_\sigma(t,t_0,x_0)\right| \le b\exp(-\beta t),$$

i.e., the nonlinear control system (3.3) with (3.4) is exponentially asymptotically stable. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have contributed in all the paper parts.

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