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A new explicit triple hierarchical problem over the set of fixed points and generalized mixed equilibrium problems

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Abstract

In this article, we introduce and consider the triple hierarchical over the fixed point set of a nonexpansive mapping and the generalized mixed equilibrium problem set of an inverse-strongly monotone napping. The strong convergence of the algorithm is proved under some mild conditions. Our results generalize and improve the results of Marino and Xu and some authors.

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. Let F be a bifunction of $H \times H$ into $\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers. A mapping A be a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP$ (F, ϕ, A). If $\phi \equiv 0$, the problem (1.1) is reduced into the *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \qquad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $GEP(F, A)$. If $A \equiv 0$, the problem (1.1) is reduced into the *mixed equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $MEP(F, \phi)$. If $A \equiv 0$ and $\phi \equiv 0$, the problem (1.1) is reduced into the *equilibrium problem* [1] is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(F)$. If $F \equiv 0$ and $\phi \equiv 0$, the problem (1.1) is reduced into the *Hartmann-Stampacchia variational inequality* [2] is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $VI(C, A)$. The variational inequality has been extensively studied in the literature [3,4]. A mapping A of C into itself is called an *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping f: $C \to C$ is called a *ρ-contraction* if there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C.$$

A mapping S: $C \to C$ is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of S provided $Sx = x$. Denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty [5]. Let A and B are two monotone operators, we consider the *hierarchical problem over generalized mixed equilibrium problem:* Find a point $x^* \in GMEP(F, \phi, B)$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in GMEP(F, \varphi, B). \tag{1.6}$$

We discuss the *hierarchical problem over fixed point*: Find a point $x^* \in F(S)$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in F(S). \tag{1.7}$$

Yao et al. [6] considered the hierarchical problem over generalize equilibrium problem and the set of fixed point, $x_{s,t}$ be defined implicitly by

$$x_{s,t} = s\left[tf\left(x_{s,t}\right) + (1-t)\left(x_{s,t} - \lambda Ax_{s,t}\right)\right] + (1-s)\,T_r\left(x_{s,t} - rBx_{s,t}\right), \quad s,\ t \in (0,1), \tag{1.8}$$

for each $(s, t) \in (0, 1)^2$. The net $x_{s,t}$ hierarchically converges to the unique solution x^* of the hierarchical problem: Find a point $x^* \in GEP(F, B)$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in GEP(F, B), \tag{1.9}$$

where A and B are two monotone operators. The solution set of (1.9) is denoted by Ω.

Marino and Xu [7] studied an explicit algorithm, which generated a sequence $\{x_n\}$ recursively by the formula: For the initial guess $x_0 \in C$ is arbitrary

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)(\alpha_n V x_n + (1-\alpha_n) T x_n), \quad \forall n \geq 0, \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in (0, 1) satisfy some conditions. Let T, V: $C \to C$ are two nonexpansive self mappings and f is a contraction on C. Then $\{x_n\}$ converges strongly to a solution, which solves another variational inequality. Recently, Jitpeera and Kumam [8] introduced and studied the iterative algorithm for solving a common element of the set of solution of fixed point for a nonexpansive mapping, the set of solution of generalized mixed equilibrium problem, and the set of solution of the variational inclusion. They proved that the sequence converges strongly to a common element of the above three sets under some mild conditions.

In this article, we consider the hierarchical problem over the set of fixed point and generalized mixed equilibrium problem, which contains (1.6) and (1.7): Find a point $x^* \in \Xi := F(S) \cap GMEP(F, \phi, B)$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi := F(S) \cap GMEP(F, \varphi, B), \tag{1.11}$$

where A and B are monotone operators. This solution set of (1.11) is denoted by Υ

We present and construct a new iterative algorithm for solving the problem (1.11). The strong convergence for the proposed algorithm to the solution is derived under some assumptions. Our results generalize and improve the results of Marino and Xu [7] and some authors.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the metric (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C. We recall some lemmas which will be needed in the rest of this article.

Lemma 2.1. *The function $x \in C$ is a solution of the variational inequality (1.5) if and only if $x \in C$ satisfies the relation $x = P_C(x - \lambda Ax)$ for all $\lambda > 0$.*

Lemma 2.2. *For a given $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0$, $\forall v \in C$. It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{2.2}$$

Lemma 2.3. *There holds the following inequality in an inner product space H*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.4. [9]*Let C be a closed convex subset of a real Hilbert space H and let $S: C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x \text{ and } x_n - Sx_n \to 0$$

imply $x = Sx$.

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction F, ϕ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;
(A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.3}$$

(B2) C is a bounded set;
(B3) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B4) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

Lemma 2.5. [10]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from* $C \times C$ *to* $\mathcal{R}$*satisfying* (A1) - (A5) *and let* $\varphi : C \to \mathcal{R}$ *be a proper lower semicontinuous and convex function. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows.*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.4}$$

for all $x \in H$. *Assume that either* (B1) *or* (B2) *holds. Then, the following results hold:*

(1) *For each* $x \in H$, $T_r(x) \neq \emptyset$;
(2) T_r *is single-valued;*
(3) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$, $||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle$;
(4) $F(T_r) = MEP(F, \phi)$;
(5) $MEP(F, \phi)$ *is closed and convex.*

Lemma 2.6. [11]*Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}, \{\beta_n\} \subset (0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathcal{R}$*such that*

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) either $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}\gamma_n|\delta_n| < \infty$;
(iii) $\sum_{n=1}^{\infty}\beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Strong convergence theorems

In this section, we introduce an iterative algorithm for solving some the hierarchical problem over the set of fixed point and generalized mixed equilibrium problem.

Theorem 3.1. *Let H be a real Hilbert space, A: $C \to C$ be an α-inverse-strongly monotone, $f: C \to C$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and S, V: $C \to C$ be two nonexpansive mappings. Let B: $C \to C$ be a β-inverse-strongly monotone and F be a bifunction from $C \times C \to \mathcal{R}$ satisfying (A1)-(A5) and let $\varphi : C \to \mathcal{R}$ is convex and lower semicontinuous with either (B1) or (B2). Assume that $\Xi := F(S) \cap GMEP(F, \phi, B)$ is nonempty. Suppose $\{x_n\}$ is a sequence generated by the following algorithm with $x_0 \in C$ arbitrarily:*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)\left[\alpha_n V(I-\lambda_n A)x_n + (1-\alpha_n) ST_{r_n}(x_n - r_n Bx_n)\right], \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$ and $\lambda_n \in (0, 2\alpha)$, $r_n \in (0, 2\beta)$ satisfy the following conditions:

(C1): $\alpha_n\lambda_n < \alpha_n < \gamma\beta_n$ *for all n and some constant γ;*

(C2): $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \frac{\beta_{n-1}}{\beta_n} = 1$;

(C3): $\lim_{n\to\infty} \frac{\alpha_{n-1}}{\alpha_n} = 1$;

(C4): $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$;

(C5): $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$, which is the unique solution of the variational inequality:

$$\left\langle \left(I - f\right) x^*, x - x^*\right\rangle \geq 0, \quad \forall x \in \Upsilon. \tag{3.2}$$

Proof. We will divide the proof into five steps.

Step 1. We will show $\{x_n\}$ is bounded. Since A, B are α, β-inverse-strongly monotone mappings, we have

$$\begin{aligned} \left\|(I-\lambda_n A)x - (I-\lambda_n A)y\right\|^2 &= \left\|\left(x-y\right) - \lambda_n\left(Ax - Ay\right)\right\|^2 \\ &= \left\|x-y\right\|^2 - 2\lambda_n\left\langle x-y, Ax-Ay\right\rangle + \lambda_n^2\left\|Ax-Ay\right\|^2 \\ &\leq \left\|x-y\right\|^2 + \lambda_n(\lambda_n - 2\alpha)\left\|Ax-Ay\right\|^2 \\ &\leq \left\|x-y\right\|^2. \end{aligned} \tag{3.3}$$

By Lemma 2.5, we have $u_n = T_{r_n}(x_n - r_n Bx_n)$ for all $n \geq 0$. Then, we have

$$\begin{aligned} \left\|u_n - q\right\|^2 &\leq \left\|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}\left(q - r_n Bq\right)\right\|^2 \\ &\leq \left\|(x_n - r_n Bx_n) - \left(q - r_n Bq\right)\right\|^2 \\ &\leq \left\|x_n - q\right\|^2 + r_n(r_n - 2\beta)\left\|Bx_n - Bq\right\|^2 \\ &\leq \left\|x_n - q\right\|^2. \end{aligned}$$

For any $q \in \Xi$. Since V, $I - \lambda_n A$ and T_{r_n} are nonexpansive mappings, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\beta_n f(x_n) + (1-\beta_n)[\alpha_n V(I-\lambda_n A)x_n + (1-\alpha_n) S T_{r_n}(x_n - r_n B x_n)] - q\| \\
&\le \beta_n \|f(x_n) - q\| + (1-\beta_n)\|\alpha_n V(I-\lambda_n A)x_n + (1-\alpha_n) S T_{r_n}(x_n - r_n B x_n) - q\| \\
&\le \beta_n \|f(x_n) - f(q)\| + \beta_n \|f(q) - q\| + (1-\beta_n)\alpha_n \|V(I-\lambda_n A)x_n - q\| \\
&+ (1-\beta_n)(1-\alpha_n)\|S T_{r_n}(x_n - r_n B x_n) - q\| \\
&\le \beta_n \rho \|x_n - q\| + \beta_n \|f(q) - q\| + (1-\beta_n)\alpha_n (\|V(I-\lambda_n A)x_n - V(I-\lambda_n A)q\| \\
&+ \|V(I-\lambda_n A)q - q\|) + (1-\beta_n)(1-\alpha_n)\|T_{r_n}(x_n - r_n B x_n) - q\| \\
&\le \beta_n \rho \|x_n - q\| + \beta_n \|f(q) - q\| + (1-\beta_n)\alpha_n (\|x_n - q\| + \|Vq - q\| + \lambda_n \|VAq\|) \\
&+ (1-\beta_n)(1-\alpha_n)\|x_n - q\| \\
&= \beta_n \rho \|x_n - q\| + \beta_n \|f(q) - q\| + (1-\beta_n)\alpha_n \|x_n - q\| + (1-\beta_n)\alpha_n \|Vq - q\| \\
&+ (1-\beta_n)\alpha_n \lambda_n \|VAq\| + (1-\beta_n)(1-\alpha_n)\|x_n - q\| \\
&\le \beta_n \rho \|x_n - q\| + \beta_n \|f(q) - q\| + (1-\beta_n)\|x_n - q\| + \alpha_n \|Vq - q\| + \alpha_n \lambda_n \|VAq\| \\
&\le [1 - (1-\rho)\beta_n]\|x_n - q\| + \beta_n (\|f(q) - q\| + \gamma \|Vq - q\| + \gamma \|VAq\|).
\end{aligned}$$

By induction, it follows that

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{1}{1-\rho}(\|f(q) - q\| + \gamma\|Vq - q\| + \gamma\|VAq\|)\right\}, \quad \forall n \ge 0.$$

Therefore $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{Ax_n\}$, $\{V x_n\}$, and $\{f(x_n)\}$.

Step 2. We claim that $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$. Setting $y_n = (I - \lambda_n A)x_n$, since $I - \lambda_n A$ be nonexpansive, we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|(I-\lambda_n A)x_n - (I-\lambda_{n-1}A)x_{n-1}\| \\
&\le \|(I-\lambda_n A)x_n - (I-\lambda_n A)x_{n-1}\| \\
&+ \|(I-\lambda_n A)x_{n-1} - (I-\lambda_{n-1}A)x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\,\|Ax_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + M_1 |\lambda_n - \lambda_{n-1}|,
\end{aligned}$$

where $M_1 = \sup\{\|Ax_n\| : n \in \mathbb{N}\}$. On the other hand, from $u_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1}Bx_{n-1})$ and $u_n = T_{r_n}(x_n - r_n B x_n)$, it follows that

$$F(u_{n-1}, y) + \langle Bx_{n-1}, y - u_{n-1}\rangle + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C \tag{3.4}$$

and

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.5}$$

Substituting $y = u_n$ into (3.4) and $y = u_{n-1}$ into (3.5), we have

$$F(u_{n-1}, u_n) + \langle Bx_{n-1}, u_n - u_{n-1}\rangle + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0$$

and

$$F(u_n, u_{n-1}) + \langle Bx_n, u_{n-1} - u_n\rangle + \varphi(u_{n-1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \ge 0.$$

From (A2), we have

$$\left\langle u_n - u_{n-1}, Bx_{n-1} - Bx_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \ge 0,$$

and then

$$\left\langle u_n - u_{n-1}, r_{n-1}\left(Bx_{n-1} - Bx_n\right) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}\left(u_n - x_n\right)\right\rangle \geq 0,$$

so

$$\left\langle u_n - u_{n-1}, r_{n-1}Bx_{n-1} - r_{n-1}Bx_n + u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}\left(u_n - x_n\right)\right\rangle \geq 0.$$

It follows that

$$\left\langle u_n - u_{n-1}, \left(I - r_{n-1}B\right)x_n - \left(I - r_{n-1}B\right)x_{n-1} + u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}\left(u_n - x_n\right)\right\rangle \geq 0,$$

$$\left\langle u_n - u_{n-1}, u_{n-1} - u_n\right\rangle + \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)\left(u_n - x_n\right)\right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &\leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)\left(u_n - x_n\right)\right\rangle \\ &\leq \|u_n - u_{n-1}\|\left\{\|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - x_n\|\right\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n}\left|r_n - r_{n-1}\right|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_2}{c}\left|r_n - r_{n-1}\right|,\end{aligned} \tag{3.6}$$

where $M_2 = \sup\left\{\|u_n - x_n\| : n \in \mathbb{N}\right\}$. From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|\beta_n f\left(x_n\right) + \left(1 - \beta_n\right)\left[\alpha_n V y_n + \left(1 - \alpha_n\right)Su_n\right]\right. \\ &\quad \left. - \beta_{n-1}f\left(x_{n-1}\right) - \left(1 - \beta_{n-1}\right)\left[\alpha_{n-1}Vy_{n-1} + \left(1 - \alpha_{n-1}\right)Su_{n-1}\right]\right\| \\ &\leq \beta_n\rho\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|f\left(x_{n-1}\right)\right\| + \left\|\left(1 - \beta_n\right)\left[\alpha_n V y_n + \left(1 - \alpha_n\right)Su_n\right]\right. \\ &\quad \left. - \left(1 - \beta_{n-1}\right)\left[\alpha_{n-1}Vy_{n-1} + \left(1 - \alpha_{n-1}\right)Su_{n-1}\right]\right\| \\ &= \beta_n\rho\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|f\left(x_n - 1\right)\right\| \\ &\quad + \left\|\left(1 - \beta_n\right)\left[\alpha_n V y_n - \alpha_n V y_{n-1} + \left(1 - \alpha_n\right)Su_n - \left(1 - \alpha_n\right)Su_{n-1}\right]\right. \\ &\quad + \left(1 - \beta_n\right)\alpha_n V y_{n-1} - \left(1 - \beta_{n-1}\right)\alpha_{n-1}Vy_{n-1} \\ &\quad \left. + \left(1 - \beta_n\right)\left(1 - \alpha_n\right)Su_{n-1} - \left(1 - \beta_{n-1}\right)\left(1 - \alpha_{n-1}\right)Su_{n-1}\right\| \\ &\leq \beta_n\rho\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|f\left(x_{n-1}\right)\right\| \\ &\quad + \left(1 - \beta_n\right)\left\|\alpha_n\left[Vy_n - Vy_{n-1}\right] + \left(1 - \alpha_n\right)\left[su_n - Su_{n-1}\right]\right\| \\ &\quad + \left\|\left(\alpha_n - \beta_n\alpha_n - \alpha_{n-1} + \beta_{n-1}\alpha_{n-1}\right)Vy_{n-1}\right. \\ &\quad \left. + \left(1 - \beta_n - \alpha_n + \beta_n\alpha_n - 1 + \beta_{n-1} + \alpha_{n-1} - \beta_{n-1}\alpha_{n-1}\right)Su_{n-1}\right\| \\ &\leq \beta_n\rho\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left\|f\left(x_{n-1}\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&+ (1-\beta_n)\alpha_n \left\|y_n - y_{n-1}\right\| + (1-\beta_n)(1-\alpha_n)\|u_n - u_{n-1}\| \\
&+ \left\|(\alpha_n - \alpha_{n-1} - \beta_n\alpha_n + \beta_n\alpha_{n-1} - \beta_n\alpha_{n-1} + \beta_{n-1}\alpha_{n-1}) V y_{n-1}\right. \\
&\left. + (-\beta_n + \beta_{n-1} - \alpha_n + \alpha_{n-1} + \beta_n\alpha_n - \beta_n\alpha_{n-1} + \beta_n\alpha_{n-1} - \beta_{n-1}\alpha_{n-1}) S u_{n-1}\right\| \\
\le\, & \beta_n\rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \left\|f(x_{n-1})\right\| \\
&+ (1-\beta_n)\alpha_n \{\|x_n - x_{n-1}\| + M_1 |\lambda_n - \lambda_{n-1}|\} \\
&+ (1-\beta_n)(1-\alpha_n)\left\{\|x_n - x_{n-1}\| + \frac{M_2}{c}|r_n - r_{n-1}|\right\} \\
&+ \left\|\left[(\alpha_n - \alpha_{n-1}) - \beta_n(\alpha_n - \alpha_{n-1}) - (\beta_n - \beta_{n-1})\alpha_{n-1}\right] V y_{n-1}\right. \\
&\left. + \left[-(\beta_n - \beta_{n-1}) - (\alpha_n - \alpha_{n-1}) + \beta_n(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})\alpha_{n-1}\right] S u_{n-1}\right\| \\
=\, & \beta_n\rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \left\|f(x_{n-1})\right\| \\
&+ (1-\beta_n)\alpha_n \|x_n - x_{n-1}\| + (1-\beta_n)\alpha_n M_1 |\lambda_n - \lambda_{n-1}| \\
&+ (1-\beta_n)(1-\alpha_n)\|x_n - x_{n-1}\| + (1-\beta_n)(1-\alpha_n)\frac{M_2}{c}|r_n - r_{n-1}| \\
&+ \left\|\left[(1-\beta_n)(\alpha_n - \alpha_{n-1}) - (\beta_n - \beta_{n-1})\alpha_{n-1}\right] V y_{n-1}\right. \\
&\left. + \left[(\beta_n - 1)(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})(\alpha_{n-1} - 1)\right] S u_{n-1}\right\| \\
\le\, & \beta_n\rho \|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}| \left\|f(x_{n-1})\right\| + (1-\beta_n)\alpha_n M_1 |\lambda_n - \lambda_{n-1}| \\
&+ (1-\beta_n)(1-\alpha_n)\frac{M_2}{c}|r_n - r_{n-1}| \\
&+ (1-\beta_n)(\alpha_n - \alpha_{n-1})\left\|V y_{n-1} - S u_{n-1}\right\| \\
&+ \left\|(\beta_n - \beta_{n-1})(\alpha_{n-1} - 1) S u_{n-1} - (\beta_n - \beta_{n-1})\alpha_{n-1} V y_{n-1}\right\|
\end{aligned}
$$

$$
\begin{aligned}
=\, & \beta_n\rho \|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&+ (1-\beta_n)\alpha_n M_1 |\lambda_n - \lambda_{n-1}| + (1-\beta_n)(1-\alpha_n)\frac{M_2}{c}|r_n - r_{n-1}| \\
&+ (1-\beta_n)(\alpha_n - \alpha_{n-1})\left\|V y_{n-1} - S u_{n-1}\right\| \\
&+ |\beta_n - \beta_{n-1}| \left(\left\|f(x_{n-1})\right\| + \alpha_{n-1}\left\|V y_{n-1}\right\| + |1 - \alpha_{n-1}| \|S u_{n-1}\|\right) \\
\le\, & \beta_n\rho \|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| + \alpha_n M_1 |\lambda_n - \lambda_{n-1}| \\
&+ \frac{M_2}{c}|r_n - r_{n-1}| + (\alpha_n - \alpha_{n-1})\left\|V y_{n-1} - S u_{n-1}\right\| \\
&+ |\beta_n - \beta_{n-1}| \left(\left\|f(x_{n-1})\right\| + \alpha_{n-1}\left\|V y_{n-1}\right\| + |1 - \alpha_{n-1}| \|S u_{n-1}\|\right) \\
\le\, & [1 - (1-\rho)\beta_n]\|x_n - x_{n-1}\| + \alpha_n M_1 |\lambda_n - \lambda_{n-1}| + \frac{M_2}{c}|r_n - r_{n-1}| \\
&+ (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) M_3 \\
=\, & [1 - (1-\rho)\beta_n]\|x_n - x_{n-1}\| + \alpha_n M_1 |\lambda_n - \lambda_{n-1}| + \frac{M_2}{c}|r_n - r_{n-1}| \\
&+ \left(\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\right)\beta_n M_3 \\
\le\, & [1 - (1-\rho)\beta_n]\|x_n - x_{n-1}\| + \alpha_n M_1 |\lambda_n - \lambda_{n-1}| + \frac{M_2}{c}|r_n - r_{n-1}| \\
&+ \left(\frac{\gamma\,|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\right)\beta_n M_3,
\end{aligned}
$$

where $M_3 = \sup\{\max\{||Vy_{n-1}||, ||Su_{n-1}||, ||f(x_{n-1})||\}\}$. Since conditions (C1)-(C5) by Lemma 2.6, we have $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$.

Step 3. We claim that $\lim_{n\to\infty}||x_n - Sx_n|| = 0$. For each $q \in \Xi$, we note that since T_{r_n} is firmly nonexpansive, then we have

$$\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(q - r_nBq)\|^2 \\
&\le \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(q - r_nBq), u_n - q\rangle \\
&= \langle (x_n - r_nBx_n) - (q - r_nBq), u_n - q\rangle \\
&= \frac{1}{2}\Big\{\|(x_n - r_nBx_n) - (q - r_nBq)\|^2 + \|u_n - q\|^2 \\
&\quad - \|(x_n - r_nBx_n) - (q - r_nBq) - (u_n - q)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n - r_n(Bx_n - Bq)\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2r_n\langle x_n - u_n, Bx_n - Bq\rangle - r_n^2\|Bx_n - Bq\|^2\Big\},
\end{aligned}$$

which imply that

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\,\|Bx_n - Bq\|. \tag{3.7}$$

From (3.1) and set $w_n := \alpha_n V(I - \lambda_n A)x_n + (1 - \alpha_n)Su_n$, when $y_n = (I - \lambda_n A)x_n$, then we have

$$\begin{aligned}
\|w_n - q\|^2 &= \|\alpha_n Vy_n + (1 - \alpha_n)Su_n - q\|^2 \\
&= \|\alpha_n Vy_n + (1 - \alpha_n)Su_n - (1 - \alpha_n)Sq + (1 - \alpha_n)Sq - q\|^2 \\
&= \|\alpha_n(Vy_n - Sq) + (1 - \alpha_n)(Su_n - Sq) + Sq - q\|^2 \\
&\le \alpha_n\|Vy_n - Sq\|^2 + (1 - \alpha_n)\|u_n - q\|^2.
\end{aligned} \tag{3.8}$$

On the other hand, we note that

$$\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(q - r_nBq)\|^2 \\
&\le \|(x_n - r_nBx_n) - (q - r_nBq)\|^2 \\
&= \|(x_n - q) - r_n(Bx_n - Bq)\|^2 \\
&\le \|x_n - q\|^2 - 2r_n\langle x_n - q, Bx_n - Bq\rangle + r_n^2\|Bx_n - Bq\|^2 \\
&\le \|x_n - q\|^2 - 2r_n\beta\|Bx_n - Bq\|^2 + r_n^2\|Bx_n - Bq\|^2.
\end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), we note that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\beta_n f(x_n) + (1 - \beta_n)w_n - q\|^2 \\
&\le \beta_n\|f(x_n) - q\|^2 + (1 - \beta_n)\|w_n - q\|^2 \\
&\le \beta_n\rho^2\|x_n - q\|^2 + (1 - \beta_n)\Big\{\alpha_n\|Vy_n - Sq\|^2 + (1 - \alpha_n)\|u_n - q\|^2\Big\} \\
&\le \beta_n\rho\|x_n - q\|^2 + (1 - \beta_n)\alpha_n\|Vy_n - Sq\|^2 + (1 - \beta_n)(1 - \alpha_n)\|u_n - q\|^2 \\
&\le \beta_n\rho\|x_n - q\|^2 + \alpha_n\|Vy_n - Sq\|^2 + (1 - \beta_n)(1 - \alpha_n) \\
&\quad \times\Big\{\|x_n - q\|^2 - 2r_n\beta\|Bx_n - Bq\|^2 + r_n^2\|Bx_n - Bq\|^2\Big\} \\
&= \beta_n\rho\|x_n - q\|^2 + \alpha_n\|Vy_n - Sq\|^2 + (1 - \beta_n)(1 - \alpha_n) \\
&\quad \times\Big\{\|x_n - q\|^2 - r_n(r_n - 2\beta)\|Bx_n - Bq\|^2\Big\} \\
&= \beta_n\rho\|x_n - q\|^2 + \alpha_n\|Vy_n - Sq\|^2 + (1 - \beta_n)(1 - \alpha_n)\|x_n - q\|^2 \\
&\quad + (1 - \beta_n)(1 - \alpha_n)r_n(r_n - 2\beta)\|Bx_n - Bq\|^2 \\
&\le \beta_n\rho\|x_n - q\|^2 + \alpha_n\|Vy_n - Sq\|^2 + (1 - \beta_n)\|x_n - q\|^2 \\
&\quad + (1 - \beta_n)(1 - \alpha_n)r_n(r_n - 2\beta)\|Bx_n - Bq\|^2 \\
&\le [1 - (1 - \rho)\beta_n]\|x_n - q\|^2 + \gamma\beta_n\|Vy_n - Sq\|^2 \\
&\quad + (1 - \beta_n)(1 - \alpha_n)r_n(r_n - 2\beta)\|Bx_n - Bq\|^2.
\end{aligned} \tag{3.10}$$

Then, we have

$$\begin{aligned}(1-\beta_n)(1-\alpha_n)c(2\beta-d)\|Bx_n-Bq\|^2 &\le \gamma\beta_n\|V\gamma_n-Sq\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\le \gamma\beta_n\|V\gamma_n-Sq\|^2\\ &\quad+\|x_n-x_{n+1}\|\left(\|x_n-q\|+\|x_{n+1}-q\|\right).\end{aligned}$$

From (C2), $\{r_n\} \subset [c, d] \subset (0, 2\beta)$ and $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$, we obtain

$$\lim_{n\to\infty}\|Bx_n-Bq\|=0. \tag{3.11}$$

Using (3.7), (3.8) and (3.10), it follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \beta_n\rho\|x_n-q\|^2+(1-\beta_n)\alpha_n\|V\gamma_n-Sq\|^2+(1-\beta_n)(1-\alpha_n)\|u_n-q\|^2\\ &\le \beta_n\rho\|x_n-q\|^2+\alpha_n\|V\gamma_n-Sq\|^2+(1-\beta_n)(1-\alpha_n)\\ &\quad\times\left\{\|x_n-q\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Bx_n-Bq\|\right\}\\ &= \beta_n\rho\|x_n-q\|^2+\alpha_n\|V\gamma_n-Sq\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-q\|^2\\ &\quad-(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2+2(1-\beta_n)(1-\alpha_n)r_n\|x_n-u_n\|\|Bx_n-Bq\|\\ &\le \beta_n\rho\|x_n-q\|^2+\alpha_n\|V\gamma_n-Sq\|^2+(1-\beta_n)\|x_n-q\|^2\\ &\quad-(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Bx_n-Bq\|\\ &\le [1-(1-\rho)\beta_n]\|x_n-q\|^2+\gamma\beta_n\|V\gamma_n-Sq\|^2\\ &\quad-(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Bx_n-Bq\|.\end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned}(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2 &\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\gamma\beta_n\|V\gamma_n-Sq\|^2\\ &\quad+2r_n\|x_n-u_n\|\|Bx_n-B_q\|\\ &\le \|x_n-x_{n+1}\|\left(\|x_n-q\|+\|x_{n+1}-q\|\right)+\gamma\beta_n\|V\gamma_n-Sq\|^2\\ &\quad+2r_n\|x_n-u_n\|\|Bx_n-Bq\|.\end{aligned}$$

From (C1), (C2), (3.13) and $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.13}$$

By (C5), we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n-u_n}{r_n}\right\|=\lim_{n\to\infty}\frac{1}{r_n}\|x_n-u_n\|=0. \tag{3.14}$$

From (3.1), it follows that

$$\begin{aligned}\|x_{n+1}-Su_n\| &= \left\|\beta_nf(x_n)+(1-\beta_n)\left[\alpha_nV\gamma_n+(1-\alpha_n)Su_n\right]-Su_n\right\|\\ &= \|\beta_nf(x_n)+(1-\beta_n)\alpha_nV\gamma_n+(1-\beta_n)(1-\alpha_n)Su_n-Su_n\|\\ &= \|\beta_nf(x_n)+(1-\beta_n)\alpha_nV\gamma_n+(1-\beta_n)Su_n+(1-\beta_n)\alpha_nSu_n-Su_n\|\\ &\le \beta_n\|f(x_n)-Su_n\|+(1-\beta_n)\alpha_n\|V\gamma_n-Su_n\|.\end{aligned} \tag{3.15}$$

By (C1) and (C2), then we get

$$\lim_{n\to\infty}\|x_{n+1}-Su_n\|=0. \tag{3.16}$$

Since

$$\begin{aligned}\|x_n-Sx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-Su_n\|+\|Su_n-Sx_n\|\\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-Su_n\|+\|u_n-x_n\|.\end{aligned}$$

By $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$, (3.13) and (3.16), so we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.17}$$

Step 4. Next, we will show that

$$\limsup_{n\to\infty} \left\langle \left(I - f\right) x^*, x_n - x^* \right\rangle \le 0.$$

Indeed, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \left\langle \left(I - f\right) x^*, x_n - x^* \right\rangle = \limsup_{n\to\infty} \left\langle \left(I - f\right) x^*, x_{n_i} - x^* \right\rangle. \tag{3.18}$$

Since ?82? is bounded, there exists a subsequence $\left\{x_{n_{i_j}}\right\}$ of ?82? which converge weakly to $z \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$. From $||x_n - Sx_n|| \to 0$, we obtain $Sx_{n_i} \rightharpoonup z$. Now, we will show that $z \in \Xi := F(S) \cap GMEP(F, \phi, B)$. Let us show $z \in F(S)$. Assume that $z \notin F(S)$. Since ?85? and $Sz \neq z$. By the Opial's condition, we obtain

$$\begin{aligned}
\liminf_{i\to\infty} \left\|x_{n_i} - z\right\| &< \liminf_{i\to\infty} \left\|x_{n_i} - Sz\right\| \\
&= \liminf_{i\to\infty} \left\|x_{n_i} - Sx_{n_i} + Sx_{n_i} - Sz\right\| \\
&\le \liminf_{i\to\infty} \left(\left\|x_{n_i} - Sx_{n_i}\right\| + \left\|Sx_{n_i} - Sz\right\|\right) \\
&= \liminf_{i\to\infty} \left\|Sx_{n_i} - Sz\right\| \\
&\le \liminf_{i\to\infty} \left\|x_{n_i} - z\right\|.
\end{aligned}$$

This is a contradiction. Thus, we have $z \in F(S)$.

Next, we will show that $z \in GMEP(F, \phi, B)$. Since $u_n = T_{r_n}(x_n - r_nBx_n)$, we have

$$F\left(u_n, y\right) + \left\langle Bx_n, y - u_n\right\rangle + \varphi\left(y\right) - \varphi\left(u_n\right) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\left\langle Bx_n, y - u_n\right\rangle + \varphi\left(y\right) - \varphi\left(u_n\right) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge F\left(y, u_n\right), \quad \forall y \in C.$$

and hence

$$\left\langle Bx_{n_i}, y - u_{n_i}\right\rangle + \varphi\left(y\right) - \varphi\left(u_{n_i}\right) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F\left(y, u_{n_i}\right), \quad \forall y \in C. \tag{3.19}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1 - t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$. So, from (3.19), we have

$$\begin{aligned}
\left\langle y_t - u_{n_i}, By_t\right\rangle &\ge \left\langle y_t - u_{n_i}, By_t\right\rangle - \varphi\left(y_t\right) + \varphi\left(u_{n_i}\right) - \left\langle y_t - u_{n_i}, Bx_{n_i}\right\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \\
&\quad + F\left(y_t, u_{n_i}\right) \\
&= \left\langle y_t - u_{n_i}, By_t - Bu_{n_i}\right\rangle + \left\langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\right\rangle - \varphi\left(y_t\right) + \varphi\left(u_{n_i}\right) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F\left(y_t, u_{n_i}\right).
\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weak lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup z$, we have at the limit

$$\langle y_t - z, By_t\rangle \geq -\varphi(y_t) + \varphi(z) + F(y_t, z) \tag{3.20}$$

as $i \to \infty$. From (A1), (A4), and (3.20), we also get

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, z) + t\varphi(y) - (1-t)\varphi(z) - \varphi(y_t) \\ &= t\left[F(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)\left[F(y_t, z) + \varphi(z) - \varphi(y_t)\right] \\ &\leq t\left[F(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)\langle y_t - z, By_t\rangle \\ &= t\left[F(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)t\langle y - z, By_t\rangle, \\ 0 &\leq F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - z, By_t\rangle. \end{aligned}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle \geq 0.$$

This implies that $z \in GMEP(F, \phi, B)$. Therefore $x^* \in \Xi$. It is easy to see that $P_\Upsilon(I - f)(x^*)$ is a contraction of H into itself. Hence H is complete, there exists a unique fixed point $x^* \in H$, such that $x^* = P_\Upsilon(I - f)(x^*)$. Since $x^* = P_\Upsilon(I - f)(x^*)$, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle (I-f)x^*, x_n - x^*\rangle &= \limsup_{n\to\infty} \langle (I-f)x^*, Sx_n - x^*\rangle \\ &= \limsup_{n\to\infty} \langle (I-f)x^*, Sx_{n_i} - x^*\rangle \\ &= \langle (I-f)x^*, z - x^*\rangle \leq 0. \end{aligned} \tag{3.21}$$

Step 5. Last, we will prove $x_n \to x^* \in \Upsilon$. It follows from (3.1) that, we compute

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)\left[\alpha_n V(I - \lambda_n A)x_n + (1-\alpha_n)ST_{r_n}(x_n - r_nBx_n)\right] - x^*\|^2 \\ &= \|\beta_n\left[f(x_n) - f(x^*)\right] + (1-\beta_n)\{\alpha_n\left[V(I-\lambda_n A)x_n - V(I-\lambda_n A)x^*\right] \\ &\quad + (1-\alpha_n)\left[ST_{r_n}(x_n - r_nBx_n) - x^*\right]\} + \beta_n\left[f(x^*) - x^*\right] \\ &\quad + (1-\beta_n)\alpha_n\left[V(I-\lambda_n A)x^* - x^*\right]\|^2 \\ &\leq \|\beta_n\left[f(x_n) - f(x^*)\right] + (1-\beta_n)\{\alpha_n\left[V(I-\lambda_n A)x_n - V(I-\lambda_n A)x^*\right] \\ &\quad + (1-\alpha_n)\left[ST_{r_n}(x_n - r_nBx_n) - x^*\right]\}\|^2 \\ &\quad + 2\langle\beta_n\left[f(x^*) - x^*\right] + (1-\beta_n)\alpha_n\left[V(I-\lambda_n A)x^* - x^*\right], x_{n+1} - x^*\rangle \\ &\leq \beta_n\|f(x_n) - f(x^*)\|^2 + (1-\beta_n)\|\alpha_n\left[V(I-\lambda_n A)x_n - V(I-\lambda_n A)x^*\right] \\ &\quad + (1-\alpha_n)\left[ST_{r_n}(x_n - r_nBx_n) - x^*\right]\|^2 \\ &\quad + 2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + 2(1-\beta_n)\alpha_n\langle V(I-\lambda_n A)x^* - x^*, x_{n+1} - x^*\rangle \\ &\leq \beta_n\rho^2\|x_n - x^*\|^2 + (1-\beta_n)\left(\alpha_n\|(I-\lambda_n A)x_n - (I-\lambda_n A)x^*\|\right. \\ &\quad \left. + (1-\alpha_n)\|T_{r_n}(x_n - r_nBx_n) - x^*\|\right)^2 \\ &\quad + 2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + 2(1-\beta_n)\alpha_n\langle Vx^* - x^*, x_{n+1} - x^*\rangle \\ &\quad - 2(1-\beta_n)\alpha_n\lambda_n\langle VAx^*, x_{n+1} - x^*\rangle \\ &\leq \beta_n\rho^2\|x_n - x^*\|^2 \\ &\quad + (1-\beta_n)\left(\alpha_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|\right)^2 \\ &\quad + 2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + 2(1-\beta_n)\alpha_n\langle Vx^* - x^*, x_{n+1} - x^*\rangle \\ &\quad - 2(1-\beta_n)\alpha_n\lambda_n\langle VAx^*, x_{n+1} - x^*\rangle \\ &\leq \beta_n\rho^2\|x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\ &\quad + 2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + 2(1-\beta_n)\alpha_n\|Vx^* - x^*\|\|x_{n+1} - x^*\| \\ &= \left[1 - (1-\rho^2)\beta_n\right]\|x_n - x^*\|^2 \\ &\quad + 2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle + 2(1-\beta_n)\gamma\beta_n\|Vx^* - x^*\|\|x_{n+1} - x^*\|. \end{aligned}$$

Setting

$$\delta_n = \frac{1}{1-\rho^2}\left\{2\left\langle f\left(x^*\right) - x^*, x_{n+1} - x^*\right\rangle + 2\left(1-\beta_n\right)\gamma \left\|Vx^* - x^*\right\| \left\|x_{n+1} - x^*\right\|\right\}.$$

By (3.18), the fact that $\limsup_{n\to\infty} \delta_n \le 0$. Therefore, by Lemma 2.6, we conclude that $x_n \to x^*$, as $n \to \infty$. This complete the proof.

Next, the following example shows that all conditions of Theorem 3.1 are satisfied.

Example 3.2. For instance, let $\alpha_n = \frac{n+1}{n^2+1}$,$\beta_n = \frac{1}{n}$,$\lambda_n = \frac{1}{2(n+1)}$ and $r_n = \frac{n}{n+1}$. Then, the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ satisfy the following condition (C1)

$$\frac{n+1}{n^2+1}\cdot\frac{1}{2(n+1)} < \frac{n+1}{n^2+1} < \gamma\frac{1}{n}.$$

We will show that the condition (C2) is achieves. Indeed, we obtain that

$$\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\frac{1}{n} = 0,$$

$$\sum_{n=1}^{\infty}\beta_n = \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{\beta_n-1}{\beta_n} &= \lim_{n\to\infty}\frac{\frac{1}{n-1}}{\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{n}{n-1}\\ &= 1.\end{aligned}$$

Next, we will show that the condition (C3) is achieves. Indeed, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{\alpha_n-1}{\alpha_n} &= \lim_{n\to\infty}\frac{\frac{(n+1)-1}{(n-1)^2+1}}{\frac{n+1}{n^2+1}}\\ &= \lim_{n\to\infty}\frac{\frac{n}{n^2-2n+1+1}}{\frac{n+1}{n^2+1}}\\ &= \lim_{n\to\infty}\frac{n}{n^2-2n+2}\cdot\frac{n^2+1}{n+1}\\ &= 1.\end{aligned}$$

Next, we will show that the condition (C4) is achieves. We observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\lambda_n - \lambda_{n-1}\right| &= \sum_{n=1}^{\infty}\left|\frac{1}{2(n+1)} - \frac{1}{2n}\right|\\ &= \left|\frac{1}{2.2}-\frac{1}{2.1}\right| + \left|\frac{1}{2.3}-\frac{1}{2.2}\right| + \left|\frac{1}{2.4}-\frac{1}{2.3}\right| + \dots\\ &= \frac{1}{2}.\end{aligned}$$

Then, the sequence $\{\lambda_n\}$ satisfy the condition (C4).

Finally, we will show that the condition (C5) is achieves. We compute

$$\begin{aligned}\sum_{n=1}^{\infty}|r_n - r_{n-1}| &= \sum_{n=1}^{\infty}\left|\frac{n}{n+1} - \frac{n-1}{(n-1)+1}\right| \\ &= \sum_{n=1}^{\infty}\left|\frac{n(n) - (n-1)(n+1)}{(n+1)n}\right| \\ &= \sum_{n=1}^{\infty}\left|\frac{n^2 - n^2 + 1}{(n+1)n}\right| \\ &= \sum_{n=1}^{\infty}\left|\frac{1}{n(n+1)}\right|\end{aligned}$$

and

$$\liminf_{n\to\infty} r_n = \liminf_{n\to\infty}\frac{n}{n+1} = 1.$$

Then, the sequence $\{r_n\}$ satisfy the condition (C5).

Corollary 3.3. *Let H be a real Hilbert space,* $f: C \to C$ *be a* ρ*-contraction with coefficient* $\rho \in [0, 1)$ *and* $S, V: C \to C$ *be two nonexpansive mappings. Let F be a bifunction from* $C \times C \to \mathcal{R}$ *satisfying (A1)-(A5) and let* $\varphi : C \to \mathcal{R}$ *is convex and lower semicontinuous with either (B1) or (B2). Assume that* $F(S) \cap MEP(F, \phi)$ *is nonempty. Suppose* $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)\left[\alpha_n V x_n + (1-\alpha_n) S T_{r_n} x_n\right], \tag{3.22}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0, 1)$ *and* $r_n \in (0, 2\beta)$ *satisfy the conditions (C1)-(C3) and (C5). Then* $\{x_n\}$ *converges strongly to* $x^* \in F(S) \cap MEP(F, \phi)$*, which is the unique solution of the variational inequality:*

$$\left\langle \left(I - f\right) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(S) \cap MEP(F, \varphi). \tag{3.23}$$

The solution of (3.23) *is denoted by* Δ*. This algorithm strongly converge to* $x^* \in \Delta$.

Proof. Putting A, $B \equiv 0$ in Theorem 3.1, we can obtain desired conclusion immediately.

Corollary 3.4. *Let H be a real Hilbert space,* $f: C \to C$ *be a* ρ*-contraction with coefficient* $\rho \in [0, 1)$ *and* $S, V: C \to C$ *be two nonexpansive mappings. Let* $A: C \to C$ *be an* α*-inverse-strongly monotone. Assume that* $F(S)$ *is nonempty. Suppose* $\{x_n\}$ *is a sequence generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)\left[\alpha_n V (I - \lambda_n A) x_n + (1-\alpha_n) S x_n\right], \tag{3.24}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0, 1)$ *and* $\lambda_n \in (0, 2\alpha)$ *satisfy the conditions (C1)-(C4).*

Then $\{x_n\}$ *converges strongly to* $x^* \in F(S)$*, which is the unique solution of the variational inequality:*

$$\left\langle \left(I - f\right) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(S). \tag{3.25}$$

The solution of (3.25) *is denoted by* Γ*. This algorithm strongly converge to* $x^* \in \Gamma$.

Proof. Putting $B \equiv 0$ and $T_{r_n} \equiv I$ in Theorem 3.1, we can obtain desired conclusion immediately.

Corollary 3.5. *Let H be a real Hilbert space,* $f: C \to C$ *be a* ρ*-contraction with coefficient* $\rho \in [0, 1)$ *and* $S, V: C \to C$ *be two nonexpansive mappings. Assume that* $F(S) \neq \emptyset$*. Suppose* $\{x_n\}$ *is a sequences generated by the following algorithm* $x_0 \in C$ *arbitrarily:*

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)[\alpha_n V x_n + (1 - \alpha_n) S x_n], \tag{3.26}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0, 1)$ *satisfy the conditions (C1)-(C3).*

Then $\{x_n\}$ *converges strongly to* $x^* \in F(S)$, *which is the unique solution of the variational inequality:*

$$\left\langle (I - f) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(S). \tag{3.27}$$

The solution of (3.27) *is denoted by* Γ'. *This algorithm strongly converge to* $x^* \in \Gamma'$.

Proof. Putting $A, B \equiv 0$ and $T_{r_n} \equiv I$ in Theorem 3.1, we can obtain desired conclusion immediately.

Remark 3.6. Corollary 3.5 generalizes and improves the result of Marino and Xu[7, Theorem 3.1].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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