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Research Article

Six-Point Subdivision Schemes with Cubic Precision

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This paper presents 6-point subdivision schemes with cubic precision. We first derive a relation between the 4-point interpolatory subdivision and the quintic *B*-spline refinement. By using the relation, we further propose the counterparts of cubic and quintic *B*-spline refinements based on 6-point interpolatory subdivision schemes. It is proved that the new family of 6-point combined subdivision schemes has higher smoothness and better polynomial reproduction property than the *B*-spline counterparts. It is also showed that, both having cubic precision, the well-known Hormann-Sabin's family increase the degree of polynomial generation and smoothness in exchange of the increase of the support width, while the new family can keep the support width unchanged and maintain higher degree of polynomial generation and smoothness.

1. Introduction

Subdivision is an efficient method for generating curves and surfaces in computer aided geometric design. In general, subdivision schemes can be divided into two categories: interpolatory schemes and approximating schemes. Interpolatory schemes get better shape control while approximating schemes have better smoothness. The most well-known interpolatory subdivision scheme is the classical 4-point binary scheme proposed by Dyn et al. [1]. In 1989, it was extended to the 6-point binary interpolatory scheme by Weissman [2]. Most approximating schemes were developed from splines. Two of the most famous approximating schemes are Chaikin's algorithm [3] and cubic *B*-spline refinement algorithm [4], which actually generate uniform quadratic and cubic *B*-spline curves with C^1 continuity and C^2 continuity, respectively.

The deep connection between interpolatory schemes and approximating schemes has been studied in many literatures [5–15]. In 2001, Maillot and Stam [5] introduced a push-back operation which is applied at each round of approximating refinement to progressively interpolate the control vertices. In 2007, Li and Ma [6] observed a relation between 4-point interpolatory subdivision and cubic *B*-spline curve refinement, and, motivated by this relation, they proposed a universal method for constructing interpolatory subdivision through the addition of weighted averaging operations to the mask of approximating subdivision. In 2008, Lin et al. [7] found another relation between 4-point interpolatory subdivision and cubic *B*-spline refinement and constructed interpolatory subdivision from approximating subdivision based on the relation. The deep connection between interpolatory and approximating schemes was also studied in [8–12] which exploited the generating functions of approximating subdivision and interpolatory subdivision. In 2012, Pan et al. [13] provided a combined ternary approximating and interpolatory subdivision scheme with C^2 continuity. Li and Zheng [14] constructed interpolatory subdivision from primal approximating subdivision with a new observation of the link between interpolatory and approximating subdivision. In 2013, Luo and Qi [15] made some theoretical analysis from the generation polynomial perspective and constructed some new interpolatory schemes from approximating schemes.

Our work is motivated by a new observation about the 4-point interpolatory subdivision and the quintic *B*-spline curve refinement. The observation gives us heuristics to construct combined subdivision schemes from existing subdivision schemes. The idea is to construct the counterparts of cubic and quintic *B*-spline refinements and make the relations between the 6-point interpolatory subdivision and the counterparts of cubic and quintic *B*-spline refinements similar to those between the 4-point interpolatory subdivision and the cubic, quintic B-spline curve refinements. Since the 6-point interpolatory subdivision from which the new subdivision scheme is deduced has good properties such as high smoothness and high accuracy, we are interested in studying which properties of the new subdivision scheme are better than their counterparts.

The new family of 6-point combined subdivision schemes is defined as follows:

$$\begin{aligned} P_{2i}^{k+1} &= \frac{1}{256}\left(\alpha\left(P_{i-2}^{k}+P_{i+2}^{k}\right)+\beta\left(P_{i-1}^{k}+P_{i+1}^{k}\right)\right. \\ &\quad \left.+\left(256-2\alpha-2\beta\right)P_{i}^{k}\right), \\ P_{2i+1}^{k+1} &= \frac{1}{256}\left(\left(3+\frac{\alpha}{2}\right)\left(P_{i-2}^{k}+P_{i+3}^{k}\right)\right. \\ &\quad +\left(\frac{\alpha+\beta}{2}-25\right)\left(P_{i-1}^{k}+P_{i+2}^{k}\right) \\ &\quad \left.+\left(150-\alpha-\frac{\beta}{2}\right)\left(P_{i}^{k}+P_{i+1}^{k}\right)\right). \end{aligned} \tag{1}$$

(1) is called the 6-point combined interpolatory and approximating binary subdivision scheme. If $\alpha = \beta = 0$, (1) generates 6-point interpolatory subdivision; otherwise, (1) produces approximating subdivision. It is proved that when suitably setting the tension parameter, all schemes from (1) are able to generate curves with C^4 continuity and reproduce cubic polynomials, whereas the B-spline refinements attain only linear precision. Moreover, we also make a comparison of properties between our family and famous Hormann-Sabin's family [16] which has the same cubic precision.

2. Preliminaries

In this section, we recall some fundamental definitions and results that are necessary to the development of the subsequent results.

Given a set of initial control points $\mathbf{P}^0 = \{p_i^0 \in \mathbb{R}\}_{i\in\mathbb{Z}}$, the set of control points $\mathbf{P}^{k+1} = \{p_i^{k+1}\}_{i\in\mathbb{Z}}$ at level $k+1$ are recursively defined by the following binary subdivision rules:

$$p_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} p_j^k, \quad i \in \mathbb{Z}. \tag{2}$$

The finite set $a = \{a_i\}_{i\in\mathbb{Z}}$ is called *mask*. The iterative algorithm based on the repeated application of (2) is termed *subdivision scheme* and is denoted by S_a. The symbol of the scheme S_a is defined as $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$.

Theorem 1 (see [17]). *Let a binary subdivision scheme S_a be convergent. Then the mask $a = \{a_i\}_{i\in\mathbb{Z}}$ satisfies*

$$\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1. \tag{3}$$

Theorem 2 (see [17]). *Let subdivision scheme S with mask $a = \{a_i\}_{i\in\mathbb{Z}}$ satisfy (3). Then there exists a subdivision scheme S_1 (first-order divided difference scheme of S) with the property*

$$d\mathbf{P}^k = S_1 d\mathbf{P}^{k-1}, \tag{4}$$

where $\mathbf{P}^k = S^k\mathbf{P}^0$ and $d\mathbf{P}^k = \{(d\mathbf{P}^k)_i = 2^k(P_{i+1}^k - P_i^k) \mid i \in \mathbb{Z}\}$. The symbol of S_1 is $a^{(1)}(z) = (2z/(1+z))a(z)$. Generally, if S_n (the nth-order divided difference scheme of S) exists with mask $a^{(n)} = \{a_i^{(n)}\}_{i\in\mathbb{Z}}$, then the symbol of S_n is $a^{(n)}(z) = (2z/(1+z))^n a(z)$.

Theorem 3 (see [17]). *(a) Let subdivision scheme S have mask $a^{(0)} = \{a_i^{(0)}\}_{i\in\mathbb{Z}}$, and its jth-order divided difference scheme S_j ($j = 1, 2, \ldots, n+1$) exists with mask $a^{(j)} = \{a_i^{(j)}\}_{i\in\mathbb{Z}}$ satisfying*

$$\sum_{i\in\mathbb{Z}} a_{2i}^{(j)} = \sum_{i\in\mathbb{Z}} a_{2i+1}^{(j)} = 1, \quad j = 0, 1, \ldots, n. \tag{5}$$

If there exists an integer $L \geq 1$, such that $\|((1/2)S_{n+1})^L\|_\infty < 1$, then the subdivision scheme S is C^n continuous, where

$$\begin{aligned} \left\|\left(\frac{1}{2}S_{n+1}\right)^L\right\|_\infty &= \max\left\{\sum_{j\in\mathbb{Z}}\left|b_{i-2^L j}^{[L]}\right| : 0 \leq i < 2^L\right\}, \\ b^{[L]}(z) &= b(z)\,b\left(z^2\right)\cdots b\left(z^{2^{L-1}}\right),\ b(z) = \frac{1}{2}a^{(n+1)}(z). \end{aligned} \tag{6}$$

In particular, when $L = 1$,

$$\left\|\frac{1}{2}S_{n+1}\right\|_\infty = \frac{1}{2}\max\left\{\sum_{i\in\mathbb{Z}}\left|a_{2i}^{(n+1)}\right|, \sum_{i\in\mathbb{Z}}\left|a_{2i+1}^{(n+1)}\right|\right\}. \tag{7}$$

(b) Let $a(z) = ((1+z)^{n+1}/2^n)b(z)$ with S_b being contractive (i.e., S_b maps any initial data to zero). Then, S_a is convergent and C^n continuous.

Theorem 4 (see [18, 19]). *Let π_d denote the space of all univariate polynomials with real coefficients up to degree d. Then a univariate subdivision scheme S_a*

(i) *generates π_d if and only if*

$$\begin{aligned} a(1) &= 2, \\ a(-1) &= 0, \\ a^{(j)}(-1) &= 0, \quad j = 1, \ldots, d; \end{aligned} \tag{8}$$

(ii) *reproduces π_d with respect to the parametrization $\{t_i^k = (i+\tau)/2^k\}_{i\in\mathbb{Z}}$ with $\tau = a^{(1)}(1)/2$ and k denoting the subdivision level, if and only if it generates π_d and*

$$a^{(j)}(1) = 2\prod_{h=0}^{j-1}(\tau - h), \quad j = 1, \ldots, d. \tag{9}$$

3. Construction of the New Family

This section first explains a new observation about the relation between 4-point interpolatory subdivision and quintic B-spline refinement. Then, a new family of 6-point combined subdivision schemes is deduced.

3.1. A New Observation. Given an initial control polygon with vertices $\{P_i^0\}$, as shown in Figure 1, the rules of 4-point interpolatory subdivision for generating $k+1$ level vertices $\{P_i^{k+1}\}$ are

$$P_{2i}^{k+1} = P_i^k,$$
$$P_{2i+1}^{k+1} = -\frac{1}{16}P_{i-1}^k + \frac{9}{16}P_i^k + \frac{9}{16}P_{i+1}^k - \frac{1}{16}P_{i+2}^k, \tag{10}$$

and quintic B-spline refinement for generating $k+1$ level vertices $\{Q_i^{k+1}\}$ is

$$Q_{2i}^{k+1} = \frac{3}{16}P_{i-1}^k + \frac{10}{16}P_i^k + \frac{3}{16}P_{i+1}^k,$$
$$Q_{2i+1}^{k+1} = \frac{1}{32}P_{i-1}^k + \frac{15}{32}P_i^k + \frac{15}{32}P_{i+1}^k + \frac{1}{32}P_{i+2}^k. \tag{11}$$

Denote by Δ_{2i}, Δ_{2i+1} the displacements of vertices from quintic B-spline refinement to 4-point interpolatory subdivision after one step of refinement, as shown in Figure 1(a), where the black lines represent the initial control polygon, the magenta lines represent the control polygon after one step of 4-point interpolatory subdivision, and the green lines represent the control polygon after one step of quintic B-spline refinement. Then, from (10) and (11), we can get

$$\Delta_{2i}^{k+1} = P_{2i}^{k+1} - Q_{2i}^{k+1} = -\frac{3}{16}P_{i-1}^k + \frac{6}{16}P_i^k - \frac{3}{16}P_{i+1}^k,$$
$$\Delta_{2i+1}^{k+1} = P_{2i+1}^{k+1} - Q_{2i+1}^{k+1} \tag{12}$$
$$= -\frac{3}{32}P_{i-1}^k + \frac{3}{32}P_i^k + \frac{3}{32}P_{i+1}^k - \frac{3}{32}P_{i+2}^k.$$

A new observation is

$$\Delta_{2i+1}^{k+1} = \frac{1}{2}\left(\Delta_{2i}^{k+1} + \Delta_{2i+2}^{k+1}\right), \tag{13}$$

which shows that the relation between 4-point interpolatory subdivision and quintic B-spline refinement is similar to the one between 4-point interpolatory subdivision and cubic B-spline refinement discovered by Lin et al. in [7]; that is, $\Delta_{2i+1}^{k+1} = (1/2)(\Delta_{2i}^{k+1} + \Delta_{2i+2}^{k+1})$, as shown in Figure 1(b), where the blue lines represent the control polygon after one step of cubic B-spline refinement. So, from the point of view of displacements, 4-point interpolatory scheme has the same connections with cubic B-spline and quintic B-spline.

We further found that though 6-point interpolatory subdivision is also constructed from polynomial interpolation just like 4-point interpolatory subdivision, analogous connection does not exist between 6-point interpolatory subdivision and quintic B-spline refinement.

3.2. Construction of the New 6-Point Combined Scheme. As is shown in [2], the rules of 6-point interpolatory subdivision for generating $k+1$ level vertices $\{P_i^{k+1}\}$ are

$$P_{2i}^{k+1} = P_i^k,$$
$$P_{2i+1}^{k+1} = \frac{3}{256}\left(P_{i-2}^k + P_{i+3}^k\right) - \frac{25}{256}\left(P_{i-1}^k + P_{i+2}^k\right) + \frac{150}{256}\left(P_i^k + P_{i+1}^k\right). \tag{14}$$

Suppose the new subdivision have the following rule:

$$\overline{P}_{2i}^{k+1} = \frac{1}{256}\left(\alpha\left(P_{i-2}^k + P_{i+2}^k\right) + \beta\left(P_{i-1}^k + P_{i+1}^k\right) + \left(256 - 2\alpha - 2\beta\right)P_i^k\right), \tag{15}$$

where α, β are tension parameters, and then

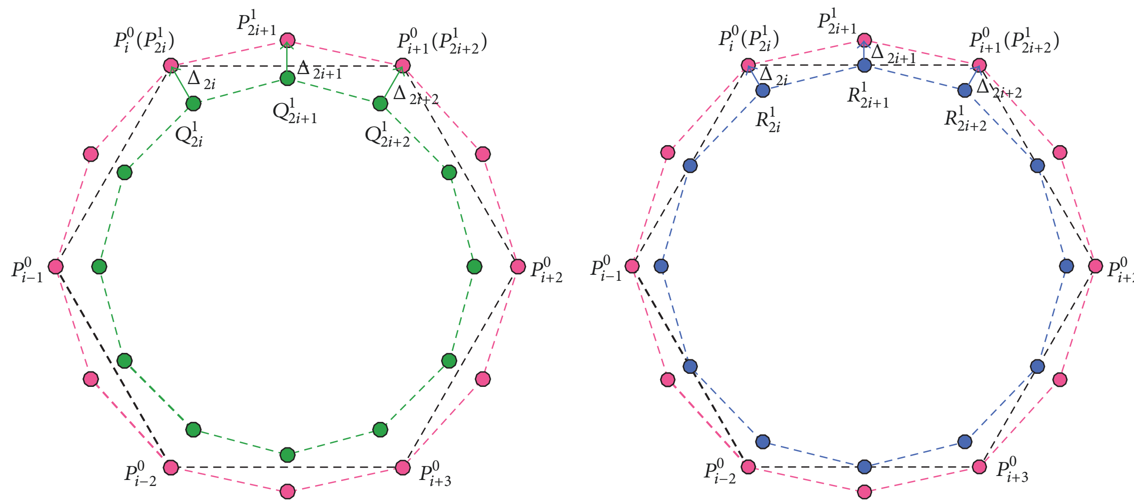
$$\overline{P}_{2i+2}^{k+1} = \frac{1}{256}\left(\alpha\left(P_{i-1}^k + P_{i+3}^k\right) + \beta\left(P_i^k + P_{i+2}^k\right) + \left(256 - 2\alpha - 2\beta\right)P_{i+1}^k\right),$$
$$\Delta_{2i}^{k+1} = P_{2i}^{k+1} - \overline{P}_{2i}^{k+1} = \frac{1}{256}\left(-\alpha\left(P_{i-2}^k + P_{i+2}^k\right) - \beta\left(P_{i-1}^k + P_{i+1}^k\right) + \left(2\alpha + 2\beta\right)P_i^k\right), \tag{16}$$
$$\Delta_{2i+2}^{k+1} = P_{2i+2}^{k+1} - \overline{P}_{2i+2}^{k+1} = \frac{1}{256}\left(-\alpha\left(P_{i-1}^k + P_{i+3}^k\right) - \beta\left(P_i^k + P_{i+2}^k\right) + \left(2\alpha + 2\beta\right)P_{i+1}^k\right).$$

Using relation (13), it can be deduced that

$$\Delta_{2i+1}^{k+1} = \frac{1}{256}\left(-\frac{\alpha}{2}\left(P_{i-2}^k + P_{i+3}^k\right) - \frac{\alpha+\beta}{2}\left(P_{i-1}^k + P_{i+2}^k\right) + \left(\alpha + \frac{\beta}{2}\right)\left(P_i^k + P_{i+1}^k\right)\right). \tag{17}$$

So, we obtain

$$\overline{P}_{2i+1}^{k+1} = \frac{1}{256}\left(\left(3 + \frac{\alpha}{2}\right)\left(P_{i-2}^k + P_{i+3}^k\right) + \left(\frac{\alpha+\beta}{2} - 25\right)\left(P_{i-1}^k + P_{i+2}^k\right) + \left(150 - \alpha - \frac{\beta}{2}\right)\left(P_i^k + P_{i+1}^k\right)\right), \tag{18}$$



(a) The relation between 4-point scheme and quintic B-spline scheme

(b) The relation between 4-point scheme and cubic B-spline scheme

FIGURE 1: The relation between 4-point interpolatory subdivision and B-spline refinement.

and then the new subdivision can be concluded from (15) and (18) as

$$\begin{aligned} P_{2i}^{k+1} &= \frac{1}{256}\left(\alpha\left(P_{i-2}^k + P_{i+2}^k\right) + \beta\left(P_{i-1}^k + P_{i+1}^k\right) + (256 - 2\alpha - 2\beta) P_i^k\right), \\ P_{2i+1}^{k+1} &= \frac{1}{256}\left(\left(3 + \frac{\alpha}{2}\right)\left(P_{i-2}^k + P_{i+3}^k\right) + \left(\frac{\alpha+\beta}{2} - 25\right)\left(P_{i-1}^k + P_{i+2}^k\right) + \left(150 - \alpha - \frac{\beta}{2}\right)\left(P_i^k + P_{i+1}^k\right)\right), \end{aligned} \tag{19}$$

which is the form of (1) in Section 1.

The mask and symbol of subdivision (1) are

$$\frac{1}{256}\left[3 + \frac{\alpha}{2}, \alpha, \frac{\alpha+\beta}{2} - 25, \beta, 150 - \alpha - \frac{\beta}{2}, 256 - 2\alpha - 2\beta, 150 - \alpha - \frac{\beta}{2}, \beta, \frac{\alpha+\beta}{2} - 25, \alpha, 3 + \frac{\alpha}{2}\right], \tag{20}$$

$$a_{\alpha,\beta}(z) = \frac{(1+z)^4}{8} \cdot \frac{1}{32}\left(3 + \frac{\alpha}{2} - (12+\alpha) z + \left(5 + \frac{3\alpha}{2} + \frac{\beta}{2}\right) z^2 + (40 - 2\alpha - \beta) z^3 + \left(5 + \frac{3\alpha}{2} + \frac{\beta}{2}\right) z^4 - (12+\alpha) z^5 + \left(3 + \frac{\alpha}{2}\right) z^6\right), \tag{21}$$

respectively. When $\beta = -4\alpha$, symbol (21) can be written as

$$a_\alpha(z) = \frac{(1+z)^6}{32} \cdot \frac{1}{8}\left(3 + \frac{\alpha}{2} - (18 + 2\alpha) z + (38 + 3\alpha) z^2 - (18 + 2\alpha) z^3 + \left(3 + \frac{\alpha}{2}\right) z^4\right). \tag{22}$$

In particular, when $\alpha = -10$,

$$a_{-10}(z) = \frac{(1+z)^7}{64} \cdot \frac{1}{2}\left(-1 + 2z + 2z^2 - z^3\right). \tag{23}$$

Denote the family of subdivision (1) by $S_{a_{\alpha,\beta}}$ and subfamily (22) by S_{a_α}. We call them the counterparts of cubic and quintic B-spline refinements based on the 6-point interpolatory subdivision. Figure 2 illustrates the limit curves of some members of $S_{a_{\alpha,\beta}}$. In Section 4, we will prove that the family $S_{a_{\alpha,\beta}}$ generates curves with C^3 continuity, and the subfamily S_{a_α} attains C^4 continuity when $\alpha \in (-14, -8)$ and reproduces cubic polynomials.

4. Analysis of the New Family

4.1. Smoothness Analysis

Proposition 5. *The scheme $S_{a_{\alpha,\beta}}$ defined by (1) converges and has smoothness C^3 when $\beta \in (32, 40)$ and $\alpha \in (-4 - \beta/4, 4 - \beta/4)$, or $\beta \in (40, 72)$ and $\alpha \in (-4 - \beta/4, 14 - \beta/2)$; and when $\alpha \in (-14, -8)$, the subfamily S_{a_α} generates C^4 continuous limit curves.*

Proof. The symbol of $S_{a_{\alpha,\beta}}$ can be written as

$$a_{\alpha,\beta}(z) = \sum_i a_i z^i = \frac{(1+z)^4}{8} \cdot b_{\alpha,\beta}(z), \tag{24}$$

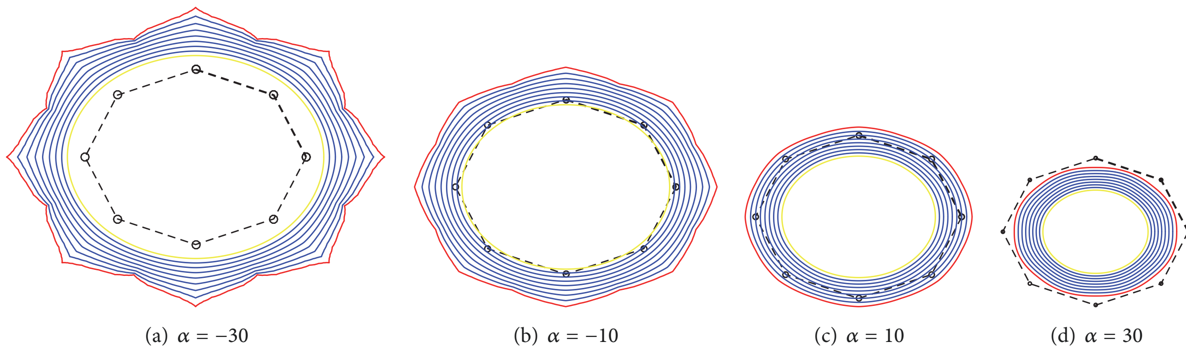


FIGURE 2: Some limit curves generated by $S_{a_{\alpha,\beta}}$ all with $\beta = -65, -50, -35, -20, -5, 10, 25, 40,$ and 55 from outside (red) to inside (yellow), respectively. The black is the initial control polygon.

where

$$b_{\alpha,\beta}(z) = \frac{1}{32}\left(3 + \frac{\alpha}{2} - (12+\alpha)z + \left(5 + \frac{3\alpha}{2} + \frac{\beta}{2}\right)z^2 + (40 - 2\alpha - \beta)z^3 + \left(5 + \frac{3\alpha}{2} + \frac{\beta}{2}\right)z^4 - (12+\alpha)z^5 + \left(3 + \frac{\alpha}{2}\right)z^6\right). \tag{25}$$

Let b_i denote the coefficients of Laurent polynomial $b_{\alpha,\beta}(z)$. By Theorem 3(b), if $S_{b_{\alpha,\beta}}$ is contractive, then $S_{a_{\alpha,\beta}}$ is C^3. When

$$\begin{gathered}\beta \in (32, 40), \\ \alpha \in \left(-4 - \frac{\beta}{4}, 4 - \frac{\beta}{4}\right) \\ \text{or } \beta \in (40, 72), \\ \alpha \in \left(-4 - \frac{\beta}{4}, 14 - \frac{\beta}{2}\right), \\ \left\|S_{b_{\alpha,\beta}}\right\|_\infty = \max\left\{\sum_i |b_{2i}|, \sum_i |b_{2i+1}|\right\} = \frac{1}{32} \cdot \max\left\{2\left(\left|5 + \frac{3\alpha}{2} + \frac{\beta}{2}\right| + \left|3 + \frac{\alpha}{2}\right|\right), |40 - 2\alpha - \beta| + 2|12+\alpha|\right\} < 1,\end{gathered} \tag{26}$$

which shows that $S_{b_{\alpha,\beta}}$ is contractive; hence, when

$$\begin{gathered}\beta \in (32, 40), \\ \alpha \in \left(-4 - \frac{\beta}{4}, 4 - \frac{\beta}{4}\right) \\ \text{or } \beta \in (40, 72), \\ \alpha \in \left(-4 - \frac{\beta}{4}, 14 - \frac{\beta}{2}\right), \\ S_{a_{\alpha,\beta}} \text{ is } C^3.\end{gathered} \tag{27}$$

When $\beta = -4\alpha$, the symbol of the subfamily S_{a_α} is

$$a_\alpha(z) = \frac{(1+z)^6}{32} \cdot \frac{1}{8}\left(3 + \frac{\alpha}{2} - (18 + 2\alpha)z + (38 + 3\alpha)z^2 - (18 + 2\alpha)z^3 + \left(3 + \frac{\alpha}{2}\right)z^4\right), \tag{28}$$

which can also be written as $a_\alpha(z) = (1+z)^5/16 \cdot b_\alpha(z)$, where

$$b_\alpha(z) = \frac{1}{16}\left(3 + \frac{\alpha}{2} - \left(15 + \frac{3\alpha}{2}\right)z + (20+\alpha)z^2 + (20+\alpha)z^3 - \left(15 + \frac{3\alpha}{2}\right)z^4 + \left(3 + \frac{\alpha}{2}\right)z^5\right). \tag{29}$$

When $\alpha \in (-14, -8)$, $\|S_{b_\alpha}\|_\infty = (1/16)(|3+\alpha/2| + |15+3\alpha/2| + |20+\alpha|) < 1$.

Hence, by Theorem 3(b), S_{b_α} is contractive and S_{a_α} is C^4 when $\alpha \in (-14, -8)$. □

4.2. Generation Degree and Reproduction Degree. Polynomial generation and polynomial reproduction are desirable properties because any convergent subdivision scheme that reproduces polynomials of degree k has approximation order $k+1$ [18]. The polynomial generation of degree k is the capability of subdivision schemes to generate the full space of polynomials of degree k [20]. The polynomial reproduction is the capability of subdivision schemes to produce in the limit exactly the same polynomial from which the initial data is sampled. The generation degree is not less than the reproduction degree. For example, the generation degree of degree-n B-spline refinement is n, but the reproduction degree of degree-n B-spline refinement only attains 1. Hormann and Sabin [16] proposed a family of subdivision schemes S_k ($k \in \mathbb{N}$) which is defined by the product of the symbol of B-spline refinement with a degree-2 polynomial and increased the degree of polynomial reproduction of B-spline schemes from 1 to 3.

Let $D_{\alpha,\beta} = \{\alpha, \beta \in \mathbb{R} \mid S_{a_{\alpha,\beta}}$ is convergent$\}$ and suppose $\alpha, \beta \in D_{\alpha,\beta}$. Using Theorem 4, we get the following results.

Proposition 6. *The subdivision scheme $S_{a_{\alpha,\beta}}$ generates*

$$\begin{aligned} &\pi_3, \quad \text{if } \beta \neq -4\alpha, \\ &\pi_5, \quad \text{if } \beta = -4\alpha. \end{aligned} \tag{30}$$

In particular, when $\alpha = -10$, S_{a_α} generates π_7.

Proof. The symbol of $S_{a_{\alpha,\beta}}$ can be written as

$$a_{\alpha,\beta}(z) = \frac{1}{256} \cdot A(z) B(z), \tag{31}$$

where $A(z) = (1+z)^4$ and $B(z) = 32 \cdot b_{\alpha,\beta}(z)$.

Then, $a^{(n)}_{\alpha,\beta}(z) = 1/256 \cdot \sum_{i=0}^{n} C_n^i A^{(i)}(z) B^{(n-i)}(z)$, and $a^{(1)}_{\alpha,\beta}(-1) = a^{(2)}_{\alpha,\beta}(-1) = a^{(3)}_{\alpha,\beta}(-1) = 0$.

Moreover, when $\beta = -4\alpha$,

$$a^{(4)}_{\alpha}(-1) = a^{(5)}_{\alpha}(-1) = 0; \tag{32}$$

and when $\alpha = -10$,

$$\begin{aligned} &a^{(6)}_{\alpha}(-1) = a^{(7)}_{\alpha}(-1) = 0, \\ &a^{(8)}_{\alpha}(-1) \neq 0. \end{aligned} \tag{33}$$

Hence, according to Theorem 4(i), we get that when $\beta \neq -4\alpha$, the subdivision scheme $S_{a_{\alpha,\beta}}$ generates π_3; when $\beta = -4\alpha$, S_{a_α} generates π_5 and when $\alpha = -10$, S_{a_α} generates π_7. □

Proposition 7. *If applying the parameter shift $\tau = 5$, the subdivision scheme $S_{a_{\alpha,\beta}}$ reproduces*

$$\begin{aligned} &\pi_1, \quad \text{if } \beta \neq -4\alpha, \\ &\pi_3, \quad \text{if } \beta = -4\alpha, \end{aligned} \tag{34}$$

with respect to the parametrization $\{t_i^k = (i+\tau)/2^k\}_{i\in\mathbb{Z}}$, where k denotes the subdivision level. In particular, when $\alpha = 0$, S_{a_α} reproduces π_5.

Proof. To consider the reproduction degree of the subdivision scheme $S_{a_{\alpha,\beta}}$, in view of Theorem 4(ii), we just need to consider $a^{(j)}_{\alpha,\beta}(1)$, $j = 1, \ldots, d$. Using the notation in Proposition 6, we get that

$$\begin{aligned} &a^{(1)}_{\alpha,\beta}(1) = 10, \\ &a^{(2)}_{\alpha,\beta}(1) = 40 + \frac{4\alpha+\beta}{16}, \\ &a^{(3)}_{\alpha,\beta}(1) = 120 + \frac{3}{8}(4\alpha+\beta), \end{aligned} \tag{35}$$

so $\tau = a^{(1)}_{\alpha,\beta}(1)/2 = 5$, and when $\beta = -4\alpha$,

$$\begin{aligned} &a^{(2)}_{\alpha}(1) = 40 = 2\tau(\tau-1), \\ &a^{(3)}_{\alpha}(1) = 120 = 2\tau(\tau-1)(\tau-2). \end{aligned} \tag{36}$$

Then $a^{(4)}_{\alpha}(1) = 240 + 3\alpha$, and when $\alpha = 0$,

$$\begin{aligned} &a^{(4)}_{\alpha}(1) = 240 = 2\prod_{h=0}^{3}(\tau-h), \\ &a^{(5)}_{\alpha}(1) = 240 = 2\prod_{h=0}^{4}(\tau-h), \\ &a^{(6)}_{\alpha}(1) \neq 0 = 2\prod_{h=0}^{5}(\tau-h). \end{aligned} \tag{37}$$

Hence, using Theorem 4(ii), we conclude that when $\beta \neq -4\alpha$, the subdivision scheme $S_{a_{\alpha,\beta}}$ reproduces π_1; when $\beta = -4\alpha$, S_{a_α} reproduces π_3 and when $\alpha = 0$, S_{a_α} reproduces π_5. □

As the new family of subdivision schemes $S_{a_{\alpha,\beta}}$ is deduced from the 6-point interpolatory scheme using the relation between 4-point interpolatory scheme and cubic, quintic B-spline, the properties of all of them are summarized in Table 1. For the new subfamily S_{a_α} and Hormann-Sabin's family S_k ($k \in \mathbb{N}$) and both have cubic precision, we list corresponding properties for a comparison in Table 2.

5. Conclusions

In this paper, we present a new family of 6-point combined subdivision schemes which provides the representation of wide variety of shapes and a subfamily of subdivision schemes with high smoothness and cubic precision. All these properties are required in many applications, such as computer aided geometric design and geometric modeling. The subfamily S_{a_α} attains cubic precision whereas the B-spline schemes have linear precision (see Figure 3). On the other hand, both having cubic precision, Hormann-Sabin' family S_k ($k \in \mathbb{N}$) increases the degree of polynomial generation and smoothness in exchange of the increase of the support width, while S_{a_α} can keep the support width unchanged and maintain higher degree of polynomial generation and smoothness. Moreover, the tension parameter α makes S_{a_α} able to provide more choices in applications (see Figures 4 and 5).

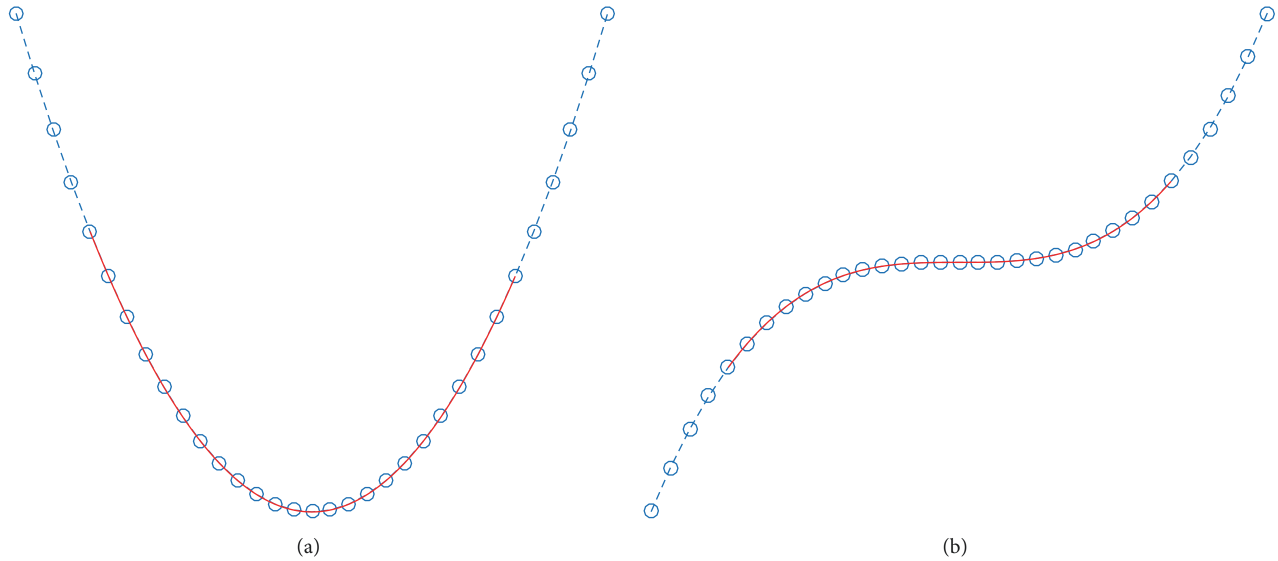


FIGURE 3: The polynomial reproduction property of S_{a_α} ($\alpha = 4$) with (a) $y = x^2$ and (b) $y = x^3$. The blue is the initial control polygon.

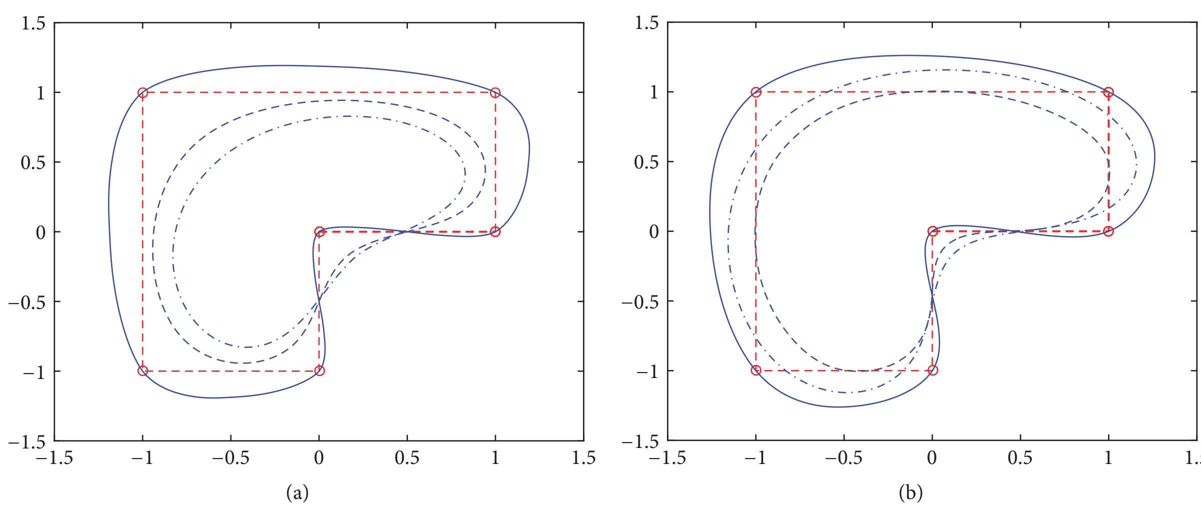


FIGURE 4: Comparison of limit curves (the blue curves) generated by (a) 4-p interpolatory scheme, cubic *B*-spline, and quintic *B*-spline refinement from outer to inner part and (b) 6-p interpolatory scheme, S_{a_α} ($\alpha = -8$) and $S_{a_{\alpha,\beta}}$ ($\alpha = 8, \beta = 10$), from outer to inner part. The red is the initial control polygon.

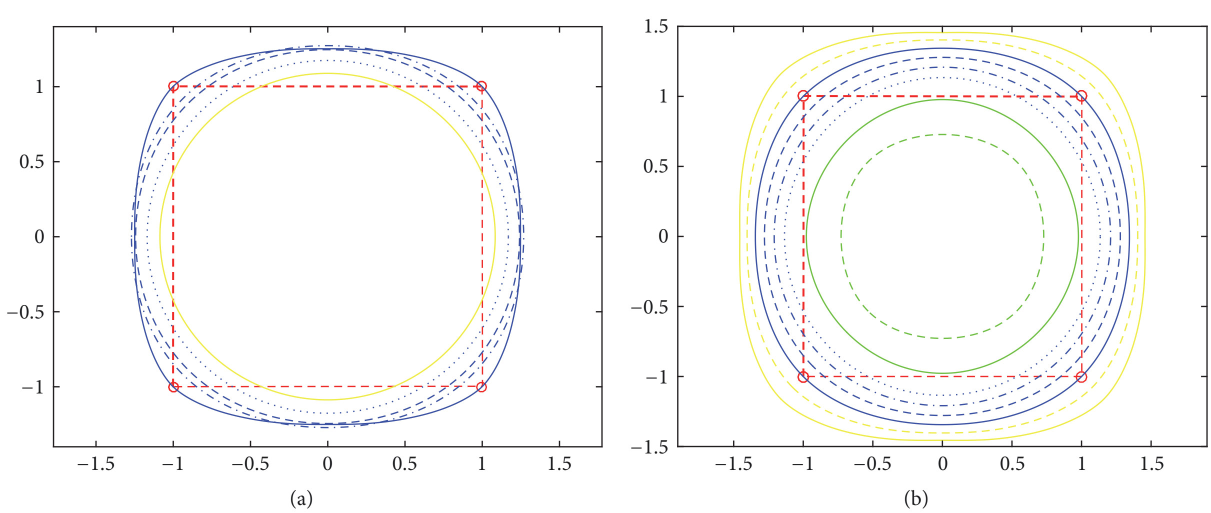


FIGURE 5: Comparison of limit curves generated by S_k (a) with $k = 4, 5, 6, 8,$ and 10 from outer to inner part and S_{a_α} (b) with $\alpha = 8, 4, 0, -4, -8, -12, -20,$ and -32 from outer to inner part. The red is the initial control polygon.

Table 1: Comparison between properties of cubic B-spline refinement, quintic B-spline refinement, 6-point interpolatory scheme, and the new family of schemes $S_{a_{\alpha,\beta}}$, S_{a_α}.

Scheme	Support	Continuity	Generation degree	Reproduction degree
4-p interpolatory scheme	6	1	3	3
Cubic B-spline	4	2	3	1
Quintic B-spline	6	4	5	1
6-p interpolatory scheme	10	2	5	5
$S_{a_{\alpha,\beta}}$	10	3	3	1
S_{a_α}	10	4	5	3
S_{a_α} $(\alpha = -10)$	10	4	7	3

Table 2: Comparison between properties of Hormann-Sabin's family S_k, $k \in \mathbb{N}$, and the new family of schemes S_{a_α}.

Scheme	Support	Continuity	Generation degree	Reproduction degree
S_4	6	1	3	3
S_5	7	2	4	3
S_6	8	3	5	3
S_7	9	4	6	3
S_8	10	5	7	3
S_9	11	6	8	3
S_k $(k \geq 4)$	$k+2$	$\lfloor k - \log_2(2+k/2) \rfloor$	$k-1$	3
S_{a_α}	10	4	5	3
S_{a_α} $(\alpha = -10)$	10	4	7	3

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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