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Research Article

Hesitant Probabilistic Fuzzy Preference Relations in Decision Making

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Preference of an alternative over another alternative is a useful way to express the opinion of decision maker. In the process of group decision making, preference relations are used in preference modelling of the alternatives under given criteria. The probability is an important tool to deal with uncertainty; in many scenarios of decision making probabilities of different events affect the decision making process directly. In order to deal with this issue, in this paper, hesitant probabilistic fuzzy preference relation (HPFPR) is defined. Furthermore, consistency of HPFPR and consensus among decision makers are studied in the hesitant probabilistic fuzzy environment. In this respect, many novel algorithms are developed to achieve consistency of HPFPRs and reasonable consensus between decision makers and a final algorithm is proposed comprehending all other algorithms, presenting a complete decision support model for group decision making. Lastly, we present a case study with complete illustration of the proposed model and discussed the effects of probabilities on decision making validating the importance of the introduction of probability in hesitant fuzzy preference relation.

1. Introduction

Fuzzy set theory was initially introduced by Zadeh [1] in 1965 as an extension of the classical set theory. In classical set theory, an element either belongs to or does not belong to the set. In fuzzy set theory, the gradual assessment of elements of set is described by the membership function that is in $[0, 1]$. Fuzzy set theory can be used in which information is vague, incomplete, or imprecise and it is successfully used in decision making problems [2]. After the popularity of this extension in set theory, several extensions and generalizations of fuzzy sets have been introduced in the literature, for example, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, trapezoidal-valued intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy multisets. These extensions have been successfully used in several practical applications of real life problems and scientific problems. Applications of these extensions can be found in artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Torra felt that there are some limitations or deficiencies in these extensions. He proposed another extension of fuzzy set theory which is named as hesitant fuzzy set (HFS) theory [3]. This extension permits the several possible membership degrees of an element in $[0, 1]$. HFS provides a much better description than the other extensions of fuzzy sets where the difficulty of establishing the membership degree and there is a specific set of possible values. Many studies on HFS have been conducted, such as extensions of HFS (see [4–9]).

Group decision making (GDM) is a procedure to find the best/optimal alternative from a set of alternatives under the basis of certain criteria and the alternatives are evaluated by the group of decision makers for the criteria [10–14]. The opinion of DMs may also be in the form of preference of each pair of alternatives and provides a comparison of one alternative over another and this comparison is a preference relation; for some basics of preference relation see [15]. Preference relations have been developed and investigated in different modes, like multiplicative preference relations [16, 17], fuzzy preference relations [18], multiplicative fuzzy preference relation [10], incomplete fuzzy preference relation [11, 19], linguistic preference relations [20, 21], intuitionistic fuzzy preference relations [22, 23], and interval-valued hesitant preference relations [6]. Due to several external nonfeasible circumstances, like shortage of time, lack of knowledge, and available data resources, DMs provided their preferences opinion over the alternatives in the form of several possible numerical values. All the discussed preference relations do not handle such kind of situations. To overcome this problem, Zhu and Xu [24] introduced hesitant fuzzy preference relation (HFPR). Here the preference of an alternative over another alternative is a HFE; this HFE shows all possible preference values that denote the hesitant degree between two alternatives. HFPRs provide a better framework for the description of the DMs' hesitation while providing their preferences among the alternatives [25–28].

In preference relations based decision making problems, the concept of consistency plays an important role. It is the level or degree of satisfaction among the values in preference relation and these values are given by each DM [29]. Consensus is also an important and valuable concept in decision making problems based on preference relations. Consensus measure is used for the mutual understanding of DMs on the finally obtained alternatives [29]. Until now, several researchers of this area successfully made some progress to convert the preference relations as consistent and generate a certain level of consensus in decision making problems [30–33]. Nowadays, hesitant probabilistic fuzzy sets received good attention in multicriteria decision making. So, it is important to discuss the consistency and consensus measure for these preference relations. In this paper, we target to develop a group decision making model for HPFPRs where the consistency of the model and consensus among the DMs are under consideration. Our proposed model is efficient and practical; furthermore, it is strictly based on theoretical foundations. Pang et al. introduce the idea of probabilistic linguistic term set [34]. They also develop the aggregation operator of this set and proposed the extended TOPSIS version to handle the multicriteria group decision making problems. By getting the motivation from the hesitant fuzzy set and probabilistic linguistic term set, Xu and Zhou proposed the concept of hesitant probabilistic fuzzy set [35]. They investigated several aggregation operators with properties for hesitant probabilistic fuzzy set. A novel algorithm was developed to handle the multicriteria group decision making problems for hesitant probabilistic fuzzy set [36]. This concept was further extended to the continuous form of hesitant probabilistic fuzzy set [37]. Distance measures were also discussed for the continuous form and it was also applied to automotive industry safety evaluation problem. In these no one discusses the preference relations of the hesitant probabilistic fuzzy set. It is worth defining the preference relations for this set and discussing the consistency and consensus measure of the decision model.

To accomplish these goals, this paper is structured in the following way. In Section 2, some preliminary concepts are discussed to understand our proposal. Section 3 is devoted to design the basic structure of hesitant probabilistic fuzzy preference relations. A distance measured between the preference relations is developed, based on it, consistency measure of preference relations is derived. Also, two novel algorithms are designed to achieve acceptable consistency. In Section 4, consensus measure is defined for the group decision making and a novel algorithm is presented for reaching acceptable consensus among decision makers. Section 5 is dedicated to presenting a complete group decision making model dealing with both issues of consistency and consensus. In Section 6, numerical analysis of the developed model through a case study is performed to understand the importance of our proposal. Section 7 is dedicated for comparison between proposed model and existing ones. Section 8 ends the paper with some concluding remarks.

2. Preliminaries

Definition 1 (see [39]). For $X = \{x_1, x_2, \ldots, x_n\}$, a fixed set, a preference relation $\succcurlyeq$ is a subset of $X \times X$, which satisfies the following two:

(1) (Completeness) for all $x_i \in X$ and for all $x_j \in X$, either $x_i \succcurlyeq x_j$ or $x_j \succcurlyeq x_i$; that is, $(x_i, x_j) \in \succcurlyeq$ or $(x_j, x_i) \in \succcurlyeq$.

(2) (Transitivity) for all $x_i \in X$, for all $x_j \in X$, and for all $x_k \in X$ if $x_i \succcurlyeq x_j$ and $x_j \succcurlyeq x_k$, then $x_i \succcurlyeq x_k$; that is, $(x_i, x_j) \in \succcurlyeq$ and $(x_j, x_k) \in \succcurlyeq \Longrightarrow (x_i, x_k) \in \succcurlyeq$.

Definition 2 (see [1, 40]). For $X = \{x_1, x_2, \ldots, x_n\}$, a fixed set, a HFPR is expressed by a matrix $A = (a_{ij})_{n\times n} \subseteq X \times X$, where $a_{ij} = \{a_{ij}^s \mid s = 1, 2, 3, \ldots, \#a_{ij}\}$ is a HFE, giving all the possible preference degrees of the alternative x_i over x_j. Also a_{ij} satisfy the following conditions for all $i, j \in N$:

$$\begin{aligned} a_{ij}^{\sigma(s)} + a_{ji}^{(\#a_{ij}-\sigma(s)+1)} &= 1, \\ a_{ii} &= \{0.5\}, \\ \#a_{ij} &= \#a_{ji}, \end{aligned} \tag{1}$$

where $a_{ij}^{\sigma(s)}$ is the s^{th} smallest value in a_{ij} and also elements of a_{ij} are arranged in increasing order.

The notion of the hesitant fuzzy set given by Torra [3] is well known and has been successfully used to model vagueness of real life. It allows decision makers to give multiple membership values, but has the deficiency to deal with probabilities of preference degrees. To make hesitant fuzzy sets more compatible with real life, Xu and Zhou [35] defined hesitant probabilistic fuzzy element (HPFE) and hesitant probabilistic fuzzy set (HPFS).

Definition 3 (see [35]). Consider a fixed set R. The HPFS on R is defined as a mathematical symbol:

$$H_p = \{h(\gamma_i \mid p_i) \mid \gamma_i, p_i\}, \tag{2}$$

where $h(\gamma_i \mid p_i)$ is HPFE comprising the elements of the form $\gamma_i \mid p_i$, expressing the hesitant fuzzy information with probabilities to the set $H_p, 0 \le \gamma_i \le 1$, $i = 1, 2, 3, \ldots, \#h$, where $\#h$ is the number of elements in $h(\gamma_i \mid p_i)$, $p_i \in [0, 1]$ is the respective hesitant probability for γ_i, and $\sum_{i=1}^{\#h} p_i = 1$.

Score function, deviation function, and comparison laws are given to compare different HPFEs.

Definition 4. For a HPFE $h(\gamma_i \mid p_i)$ where $i = 1, 2, \ldots, \#h$, $s(h) = \sum_{i=1}^{\#h} \gamma_i p_i$ is called the score function of $h(\gamma_i \mid pi)$, where $\#h$ is the number of possible elements in $h(\gamma_i \mid pi)$.

Definition 5. For a HPFE $h(\gamma_i \mid p_i)$ where $i = 1, 2, \ldots, \#h$, $d(h) = \sum_{i=1}^{\#h} (\gamma_i - s(h))^2 p_i$ is called the deviation function of $h(\gamma_i \mid pi)$, where $s(h) = \sum_{i=1}^{\#h} \gamma_i p_i$ is the score function of $h(\gamma_i \mid pi)$ and $\#h$ is the number of possible elements in $h(\gamma_i \mid pi)$.

The score and deviation functions are similar to the expectation and variance of the random variable, respectively, and, thus, the comparison laws for two HPFEs h_1 and h_2 can be presented as follows:

$$\begin{aligned}
&\text{If } s(h_1) > s(h_2), \text{ then } h_1 > h_2,\\
&\text{If } s(h_1) = s(h_2) \text{ and } d(h_1) > d(h_2), \text{ then } h_1 < h_2,\\
&\text{If } s(h_1) = s(h_2) \text{ and } d(h_1) = d(h_2), \text{ then } h_1 = h_2,\\
&\text{If } s(h_1) = s(h_2) \text{ and } d(h_1) < d(h_2), \text{ then } h_1 > h_2.
\end{aligned}$$

3. Hesitant Probabilistic Fuzzy Preference Relation

In order to build a complete model for group decision making first some operations are defined for HPFEs of the same length. Let $h(\gamma_i \mid p_i)$, $h_1(\gamma_i' \mid p_i')$, and $h_2(\gamma_i'' \mid p_i'')$ be HPFEs with $\#h = \#h_1 = \#h_2$. Then

$$h_1 \oplus h_2 = \bigcup_{\gamma'_{\sigma(s)} \mid p'_{\sigma(s)} \in h_1, \gamma''_{\sigma(s)} \mid p''_{\sigma(s)} \in h_2} \left\{ \gamma'_{\sigma(s)} + \gamma''_{\sigma(s)} \mid p'_{\sigma(s)} + p''_{\sigma(s)} \right\} \quad (3)$$

$$h_1 \ominus h_2 = \bigcup_{\gamma'_{\sigma(s)} \mid p'_{\sigma(s)} \in h_1, \gamma''_{\sigma(s)} \mid p''_{\sigma(s)} \in h_2} \left\{ \gamma'_{\sigma(s)} - \gamma''_{\sigma(s)} \mid p'_{\sigma(s)} + p''_{\sigma(s)} \right\} \quad (4)$$

$$\max(h_1, h_2) = \bigcup_{\gamma'_{\sigma(s)} \mid p'_{\sigma(s)} \in h_1, \gamma''_{\sigma(s)} \mid p''_{\sigma(s)} \in h_2} \left\{ \max\left(\gamma'_{\sigma(s)}, \gamma''_{\sigma(s)}\right) \mid \max\left(p'_{\sigma(s)} + p''_{\sigma(s)}\right) \right\} \quad (5)$$

$$\min(h_1, h_2) = \bigcup_{\gamma'_{\sigma(s)} \mid p'_{\sigma(s)} \in h_1, \gamma''_{\sigma(s)} \mid p''_{\sigma(s)} \in h_2} \left\{ \min\left(\gamma'_{\sigma(s)}, \gamma''_{\sigma(s)}\right) \mid \min\left(p'_{\sigma(s)} + p''_{\sigma(s)}\right) \right\} \quad (6)$$

$$\omega h = \bigcup_{\gamma_{\sigma(s)} \mid p_{\sigma(s)} \in h} \left\{ \omega \gamma_{\sigma(s)} \mid \omega p_{\sigma(s)} \right\}: \quad \omega \ge 0, \quad (7)$$

where $\gamma'_{\sigma(s)} \mid p'_{\sigma(s)}$ and $\gamma''_{\sigma(s)} \mid p''_{\sigma(s)}$ are s^{th} elements of $h_1(\gamma_i' \mid p_i')$ and $h_2(\gamma_i'' \mid p_i'')$, respectively.

For simplicity $h(\gamma_i \mid p_i)$ will be written as h throughout this paper.

To allow decision makers to provide the preferences in hesitant probabilistic environment, we define hesitant probabilistic fuzzy preference relation (HPFPR).

Definition 6 (HPFPR). Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be the set of alternatives. The HPFPR is a matrix $H = (h_{ij})_{n \times n}$, where $h_{ij} = \{h_{ij}^s \mid p_{ij}^s : s = 1, 2, 3, \ldots, \#h_{ij}\}$ is the HPFE expressing the possible preference degrees of the alternative x_i over x_j with probabilities and with $j > i$ satisfying the following conditions:

$$\begin{aligned}
h_{ij}^{\sigma(s)} + h_{ji}^{\sigma(s)} &= 1,\\
p_{ij}^{\sigma(s)} &= p_{ji}^{\sigma(s)},\\
\#h_{ij} &= \#h_{ji},\\
h_{ii}^{\sigma(s)} &= \frac{1}{2},\\
h_{ij}^{\sigma(s)} &< h_{ij}^{\sigma(s+1)},\\
h_{ji}^{\sigma(s+1)} &< h_{ji}^{\sigma(s)},
\end{aligned} \quad (8)$$

where $h_{ij}^{\sigma(s)} \mid p_{ij}^{\sigma(s)}$ and $h_{ji}^{\sigma(s)} \mid p_{ji}^{\sigma(s)}$ are the s^{th} elements in h_{ij} and h_{ji}, respectively.

Remark 7. The above definition is very much alike to the definition of probabilistic hesitant fuzzy preference relation proposed by Zhou and Xu [41, Definition 4]. They take

$$h_{ii} = \left\{ \frac{1}{2} \mid 1 \right\}: \quad i = 1, 2, \ldots, n. \quad (9)$$

But if one adopted the technique of β-normalization, that is, making the length of h_{ij} the same by adding elements to HPFEs of shorter length, it implicates numerous errors and difficulties when dealing with consistency of HPFPRs and consensus among decision makers based on β-normalization. So fixing the length of diagonal elements to one does not match with the condition $\sum_{s=1}^{\#h} p_{ii}^{\sigma(s)} = 1$. So by allowing variation, the probabilities of diagonal HPFEs helps us to maintain the spirit of HPFE in discussing consistency and consensus in the context of β-normalization. The diversity in probabilities of diagonal preference degrees does not cause any harm; the net impact remains the same as the sum of all probabilities is 1.

Often the length of HPFEs has been different, but to apply the above defined operations (3), (4), and (7), the length is needed to be equal for all HPFEs. Some elements will be added to HPFE who has less elements, but the information it provides that should not be changed. Now, the definition of normalized hesitant probabilistic fuzzy preference relation (NHPFPR) is proposed.

Definition 8 (normalized HPFPR). A HPFPR $H = (h_{ij})_{n\times n}$ is called NHPFPR if the length of all h_{ij} is the same for all $i, j = 1, 2, \ldots, n$.

Let $h = \{h_i \mid p_i : i = 1, 2, 3, \ldots, \#h\}$ be a HPFE. For preference degrees h_i Zhu et al. [28] define a way to add elements in HFE; for an optimized parameter $0 \le \zeta \le 1$ the preference degree that will be added to h_i is $\zeta h^+ + (1-\zeta)h^-$, where h^+ is the largest and h^- is the smallest among h_i. The decision maker can choose the value of ζ according to his risk preferences. The added element will be h^+ and h^- for $\zeta = 1$ and $\zeta = 0$, respectively, which demonstrate the optimistic and pessimistic approach of decision maker proposed by Xu and Xia [42]. In hesitant probabilistic fuzzy environment, some way is needed to assign probability to the added preference degree such that the information of HPFPR is not changed. There are many ways to do it; one option is to assign 0 to added preference degree $\zeta h^+ + (1-\zeta)h^-$, but for the extreme cases pessimistic approach, that is, $\zeta = 0$, and optimistic approach, that is, $\zeta = 1$, the added element in HPFE is $h^- \mid p^-/(d - \#h + 1)$ and $h^+ \mid p^+/(d - \#h + 1)$, respectively, where d is the required length of HPFEs and p^- and p^+ are the probabilities of h^- and h^+, respectively.

For a given HPFPR $H = (h_{ij})_{n\times n}$, we normalize it as follows. Let $d = \max\{\#h_{ij}\}$ and $i, j = 1, 2, \ldots, n$. For optimized parameter $\zeta = 0$,

$$\overline{h}_{ij} = \begin{cases} h_{ij}, & \#h_{ij} = d; \\ \left\{ \underbrace{h_{ij}^{\sigma(1)} \mid \frac{p_{ij}^{\sigma(1)}}{d - \#h_{ij} + 1}, \ldots, h_{ij}^{\sigma(1)} \mid \frac{p_{ij}^{\sigma(1)}}{d - \#h_{ij} + 1}}_{d-\#h_{ij}+1}, \underbrace{h_{ij}^{\sigma(2)} \mid p_{ij}^{\sigma(2)}, h_{ij}^{\sigma(3)} \mid p_{ij}^{\sigma(3)}, \ldots, h_{ij}^{\sigma(\#h_{ij})} \mid p_{ij}^{\sigma(\#h_{ij})}}_{\#h_{ij}-1} \right\}, & \#h_{ij} < d. \end{cases} \tag{10}$$

For $\zeta = 1$,

$$\overline{h}_{ij} = \begin{cases} h_{ij}, & \#h_{ij} = d; \\ \left\{ \underbrace{h_{ij}^{\sigma(1)} \mid p_{ij}^{\sigma(1)}, h_{ij}^{\sigma(2)} \mid p_{ij}^{\sigma(2)}, \ldots, h_{ij}^{\sigma(\#h_{ij}-1)} \mid p_{ij}^{\sigma(\#h_{ij}-1)}}_{\#h_{ij}-1}, \underbrace{h_{ij}^{\sigma(\#h_{ij})} \mid \frac{p_{ij}^{\sigma(\#h_{ij})}}{d - \#h_{ij} + 1}, \ldots, h_{ij}^{\sigma(\#h_{ij})} \mid \frac{p_{ij}^{\sigma(\#h_{ij})}}{d - \#h_{ij} + 1}}_{d-\#h_{ij}+1} \right\}, & \#h_{ij} < d. \end{cases} \tag{11}$$

For $0 < \zeta < 1$, for $i < j$

$$\begin{aligned} \overline{h}_{ij} &= \begin{cases} h_{ij}, & \#h_{ij} = d; \\ \left\{ \underbrace{h_{ij}^{\sigma(1)} \mid p_{ij}^{\sigma(1)}, \ldots, h_{ij}^{\sigma(t_{ij})} \mid p_{ij}^{\sigma(t_{ij})}}_{t_{ij}}, \underbrace{\zeta h_{ij}^{\sigma(\#h_{ij})} + (1-\zeta)\, h_{ij}^{\sigma(1)} \mid 0, \ldots, \zeta h_{ij}^{\sigma(\#h_{ij})} + (1-\zeta)\, h_{ij}^{\sigma(1)} \mid 0}_{d-(\#h_{ij})}, \underbrace{h_{ij}^{\sigma(t_{ij}+1)} \mid p_{ij}^{\sigma(t_{ij}+1)}, \ldots, h_{ij}^{\sigma(\#h_{ij})} \mid p_{ij}^{\sigma(\#h_{ij})}}_{(\#h_{ij})-t_{ij}} \right\}, & \#h_{ij} < d, \end{cases} \\ \overline{h}_{ji} &= \begin{cases} h_{ji}, & \#h_{ji} = d; \\ \left\{ \underbrace{h_{ji}^{\sigma(1)} \mid p_{ji}^{\sigma(1)}, \ldots, h_{ji}^{\sigma(t_{ji})} \mid p_{ji}^{\sigma(t_{ji})}}_{t_{ji}}, \underbrace{(1-\zeta)\, h_{ji}^{\sigma(1)} + \zeta h_{ji}^{\sigma(\#h_{ji})} \mid 0, \ldots, (1-\zeta)\, h_{ji}^{\sigma(1)} + \zeta h_{ji}^{\sigma(h_{ji})} \mid 0}_{d-(\#h_{ji})}, \underbrace{h_{ji}^{\sigma(t_{ji}+1)} \mid p_{ji}^{\sigma(t_{ji}+1)}, \ldots, h_{ji}^{\sigma(\#h_{ji})} \mid p_{ji}^{\sigma(\#h_{ji})}}_{(\#h_{ji})-t_{ji}} \right\}, & \#h_{ji} < d, \end{cases} \end{aligned} \tag{12}$$

where $t_{ij} = t_{ji} = \max_{s\in\{1,2,\ldots,\#h_{ij}\}} h_{ij}^{\sigma(s)} \le \zeta h_{ij}^{\sigma(\#h_{ij})} + (1-\zeta) h_{ij}^{\sigma(1)}$.

Now $\overline{H} = (\overline{h}_{ij})_{n\times n}$ is NHPFPR; next we deal with consistency.

Example 9. Let

$$\begin{aligned}
H_1 &= \begin{pmatrix} \{.5 \mid 1\} & \{0.1196 \mid 0.4481, 0.3567 \mid 0.5519\} & \{0.2898 \mid 1\} & \{0.1702 \mid 0.1906, 0.231 \mid 0.4432, 0.9392 \mid 0.3662\} \\ \{0.8804 \mid 0.4481, 0.6433 \mid 0.5519\} & \{.5 \mid 1\} & \{0.0712 \mid 0.0575, 0.566 \mid 0.9425\} & \{0.3587 \mid 0.2586, 0.7928 \mid 0.7414\} \\ \{0.7102 \mid 1\} & \{0.9288 \mid 0.0575, 0.434 \mid 0.9425\} & \{.5 \mid 1\} & \{0.5053 \mid 0.9692, 0.7557 \mid 0.0308\} \\ \{0.8298 \mid 0.1906, 0.769 \mid 0.4432, 0.0608 \mid 0.3662\} & \{0.6413 \mid 0.2586, 0.2072 \mid 0.7414\} & \{0.4947 \mid 0.9692, 0.2443 \mid 0.0308\} & \{.5 \mid 1\} \end{pmatrix}, \\
H_2 &= \begin{pmatrix} \{.5 \mid 1\} & \{0.331 \mid 0.4663, 0.4815 \mid 0.53, 0.7136 \mid 0.0036\} & \{0.3152 \mid 0.5797, 0.3269 \mid 0.1281, 0.5828 \mid 0.2921\} & \{0.3593 \mid 0.5865, 0.5943 \mid 0.2853, 0.8714 \mid 0.1281\} \\ \{0.669 \mid 0.4663, 0.5185 \mid 0.53, 0.2864 \mid 0.0036\} & \{.5 \mid 1\} & \{0.4762 \mid 0.6629, 0.7996 \mid 0.3371\} & \{0.2436 \mid 1\} \\ \{0.6848 \mid 0.5797, 0.6731 \mid 0.1281, 0.4172 \mid 0.2921\} & \{0.5238 \mid 0.6629, 0.2004 \mid 0.3371\} & \{.5 \mid 1\} & \{0.1829 \mid 0.1629, 0.3216 \mid 0.404, 0.8871 \mid 0.4331\} \\ \{0.6407 \mid 0.5865, 0.4057 \mid 0.2853, 0.1286 \mid 0.1281\} & \{0.7564 \mid 1\} & \{0.8171 \mid 0.1629, 0.6784 \mid 0.404, 0.1129 \mid 0.4331\} & \{.5 \mid 1\} \end{pmatrix}.
\end{aligned} \tag{13}$$

Then H_1 and H_2 are HPFPRs. Now by taking optimized parameters 0, 1 for H_1 and H_2, respectively, the following NHPFPR are obtained.

$$\begin{aligned}
\overline{H}_1 &= \begin{pmatrix} \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.1196 \mid 0.2241, 0.1196 \mid 0.2241, 0.3567 \mid 0.5519\} & \{0.2898 \mid .3333, 0.2898 \mid .3333, 0.2898 \mid .3333\} & \{0.1702 \mid 0.1906, 0.231 \mid 0.4432, 0.9392 \mid 0.3662\} \\ \{0.8804 \mid 0.2241, 0.8804 \mid 0.2241, 0.6433 \mid 0.5519\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.0712 \mid 0.0287, 0.0712 \mid 0.0287, 0.566 \mid 0.9425\} & \{0.3587 \mid 0.1293, 0.3587 \mid 0.1293, 0.7928 \mid 0.7414\} \\ \{0.7102 \mid .3333, 0.7102 \mid .3333, 0.7102 \mid .3333\} & \{0.9288 \mid 0.0287, 0.9288 \mid 0.0287, 0.434 \mid 0.9425\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.5053 \mid 0.4846, 0.5053 \mid 0.4846, 0.7557 \mid 0.0308\} \\ \{0.8298 \mid 0.1906, 0.769 \mid 0.4432, 0.0608 \mid 0.3662\} & \{0.6413 \mid 0.1293, 0.6413 \mid 0.1293, 0.2072 \mid 0.7414\} & \{0.4947 \mid 0.4846, 0.4947 \mid 0.4846, 0.2443 \mid 0.0308\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} \end{pmatrix}, \\
\overline{H}_2 &= \begin{pmatrix} \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.331 \mid 0.4663, 0.4815 \mid 0.53, 0.7136 \mid 0.0036\} & \{0.3152 \mid 0.5797, 0.3269 \mid 0.1281, 0.5828 \mid 0.2921\} & \{0.3593 \mid 0.5865, 0.5943 \mid 0.2853, 0.8714 \mid 0.1281\} \\ \{0.669 \mid 0.4663, 0.5185 \mid 0.53, 0.2864 \mid 0.0036\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.4762 \mid 0.6629, 0.7996 \mid 0.1686, 0.7996 \mid 0.1686\} & \{0.2436 \mid .3333, 0.2436 \mid .3333, 0.2436 \mid .3333\} \\ \{0.6848 \mid 0.5797, 0.6731 \mid 0.1281, 0.4172 \mid 0.2921\} & \{0.5238 \mid 0.6629, 0.2004 \mid 0.1686, 0.2004 \mid 0.1686\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} & \{0.1829 \mid 0.1629, 0.3216 \mid 0.404, 0.8871 \mid 0.4331\} \\ \{0.6407 \mid 0.5865, 0.4057 \mid 0.2853, 0.1286 \mid 0.1281\} & \{0.7564 \mid .3333, 0.7564 \mid .3333, 0.7564 \mid .3333\} & \{0.8171 \mid 0.1629, 0.6784 \mid 0.404, 0.1129 \mid 0.4331\} & \{.5 \mid .3333, .5 \mid .3333, .5 \mid .3333\} \end{pmatrix}.
\end{aligned} \tag{14}$$

3.1. Consistency Measure of Hesitant Probabilistic Fuzzy Preference Relation. In order to obtain valuable decision from preference relations they should be consistent in a sense that, let us say, x_1 is preferable to x_2 and x_2 is preferable to x_3 then x_1 must be preferable to x_3. Several authors have pursued consistency issues for preference relations [1, 21, 32, 33, 43].

Additive consistency for fuzzy preference degrees is well known; actually Tanino [44] defined the additive consistency for fuzzy preference relation based on moderate stochastic transitivity, well known in the probabilistic choice theory [45, page 27]. Furthermore, many kinds of transitivity are proposed and studied for probabilities in comparing the preferences in the choice theory.

Considering HPFPR $H = (h_{ij})_{n\times n}$ by (10), (11) calculate NHPFPR $\overline{H} = (\overline{h}_{ij})_{n\times n}$. The weak stochastic transitivity for probability means

$$\begin{aligned} &\overline{p}_{ik}^{\sigma(s)} \geq \frac{1}{2} \wedge \overline{p}_{kj}^{\sigma(s)} \geq \frac{1}{2} \Longrightarrow \\ &\overline{p}_{ij}^{\sigma(s)} \geq \frac{1}{2}. \end{aligned} \tag{15}$$

This will provide a platform to define consistency for HPFPR.

Definition 10 (consistency). For a given HPFPR, $H = (h_{ij})_{n\times n}$ and its NHFPR $\overline{H} = (\overline{h}_{ij})_{n\times n}$ with optimized parameter ζ. If

$$\overline{h}_{ij}^{\sigma(s)} = \overline{h}_{ik}^{\sigma(s)} - \overline{h}_{jk}^{\sigma(s)} + \frac{1}{2}, \tag{16}$$

$$\begin{aligned} &\overline{p}_{ik}^{\sigma(s)} \geq \frac{1}{2} \wedge \overline{p}_{kj}^{\sigma(s)} \geq \frac{1}{2} \Longrightarrow \\ &\overline{p}_{ij}^{\sigma(s)} \geq \frac{1}{2}, \end{aligned} \tag{17}$$

for all $i, j, k = 1, 2, 3, \ldots, n$, then H is called consistent HPFPR with optimized parameter ζ.

But many times preference relations are not consistent and, for meaningful decision making, some level of consistency is required in the least if it is not fully consistent. For preference degrees, take the summation of (16) for all k

$$n\overline{h}_{ij}^{\sigma(s)} = \sum_{k=1}^{n} \left(\overline{h}_{ik}^{\sigma(s)} - \overline{h}_{jk}^{\sigma(s)}\right) + \frac{n}{2}; \tag{18}$$

therefore,

$$\overline{h}_{ij}^{\sigma(s)} = \frac{1}{n}\sum_{k=1}^{n} \left(\overline{h}_{ik}^{\sigma(s)} - \overline{h}_{jk}^{\sigma(s)}\right) + \frac{1}{2}. \tag{19}$$

Thus, (19) is satisfied by a consistent HPFPR, if not put

$$\widetilde{h}_{ij}^{\sigma(s)} = \frac{1}{n}\sum_{k=1}^{n} \left(\overline{h}_{ik}^{\sigma(s)} - \overline{h}_{jk}^{\sigma(s)}\right) + \frac{1}{2}; \tag{20}$$

one can check that the preference degrees $\widetilde{h}_{ij}^{\sigma(s)}$ obtained from the above equation are consistent. For probabilities, matter is not that simple; some mechanism is needed to make consistent probabilities with all the restrictions of HPFPR like $\overline{p}_{ij}^{\sigma(s)} = \overline{p}_{ji}^{\sigma(s)}$ and $\sum_{s=1}^{\#\overline{h}_{ij}} \overline{p}_{ij}^{\sigma(s)} = 1$. Let $\overline{p}_{ik}^{\sigma(s)} \geq 1/2$ and $\overline{p}_{kj}^{\sigma(s)} \geq 1/2$. Then define

$$\widetilde{p}_{ij}^{\sigma(s)} := \frac{\overline{p}_{ik}^{\sigma(s)} + \overline{p}_{kj}^{\sigma(s)}}{2} \geq \frac{1}{2}, \tag{21}$$

to keep account for all $k = 1, 2, \ldots, n$ and keeping in mind $\sum_{s=1}^{\#\overline{h}_{ij}} \overline{p}_{ij}^{\sigma(s)} = 1$ we modify (21) as

$$\widetilde{p}_{ij}^{\sigma(s)} = \frac{\left(\overline{p}_{i1}^{\sigma(s)} + \overline{p}_{1j}^{\sigma(s)}\right)/2 + \left(\overline{p}_{i2}^{\sigma(s)} + \overline{p}_{2j}^{\sigma(s)}\right)/2 + \cdots + \left(\overline{p}_{in}^{\sigma(s)} + \overline{p}_{nj}^{\sigma(s)}\right)/2}{n}. \tag{22}$$

Hence $\sum_{s=1}^{\#\overline{h}_{ij}} \overline{p}_{ij}^{\sigma(s)} = 1$ and $\widetilde{p}_{ij}^{\sigma(s)} = \widetilde{p}_{ji}^{\sigma(s)}$ and if $\overline{p}_{ik}^{\sigma(s)} \geq 1/2$ and $\overline{p}_{kj}^{\sigma(s)} \geq 1/2$ for all $k = 1, 2, \ldots, n$ then surely $\widetilde{p}_{ij}^{\sigma(s)} \geq 1/2$. But, it is possible that $\overline{p}_{ik}^{\sigma(s)} \geq 1/2$ and $\overline{p}_{kj}^{\sigma(s)} \geq 1/2$ are not true for some k that will lead to a situation where $\widetilde{p}_{ik}^{\sigma(s)} \geq (1/2) \wedge \widetilde{\widetilde{p}}_{kj}^{\sigma(s)} \geq 1/2$ and $\widetilde{p}_{ij}^{\sigma(s)} < 1/2$. Now if another convex combination is calculated by (22) then obtained probability will increase. These observations lead to the following novel algorithm producing a sequence of HPFPRs convergent to fully consistent HPFPR.

Algorithm 11 (consistent HPFPR calculator).

Input. HPFPR H and optimized parameter ζ.

Output. NHPFPR $\overline{H}$, consistent HPFPR $\widetilde{H}$, and number of iterations t.

Step 1. Compute NHPFPR $\overline{H}$ by (10) or (11). Let $t = 0$ and $\widetilde{H}^{(0)} = (h_{ij}^{(0)})_{n\times n}$ be defined as

$$\widetilde{h}_{ij}^{(0)} = \left\{ \frac{1}{n}\sum_{k=1}^{n}\left(\overline{h}_{ik}^{\sigma(s)} - \overline{h}_{jk}^{\sigma(s)}\right) + \frac{1}{2} \mid \overline{p}_{ij}^{\sigma(s)} : s = 1, 2, 3, \ldots, d \right\}. \tag{23}$$

Step 2. If the following condition is true, then go to Step 4; otherwise, go to Step 3.

$$\left(\widetilde{p}_{ik}^{(t)}\right)^{\sigma(s)} \geq \frac{1}{2} \wedge \left(\widetilde{p}_{kj}^{(t)}\right)^{\sigma(s)} \geq \frac{1}{2} \Longrightarrow \left(\widetilde{p}_{ij}^{(t)}\right)^{\sigma(s)} \geq \frac{1}{2}, \quad \forall \left(i, j, k = 1, 2, \ldots, n\right). \tag{24}$$

Step 3. $\widetilde{H}^{(t+1)} = (\widetilde{h}_{ij}^{(t+1)})_{n\times n}$ is defined as

$$\widetilde{h}_{ij}^{(t+1)} = \left\{ \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)} \mid \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\left(\widetilde{p}_{ik}^{(t)}\right)^{\sigma(s)} + \left(\widetilde{p}_{kj}^{(t)}\right)^{\sigma(s)}}{2}\right) : s = 1, 2, 3, \ldots, d \right\}; \tag{25}$$

put $t = t + 1$. Go to Step 2.

Step 4. Output NHPFPR $\overline{H}$, consistent HPFPR $\widetilde{H}^{(t)}$, and number of iterations t.

Step 5. End.

Proposition 12. *Let $H = (h_{ij})_{n\times n}$ be a HPFPR with its NHFPR $\overline{H} = (\overline{h}_{ij})_{n\times n}$ with optimized parameter ζ. Then $\widetilde{H} = (\widetilde{h}_{ij})_{n\times n}$ output of Algorithm 11 is consistent HPFPR.*

Proof. Definition 10, (23) and (24) directly imply this proposition. □

This result also gives the following theorem.

Proposition 13. *Consider a HPFPR $H = (h_{ij})_{n\times n}$, its NHFPR $\overline{H} = (\overline{h}_{ij})_{n\times n}$, and $\widetilde{H} = (\widetilde{h}_{ij})_{n\times n}$ consistent HPFPR with optimized parameter ζ. Then, H is consistent if and only if $\overline{H} = \widetilde{H}$.*

The above algorithm is quite efficient; to see this fact, we generate 1000 random HPFPRs with different values of n, d and apply Algorithm 11 to find their consistent HPFPRs. Table 1 shows the average value of the number of iterations in Algorithm 11.

Example 14. Take H_1, H_2 and their optimized parameter the same as in Example 9. Then by Algorithm 11, the following consistent HPFPRs $\widetilde{H}_1$ and $\widetilde{H}_2$ are obtained.

$$\widetilde{H}_1 = \begin{pmatrix} \{.5 \mid 0.2574, .5 \mid 0.2894, .5 \mid 0.4532\} & \{0.3173 \mid 0.2567, 0.3325 \mid 0.2882, 0.3959 \mid 0.4551\} & \{0.1088 \mid 0.2576, 0.124 \mid 0.2891, 0.4214 \mid 0.4533\} & \{0.1535 \mid 0.2575, 0.1839 \mid 0.2895, 0.7684 \mid 0.453\} \\ \{0.6827 \mid 0.2567, 0.6675 \mid 0.2882, 0.6041 \mid 0.4551\} & \{.5 \mid 0.2559, .5 \mid 0.287, .5 \mid 0.457\} & \{0.2915 \mid 0.2569, 0.2915 \mid 0.2879, 0.5256 \mid 0.4552\} & \{0.3361 \mid 0.2568, 0.3513 \mid 0.2883, 0.8725 \mid 0.4549\} \\ \{0.8912 \mid 0.2576, 0.876 \mid 0.2891, 0.5786 \mid 0.4533\} & \{0.7085 \mid 0.2569, 0.7085 \mid 0.2879, 0.4744 \mid 0.4552\} & \{.5 \mid 0.2578, .5 \mid 0.2888, .5 \mid 0.4534\} & \{0.5447 \mid 0.2577, 0.5599 \mid 0.2892, 0.8469 \mid 0.4531\} \\ \{0.8465 \mid 0.2575, 0.8161 \mid 0.2895, 0.2316 \mid 0.453\} & \{0.6639 \mid 0.2568, 0.6487 \mid 0.2883, 0.1275 \mid 0.4549\} & \{0.4553 \mid 0.2577, 0.4401 \mid 0.2892, 0.1531 \mid 0.4531\} & \{.5 \mid 0.2576, .5 \mid 0.2897, .5 \mid 0.4527\} \end{pmatrix},$$

$$\widetilde{H}_2 = \begin{pmatrix} \{.5 \mid 0.4915,\ .5 \mid 0.3192,\ .5 \mid 0.1893\} & \{0.4042 \mid 0.4702, 0.4603 \mid 0.3303, 0.7095 \mid 0.1995\} & \{0.4035 \mid 0.4631, 0.5519 \mid 0.2889, 0.6658 \mid 0.248\} & \{0.1978 \mid 0.4228, 0.3906 \mid 0.3291, 0.7925 \mid 0.2481\} \\ \{0.5958 \mid 0.4702, 0.5397 \mid 0.3303, 0.2905 \mid 0.1995\} & \{.5 \mid 0.449,\ .5 \mid 0.3413,\ .5 \mid 0.2097\} & \{0.4994 \mid 0.4418, 0.5916 \mid 0.2999, 0.4563 \mid 0.2582\} & \{0.2937 \mid 0.4015, 0.4303 \mid 0.3402, 0.5829 \mid 0.2583\} \\ \{0.5965 \mid 0.4631, 0.4481 \mid 0.2889, 0.3342 \mid 0.248\} & \{0.5006 \mid 0.4418, 0.4084 \mid 0.2999, 0.5437 \mid 0.2582\} & \{.5 \mid 0.4347,\ .5 \mid 0.2585,\ .5 \mid 0.3068\} & \{0.2943 \mid 0.3944, 0.3387 \mid 0.2987, 0.6267 \mid 0.3069\} \\ \{0.8022 \mid 0.4228, 0.6094 \mid 0.3291, 0.2075 \mid 0.2481\} & \{0.7063 \mid 0.4015, 0.5697 \mid 0.3402, 0.4171 \mid 0.2583\} & \{0.7057 \mid 0.3944, 0.6613 \mid 0.2987, 0.3733 \mid 0.3069\} & \{.5 \mid 0.354,\ .5 \mid 0.339,\ .5 \mid 0.307\} \end{pmatrix}. \quad (26)$$

Remark 15. To see the consistency of HPFPR geometrically, three area graphs of fuzzy preference degrees $[h_{ij}^{\sigma(s)}]$, probabilities of preference degrees $[p_{ij}^{\sigma(s)}]$, and score values $[h_{ij}^{\sigma(s)} * p_{ij}^{\sigma(s)}]$ are made. The procedure to make these graphs is explained for $\widetilde{H}_1$, as follows.

Three matrices , P, and S of order 4×12 of fuzzy preference values, probability values, and score values are made from HPFPR $\widetilde{H}_1$,

$$F = \begin{bmatrix} .5 & .5 & .5 & .3959 & .3325 & .3173 & .4214 & .124 & .1088 & .7684 & .1839 & .1535 \\ .6827 & 0.6675 & .6041 & .5 & .5 & .5 & .5256 & .2915 & .2915 & .8725 & .3513 & .3361 \\ .8912 & .876 & .5786 & .7085 & .7085 & .4744 & .5 & .5 & .5 & .8469 & .5599 & .5447 \\ .8465 & .8161 & .2316 & .6639 & .6487 & .1275 & .4553 & .4401 & .1531 & .5 & .5 & .5 \end{bmatrix},$$

$$P = \begin{bmatrix} .4532 & .2894 & .2573 & .4551 & .2882 & .2567 & .4533 & .2891 & .2576 & .453 & .2895 & .2575 \\ .4551 & .2882 & .2567 & .457 & .287 & .2559 & .4552 & .2879 & .2569 & .4549 & .2883 & .2568 \\ .4533 & .2891 & .2576 & .4552 & .2879 & .2569 & .4534 & .2888 & .2578 & .4531 & .2892 & .2577 \\ .453 & .2895 & .2575 & .4549 & .2883 & .2568 & .4531 & .2892 & .2577 & .4527 & .2897 & .2576 \end{bmatrix}, \quad (27)$$

$$S = \begin{bmatrix} .2266 & .1447 & .1287 & .1802 & .0958 & .0814 & .191 & .0359 & .028 & .348 & .0532 & .0395 \\ .2749 & .1924 & .1752 & .2285 & .1435 & .128 & .2392 & .0839 & .0749 & .3969 & .1013 & .0863 \\ .2623 & .2533 & .2295 & .216 & .204 & .0182 & .2267 & .1444 & .1289 & .3837 & .1619 & .1403 \\ .2363 & .218 & .1049 & .187 & .1705 & .058 & .1273 & .1173 & .0694 & .2264 & .1448 & .1288 \end{bmatrix}.$$

The area graphs are made of the above matrices by using Matlab drawing tool bar. Figures 1, 2, and 3 and Figures 4, 5, and 6 show the comparison of area graphs for fuzzy preference degrees, probability values, and score values between $\overline{H}_1$, $\widetilde{H}_1$ and $\overline{H}_2$, $\widetilde{H}_2$, respectively. The areas are more smooth for consistent HPFPRs $\widetilde{H}_1$ and $\widetilde{H}_2$.

Once the consistent HPFPR is found, we are in a position to make HPFPR acceptably consistent by defining consistency measure. In order to define consistency measure, first distance between two HPFEs is defined. Let $h_1 = \{\gamma_1^s \mid p_1^s : s = 1, 2, 3, \ldots, \#h\}$ and $h_2 = \{\gamma_2^s \mid p_2^s : s = 1, 2, 3, \ldots, \#h_2\}$ with $\#h_1 = \#h_2 = \#h$. Then

$$D(h_1, h_2) = \frac{\sum_{\gamma_1^{\sigma(s)} | p_1^{\sigma(s)} \in h_1, \gamma_2^{\sigma(s)} | p_2^{\sigma(s)} \in h_2} \max\left(\left|\gamma_1^{\sigma(s)} - \gamma_2^{\sigma(s)}\right|, \left|p_1^{\sigma(s)} - p_2^{\sigma(s)}\right|\right)}{\#h}. \quad (28)$$

Based on the above equation the distance between two HPFPRs is defined as follows.

Definition 16 (distance). Consider two HPFPRs $H_1 = (h_{ij,1})_{n\times n}$ and $H_2 = (h_{ij,2})_{n\times n}$ and their NHPFPRs $\overline{H}_1 = (\overline{h}_{ij,1})_{n\times n}$ and $\overline{H}_2 = (\overline{h}_{ij,2})_{n\times n}$. The distance is defined as

$$D(H_1, H_2) = \frac{2}{n(n+1)} \sum_{i\leq j}^{n} D\left(\overline{h}_{ij,1}, \overline{h}_{ij,2}\right). \quad (29)$$

It is clear that the following properties are satisfied for $D(H_1, H_2)$:

(1) $0 \leq D(H_1, H_2) \leq 1$;

(2) $D(H_1, H_2) = D(H_2, H_1)$;

(3) $H_1 = H_2$ if and only if $D(H_1, H_2) = 0$.

While the decision maker provides its preference in form of HPFPR, it should be noted that it can be used for decision making with good results only if it has sufficient consistency.

Definition 17 (consistency index). For a given HPFPR $H = (h_{ij})_{n\times n}$, its NHPFPR $\overline{H} = (\overline{h}_{ij})_{n\times n}$ and consistent HPFPR $\widetilde{H} = (\widetilde{h}_{ij})_{n\times n}$ with optimized parameter $0 \leq \zeta \leq 1$ obtained from Algorithm 11. The consistency index of H is defined to be the distance between $\overline{H}$ and $\widetilde{H}$, denoted as

$$\mathrm{CI}(H) = D\left(\overline{H}, \widetilde{H}\right). \quad (30)$$

It is clear that $\mathrm{CI}(H) = 0$ if and only if H is consistent. Ideally, the decision maker should provide consistent HPFPR

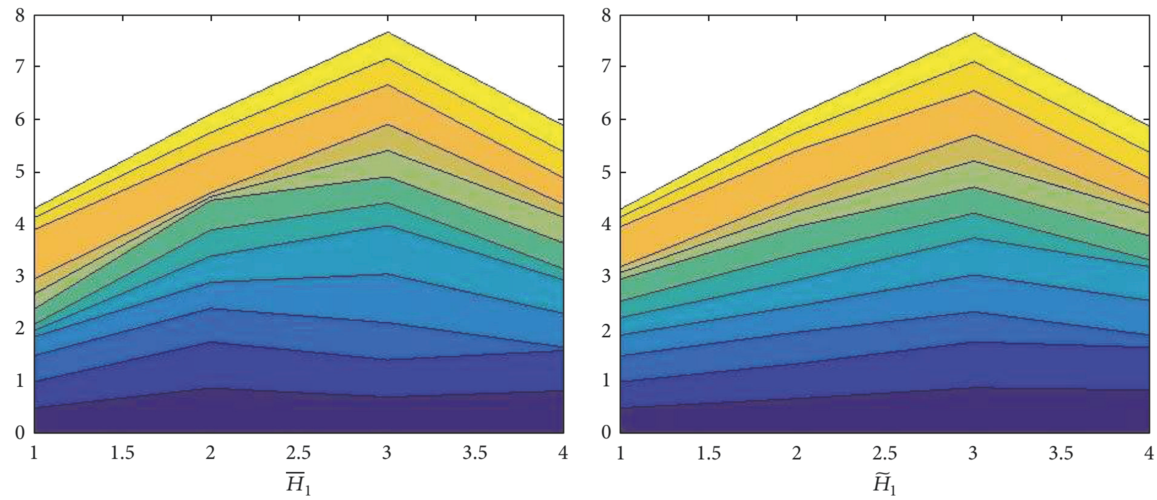


Figure 1: Area graphs of fuzzy preference degrees for H_1.

Table 1: Average value of iterations in Algorithm 11.

n	d	$\zeta = 0$	$\zeta = 1$
4	4	1.684	1.679
	3	1.357	1.373
	2	1.068	1.069
5	5	2.527	2.429
	4	1.988	1.958
	3	1.405	1.439
6	6	2.924	2.948
	5	2.824	2.796
	4	2.242	2.25
7	7	2.664	2.839
	6	3.14	3.17
	5	3.148	3.168
8	8	2.168	2.192
	7	2.777	2.732
	6	3.254	3.38
9	9	1.65	1.644
	8	2.067	2.005
	7	2.634	2.653
10	10	1.301	1.345
	9	1.56	1.496
	8	1.884	1.825

so that it can be used for meaningful decision making. However, some margin of error should be provided to decision maker relative to the practical problems.

Definition 18 (acceptably consistent HPFPR). Consider a HPFPR $H = (h_{ij})_{n\times n}$. For a given tolerance value CI_r HPFPR H is said to be acceptably consistent HPFPR if

$$\mathrm{CI}(H) \le \mathrm{CI}_r. \tag{31}$$

If a HPFPR is not even acceptably consistent, then the decision maker should revisit and modify it. A novel algorithm is proposed to make a HPFPR acceptably consistent.

Algorithm 19 (acceptably consistent HPFPR calculator).

Input. The HPFPR $H = (h_{ij})_{n\times n}$, the consistency tolerance value CI_r, and the parameter $\alpha \in (0,1)$.

Output. The acceptably consistent HPFPR $H^{(t)}$, consistency index $\mathrm{CI}(H^{(t)})$, and number of iterations value t.

Step 1. Compute $\overline{H} = (\overline{h}_{ij})_{n\times n}$ with optimized parameter ζ. Let $t = 0$, $H^{(0)} = (h_{ij}^{(0)})_{n\times n} = \overline{H} = (\overline{h}_{ij})_{n\times n}$.

Step 2. Calculate the consistent HPFPR $\widetilde{H}^{(t)} = (\widetilde{h}_{ij}^{(t)})_{n\times n}$ by applying Algorithm 11 to $H^{(t)}$ and consistency index $\mathrm{CI}(H^{(t)})$, where

$$\mathrm{CI}\left(H^{(t)}\right) = \frac{2}{n(n+1)}\sum_{i\le j}^{n} D\left(h_{ij}^{(t)}, \widetilde{h}_{ij}^{(t)}\right). \tag{32}$$

Step 3. If $\mathrm{CI}(H^{(t)}) \le \mathrm{CI}_r$, then go to Step 5; otherwise, go to Step 4.

Step 4. Make the adjusted HPFPR $H^{(t+1)} = (h_{ij}^{(t+1)})$, where

$$h_{ij}^{(t+1)} = \alpha h_{ij}^{(t)} \oplus (1-\alpha)\,\widetilde{h}_{ij}^{(t)}. \tag{33}$$

Let $t = t+1$; now return to Step 2.

Step 5. Output , $H^{(t)}$, and $\mathrm{CI}(H^{(t)})$.

Step 6. End.

This proposed algorithm will be convergent as the following result shows.

Theorem 20. *Consider a HPFPR H, consistency tolerance value CI_r, and the sequence $\{H^{(t)}\}$ of HPFPRs generated by Algorithm 19. Then*

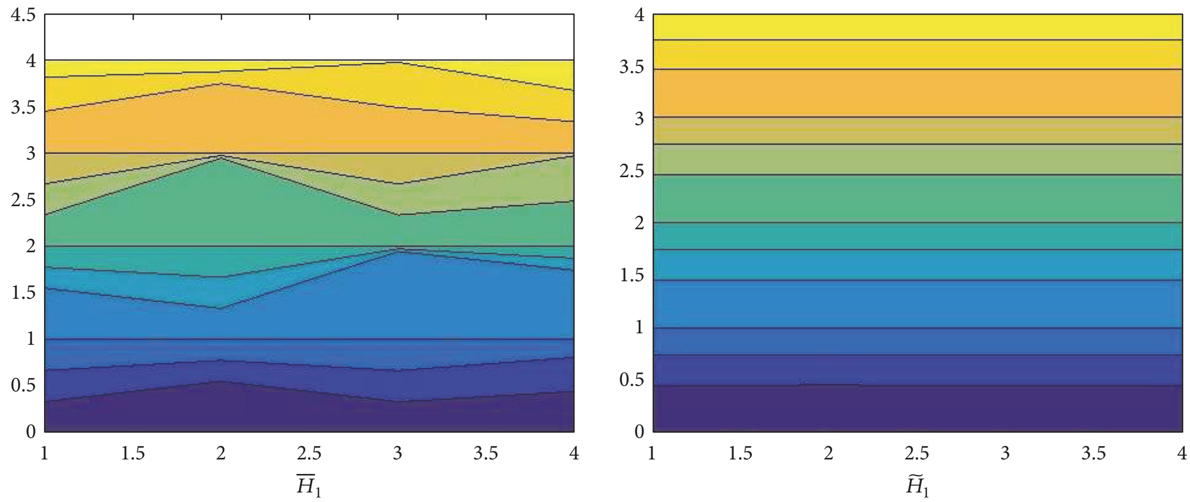


FIGURE 2: Area graphs of probabilities for H_1.

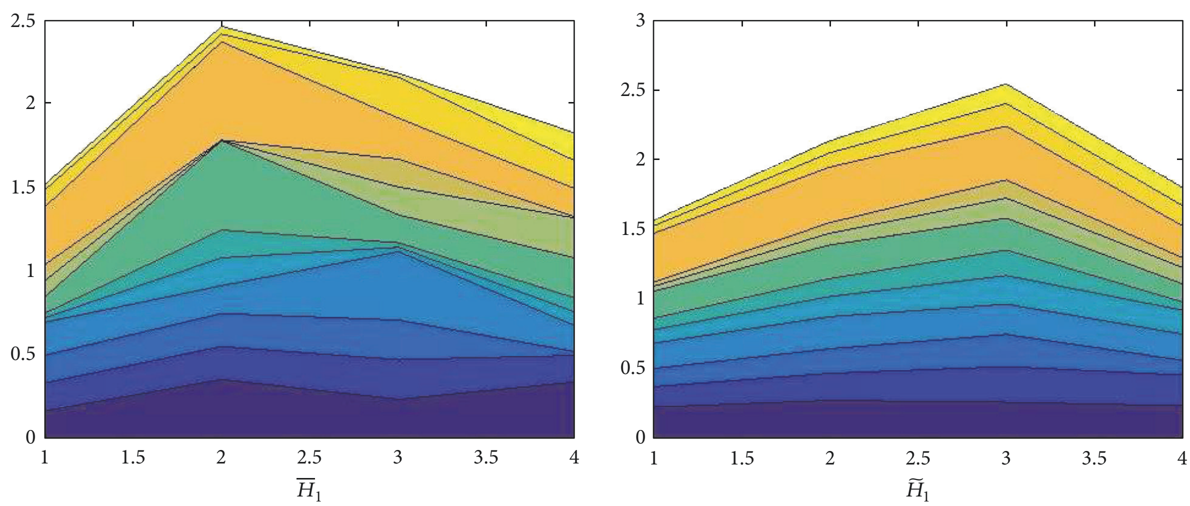


FIGURE 3: Area graphs of score values for H_1.

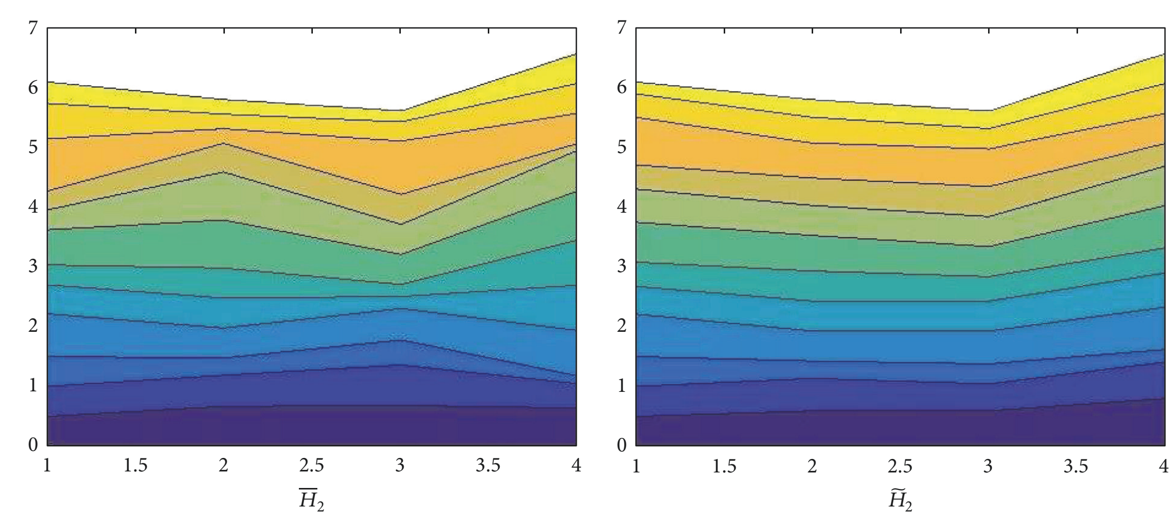


FIGURE 4: Area graphs of fuzzy preference degrees for H_2.

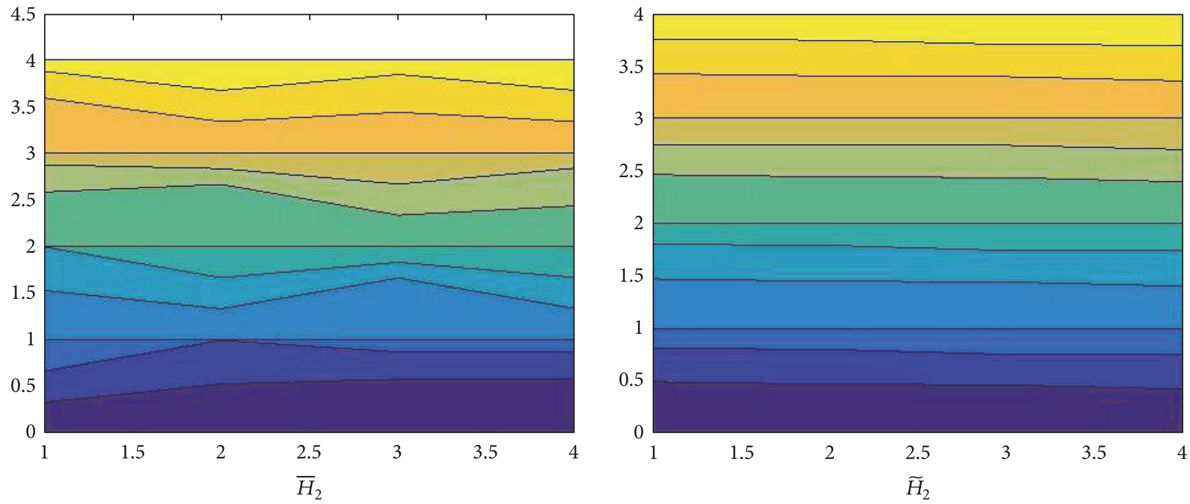


FIGURE 5: Area graphs of probabilities for H_2.

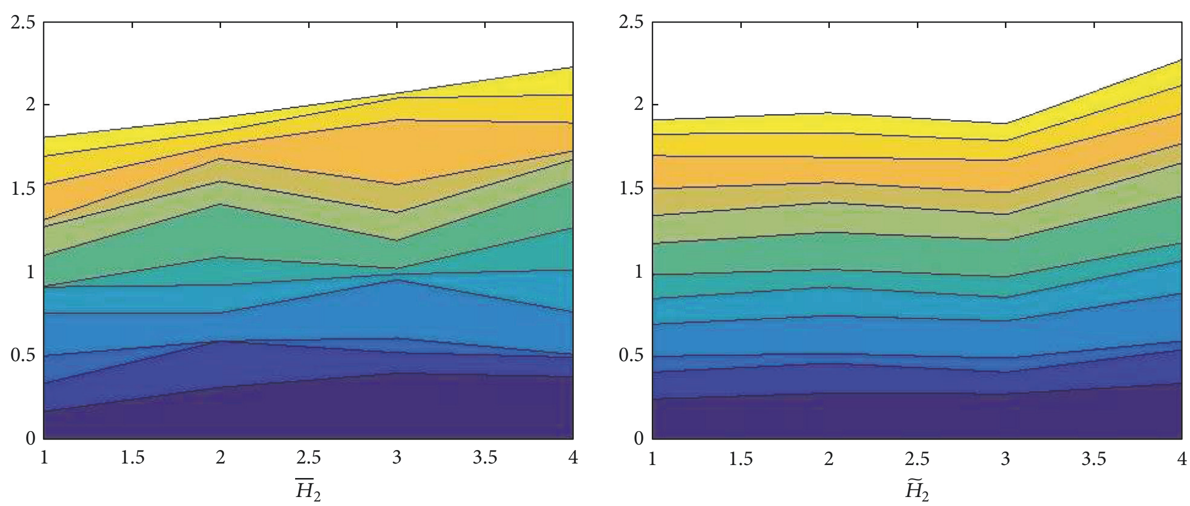


FIGURE 6: Area graphs of score values for H_2.

$$CI\left(H^{(t+1)}\right) < CI\left(H^{(t)}\right) \quad \forall t, \qquad \lim_{t\to\infty} CI\left(H^{(t)}\right) = 0. \tag{34}$$

Proof. From (23) and (33), it follows that

$$\begin{aligned}\left(\widetilde{h}_{ij}^{(t+1)}\right)^{\sigma(s)} &= \frac{1}{n}\sum_{k=1}^{n}\left(\left(h_{ik}^{(t+1)}\right)^{\sigma(s)} - \left(h_{jk}^{(t+1)}\right)^{\sigma(s)}\right) + \frac{1}{2} \\ &= \frac{1}{n}\sum_{k=1}^{n}\left(\alpha\left(h_{ik}^{(t)}\right)^{\sigma(s)} + (1-\alpha)\left(\widetilde{h}_{ik}^{(t)}\right)^{\sigma(s)} - \alpha\left(h_{jk}^{(t)}\right)^{\sigma(s)} - (1-\alpha)\left(\widetilde{h}_{jk}^{(t)}\right)^{\sigma(s)}\right) + \frac{1}{2} = \alpha\left(\frac{1}{n}\cdot\sum_{k=1}^{n}\left(\left(h_{ik}^{(t)}\right)^{\sigma(s)} - \left(h_{jk}^{(t)}\right)^{\sigma(s)}\right) + \frac{1}{2}\right) + (1-\alpha)\left(\frac{1}{n}\cdot\sum_{k=1}^{n}\left(\left(\widetilde{h}_{ik}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{h}_{jk}^{(t)}\right)^{\sigma(s)}\right) + \frac{1}{2}\right) = \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)};\end{aligned} \tag{35}$$

also

$$\left(h_{ij}^{(t+1)}\right)^{\sigma(s)} = \alpha\left(h_{ij}^{(t)}\right)^{\sigma(s)} + (1-\alpha)\left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)}. \tag{36}$$

Thus,

$$\begin{aligned}&\left|\left(h_{ij}^{(t+1)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t+1)}\right)^{\sigma(s)}\right| \\ &\quad = \left|\alpha\left(h_{ij}^{(t)}\right)^{\sigma(s)} + (1-\alpha)\left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)}\right|\end{aligned}$$

$$\begin{aligned} &= \alpha \left| \left(h_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)} \right| \\ &< \left| \left(h_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)} \right| ; \end{aligned} \tag{37}$$

therefore,

$$\begin{aligned} &\lim_{t\to\infty} \left| \left(h_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t)}\right)^{\sigma(s)} \right| \\ &\quad = \lim_{t\to\infty} \alpha \left| \left(h_{ij}^{(t-1)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(t-1)}\right)^{\sigma(s)} \right| = \cdots \\ &\quad = \lim_{t\to\infty} \alpha^{t} \left| \left(h_{ij}^{(0)}\right)^{\sigma(s)} - \left(\widetilde{h}_{ij}^{(0)}\right)^{\sigma(s)} \right| = 0. \end{aligned} \tag{38}$$

By (33),

$$\left(p_{ij}^{(t)}\right)^{\sigma(s)} < \left(p_{ij}^{(t+1)}\right)^{\sigma(s)} < \left(\widetilde{p}_{ij}^{(t)}\right)^{\sigma(s)}, \tag{39}$$

with each iteration $(p_{ij}^{(t)})^{\sigma(s)}$, will come closer to $(\widetilde{p}_{ij}^{(t)})^{\sigma(s)}$. As $\widetilde{H}^{(t)}$ is consistent; therefore, each iteration will make $H^{(t)}$ more consistent; that is, if $((p_{ik}^{(t)})^{\sigma(s)} \geq (1/2) \wedge (p_{kj}^{(t)})^{\sigma(s)} \geq (1/2) \wedge (p_{ij}^{(t)})^{\sigma(s)} < 1/2)$, then by (33) and (25)

$$\left| \left(p_{ij}^{(t+1)}\right)^{\sigma(s)} - \left(\widetilde{p}_{ij}^{(t+1)}\right)^{\sigma(s)} \right| < \left| \left(p_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{p}_{ij}^{(t)}\right)^{\sigma(s)} \right|, \tag{40}$$

and each iteration will increase $(p_{ij}^{(t)})^{\sigma(s)}$; thus eventually it will be $\geq 1/2$. Hence $H^{(t)}$ will become consistent when $t \to \infty$; that is,

$$\lim_{t\to\infty} \left| \left(p_{ij}^{(t)}\right)^{\sigma(s)} - \left(\widetilde{p}_{ij}^{(t)}\right)^{\sigma(s)} \right| = 0. \tag{41}$$

Finally, (37) and (40) imply

$$\begin{aligned} &\mathrm{CI}\left(H^{(t+1)}\right) < \mathrm{CI}\left(H^{(t)}\right) \quad \forall t, \\ &\lim_{t\to\infty} \mathrm{CI}\left(H^{(t)}\right) = 0, \end{aligned} \tag{42}$$

following from (38) and (41). □

The choice of parameters n, d, α directly affects the performance of Algorithm 19. For performance measurement of Algorithm 19, 1000 random HPFPRs are generated and their acceptably consistent HPFPRs are computed by Algorithm 19, the average iteration in Algorithm 19 with respect to different parameters shown in Table 2. It is apparent that Algorithm 19 is quite efficient and, as for the effects of parameters, the increase in value α leads to more iteration in Algorithm 19 to compute consistent HPFPR; therefore, it is suggested to choose α small. Also, the number of iterations in Algorithm 19 is inversely proportional to the consistency index CI_r.

4. Consensus Measure in Group Decision Making

For group decision making, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives and consider $E = \{e_1, e_2, e_3, \ldots, e_m\}$, the

Table 2: Average iterations values in Algorithm 19.

n	d	CI_r	α			
			0.1	0.3	0.6	0.8
		0.05	1.001	1.939	3.226	6.88
	4	0.1	0.996	0.999	1.898	3.659
		0.15	0.939	0.959	0.998	1.824
		0.05	1.001	1.939	3.528	7.326
4	3	0.1	0.991	0.997	2	4.084
		0.15	0.954	0.95	1.208	2.169
		0.05	1.05	1.882	3.527	7.415
	2	0.1	0.982	1.054	2.089	4.1
		0.15	0.868	0.891	1.297	2.404
		0.05	1	1.976	3.095	6.851
	5	0.1	1	1	1.978	3.71
		0.15	0.989	0.994	1.017	1.854
		0.05	1.004	1.992	3.418	7.266
5	4	0.1	1	1	1.989	4.021
		0.15	0.999	0.997	1.123	2.214
		0.05	1.001	1.992	3.767	7.709
	3	0.1	1	1.002	2.123	4.43
		0.15	0.994	0.997	1.403	2.575
		0.05	1	1.994	3.028	6.776
	6	0.1	1	1	1.991	3.654
		0.15	1	1	1	1.818
		0.05	1.001	1.999	3.2	7.025
6	5	0.1	1	1	1.997	3.942
		0.15	1	1	1.016	2.074
		0.05	1.001	2	3.606	7.446
	4	0.1	1	1	2.011	4.249
		0.15	0.999	1	1.178	2.443
		0.05	1	1.993	3.009	6.659
	7	0.1	1	1	1.991	3.487
		0.15	0.998	1	0.999	1.732
		0.05	1	1.998	3.05	6.912
7	6	0.1	1	1	1.999	3.816
		0.15	1	1	1.001	1.972
		0.05	1	2	3.261	7.125
	5	0.1	1	1	2	4.023
		0.15	1	1	1.022	2.168
		0.05	1	1.993	3.002	6.526
	8	0.1	1	1	1.984	3.317
		0.15	1	1	1	1.599
		0.05	1	2	3.009	6.831
8	7	0.1	1	1	1.997	3.658
		0.15	1	1	1	1.867
		0.05	1	2	3.078	7.003
	6	0.1	1	1	2	3.927
		0.15	1	1	1	2.026
		0.05	1	1.993	3	6.396
	9	0.1	1	1	1.978	3.229
		0.15	1	1	1	1.474
		0.05	1	2	3	6.676
9	8	0.1	1	1	1.998	3.47
		0.15	1	1	1	1.74
		0.05	1	2	3.01	6.92
	7	0.1	1	1	2	3.786
		0.15	1	1	1	1.942

TABLE 2: Continued.

n	d	CI_r	α 0.1	0.3	0.6	0.8
		0.05	1	1.986	3	6.326
	10	0.1	1	1	1.973	3.164
		0.15	1	1	1	1.367
		0.05	1	1.994	3	6.543
10	9	0.1	1	1	1.995	3.308
		0.15	1	1	1	1.58
		0.05	1	2	3.002	6.798
	8	0.1	1	1	2	3.618
		0.15	1	1	1	1.864

set of decision makers providing their preferences in HPFPRs $H_1 = (h_{ij,1}), H_2 = (h_{ij,2}), \ldots, H_m = (h_{ij,m})$, respectively. Also, let $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ be the importance/weight vector of decision makers in normalized form; that is, $\sum_{i=1}^{m} \omega_i = 1$. The Algorithm 19 provides the solution of consistency issue. In order to make group decision, the following aggregation operator is defined, which will be used to fuse all respective NHPFPRs of decision makers.

Definition 21. Take a collection of HPFEs h_i $(i = 1, 2, 3, \ldots, n)$ with equal length and their weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in normalized form; that is, $\sum_{i=1}^{n} \lambda_i = 1$. The hesitant probabilistic fuzzy weighted averaging (HPFWA) operator is defined as follows:

$$\begin{aligned}\mathrm{HPFWA}(h_1, h_2, \ldots, h_n) &= \bigoplus_{i=1}^{n} (\lambda_i h_i) \\ &= \bigcup_{\gamma_1^{\sigma(s)} | p_1^{\sigma(s)} \in h_1, \gamma_2^{\sigma(s)} | p_2^{\sigma(s)} \in h_2, \ldots, \gamma_n^{\sigma(s)} | p_n^{\sigma(s)} \in h_n} \left\{ \sum_{i=1}^{n} \lambda_i \gamma_i^{\sigma(s)} \mid \sum_{i=1}^{n} \lambda_i p_i^{\sigma(s)} \right\}.\end{aligned} \tag{43}$$

Also, if we take weight vector $\lambda = (1/n, 1/n, \ldots, 1/n)$ then the above operator is reduced to hesitant probabilistic fuzzy averaging (HPFA) operator as

$$\begin{aligned}\mathrm{HPFA}(h_1, h_2, \ldots, h_n) &= \bigoplus_{i=1}^{n} \left(\frac{1}{n} h_i\right) \\ &= \bigcup_{\gamma_1^{\sigma(s)} | p_1^{\sigma(s)} \in h_1, \gamma_2^{\sigma(s)} | p_2^{\sigma(s)} \in h_2, \ldots, \gamma_n^{\sigma(s)} | p_n^{\sigma(s)} \in h_n} \left\{ \sum_{i=1}^{n} \frac{1}{n} \gamma_i^{\sigma(s)} \mid \sum_{i=1}^{n} \frac{1}{n} p_i^{\sigma(s)} \right\}.\end{aligned} \tag{44}$$

Theorem 22 (boundedness). *If HPFEs h_i $(i = 1, 2, 3, \ldots, n)$ with equal length and their weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in normalized form, that is, $\sum_{i=1}^{n} \lambda_i = 1$, then*

$$\begin{aligned}&\min(\lambda_1 h_1, \lambda_2 h_2, \ldots, \lambda_n h_n) \\ &\quad \leq HPFWA(h_1, h_2, \ldots, h_n) \\ &\quad \leq (\max(\lambda_1 h_1, \lambda_2 h_2, \ldots, \lambda_n h_n)).\end{aligned} \tag{45}$$

Proof. From (5), (6), and (43), it is easy to yield the desired result. □

Theorem 23 (commutativity). *If HPFEs h_i $(i = 1, 2, 3, \ldots, n)$ is a permutation of h_i' $(i = 1, 2, 3, \ldots, n)$ with equal length and weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of h_i is a permutation of $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_n')$ of h_i' in normalized form, that is, $\sum_{i=1}^{n} \lambda_i = 1$ and $\sum_{i=1}^{n} \lambda_i' = 1$, then*

$$HPFWA(h_1, h_2, \ldots, h_n) = HPFWA(h_1', h_2', \ldots, h_n'). \tag{46}$$

Proof. Since multiplication is commutative, so it is easy to proof that HPFWA operator is also commutative. □

Theorem 24 (monotonicity). *If HPFEs $h_i \leq h_i'$ for $i = 1, 2, 3, \ldots, n$ with equal length and weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, that is, $\sum_{i=1}^{n} \lambda_i = 1$, then*

$$HPFWA(h_1, h_2, \ldots, h_n) \leq HPFWA(h_1', h_2', \ldots, h_n'). \tag{47}$$

Proof. Since $h_i \leq h_i'$ and $\lambda_i \in [0, 1]$, then $\lambda_i h_i \leq \lambda_i h_i'$.

Hence $\mathrm{HPFWA}(h_1, h_2, \ldots, h_n) \leq \mathrm{HPFWA}(h_1', h_2', \ldots, h_n')$. □

Proposition 25. *Take $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, \ldots, m)$ HPFPRs given by decision makers and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weight vector of decision makers with $\sum_{i=}^{m} \omega_i = 1$. The NHPFPRs $\overline{H}_k = (\overline{h}_{ij,k})$ are computed with optimized parameters ζ_k $(k = 1, 2, \ldots, m)$. Then the HPFPR $H_g = (h_{ij,g})_{n\times n}$ calculated as*

$$H_g = \left(\bigoplus_{k=1}^{m} \omega_k \overline{h}_{ij,k}\right)_{n\times n} \tag{48}$$

is HPFPR.

Proof. It follows from (3), (7), and (43). □

The next result shows that the aggregated group HPFPR H_g obtained by (48) is consistent, provided all the individual HPFPRs are consistent.

Theorem 26. *Consider $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ HPFPRs given by decision makers and group HPFPR H_g is computed by (48). Then*

$$CI(H_g) \leq \max_{k} \{CI(H_k)\}. \tag{49}$$

Proof. Let $\widetilde{H}_g^{(t)} = (\widetilde{h}_{ij,g}^{(t)})$ and $\widetilde{H}_k^{(t)} = (\widetilde{h}_{ij,k}^{(t)})$ $(k = 1, 2, \ldots, m)$ be the sequences of HPFPRs generated by Algorithm 11 applied to H_g and H_k $(k = 1, 2, \ldots, m)$, respectively. By (28)

$$D\left(h_{ij,g},\widetilde{h}_{ij,g}^{(t)}\right)=\frac{\sum_{h_{ij,g}^{\sigma(s)}|p_{ij,g}^{\sigma(s)}\in h_{ij,g},\widetilde{h}_{ij,g}^{\sigma(s)}|\widetilde{p}_{ij,g}^{\sigma(s)}\in\widetilde{h}_{ij,g}}\max\left(\left|h_{ij,g}^{\sigma(s)}-\widetilde{h}_{ij,g}^{\sigma(s)}\right|,\left|p_{ij,g}^{\sigma(s)}-\left(\widetilde{p}_{ij,g}^{(t)}\right)^{\sigma(s)}\right|\right)}{d}. \tag{50}$$

Now

$$\begin{aligned}
&\left|h_{ij,g}^{\sigma(s)}-\left(\widetilde{h}_{ij,g}^{(t)}\right)^{\sigma(s)}\right| \\
&=\left|\sum_{k=1}^{m}\omega_k\overline{h}_{ij,k}^{\sigma(s)}-\frac{1}{n}\sum_{l=1}^{n}\left(h_{il,g}^{\sigma(s)}-h_{jl,g}^{\sigma(s)}\right)+\frac{1}{2}\right| \\
&=\left|\sum_{k=1}^{m}\omega_k\overline{h}_{ij,k}^{\sigma(s)}-\frac{1}{n}\sum_{l=1}^{n}\left(\sum_{k=1}^{m}\omega_k\overline{h}_{il,k}^{\sigma(s)}-\sum_{k=1}^{m}\omega_k\overline{h}_{jl,k}^{\sigma(s)}\right)+\frac{1}{2}\right| \\
&=\left|\sum_{k=1}^{m}\omega_k\left(\overline{h}_{ij,k}^{\sigma(s)}-\frac{1}{n}\sum_{l=1}^{n}\left(\overline{h}_{il,k}^{\sigma(s)}-\overline{h}_{jl,k}^{\sigma(s)}\right)+\frac{1}{2}\right)\right| \\
&\left|h_{ij,g}^{\sigma(s)}-\left(\widetilde{h}_{ij,g}^{(t)}\right)^{\sigma(s)}\right|=\left|\sum_{k=1}^{m}\omega_k\left(\overline{h}_{ij,k}^{\sigma(s)}-\widetilde{h}_{ij,k}^{\sigma(s)}\right)\right| \\
&\le\max_k\left|\overline{h}_{ij,k}^{\sigma(s)}-\widetilde{h}_{ij,k}^{\sigma(s)}\right|\le d\max_k D\left(\overline{h}_{ij,k},\widetilde{h}_{ij,k}^{(t)}\right).
\end{aligned} \tag{51}$$

Consider

$$\begin{aligned}
\left(\widetilde{p}_{ij,g}^{(1)}\right)^{\sigma(s)}&=\frac{1}{2n}\sum_{l=1}^{n}\left(p_{il,g}^{\sigma(s)}+p_{lj,g}^{\sigma(s)}\right) \\
&=\frac{1}{2n}\sum_{l=1}^{n}\left(\sum_{k=1}^{m}\omega_k\overline{p}_{il,k}^{\sigma(s)}+\sum_{k=1}^{m}\omega_k\overline{p}_{lj,k}^{\sigma(s)}\right) \\
&=\sum_{k=1}^{m}\omega_k\left(\frac{1}{2n}\sum_{l=1}^{n}\left(\overline{p}_{il,k}^{\sigma(s)}+\overline{p}_{lj,k}^{\sigma(s)}\right)\right) \\
&=\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{ij,k}^{(1)}\right)^{\sigma(s)}.
\end{aligned} \tag{52}$$

For the inductive step, suppose the following:

$$\left(\widetilde{p}_{ij,g}^{(r)}\right)^{\sigma(s)}=\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{ij,k}^{(r)}\right)^{\sigma(s)},\quad(i,j=1,2,\ldots,n). \tag{53}$$

Then

$$\left(\widetilde{p}_{ij,g}^{(r+1)}\right)^{\sigma(s)}=\frac{1}{2n}\sum_{l=1}^{n}\left(\left(\widetilde{p}_{il,g}^{(r)}\right)^{\sigma(s)}+\left(\widetilde{p}_{lj,g}^{(r)}\right)^{\sigma(s)}\right); \tag{54}$$

by supposition,

$$\begin{aligned}
&\left(\widetilde{p}_{ij,g}^{(r+1)}\right)^{\sigma(s)} \\
&=\frac{1}{2n}\sum_{l=1}^{n}\left(\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{il,k}^{(r)}\right)^{\sigma(s)}+\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{lj,k}^{(r)}\right)^{\sigma(s)}\right) \\
&=\sum_{k=1}^{m}\omega_k\left(\frac{1}{2n}\sum_{l=1}^{n}\left(\left(\widetilde{p}_{il,k}^{(r)}\right)^{\sigma(s)}+\left(\widetilde{p}_{lj,k}^{(r)}\right)^{\sigma(s)}\right)\right) \\
&=\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{ij,k}^{(r+1)}\right)^{\sigma(s)}.
\end{aligned} \tag{55}$$

Hence, by principle of mathematical induction for all $t\in\mathbb{N}$

$$\left(\widetilde{p}_{ij,g}^{(t)}\right)^{\sigma(s)}=\sum_{k=1}^{m}\omega_k\left(\widetilde{p}_{ij,k}^{(t)}\right)^{\sigma(s)},\quad(i,j=1,2,\ldots,n). \tag{56}$$

Therefore,

$$\begin{aligned}
\left|p_{ij,g}^{\sigma(s)}-\left(\widetilde{p}_{ij,g}^{(t)}\right)^{\sigma(s)}\right|&\le\max_k\left|\overline{p}_{ij,k}^{\sigma(s)}-\left(\widetilde{p}_{ij,k}^{(t)}\right)^{\sigma(s)}\right| \\
&\le d\max_k D\left(\overline{h}_{ij,k},\widetilde{h}_{ij,k}^{(t)}\right).
\end{aligned} \tag{57}$$

Equations (51) and (57) imply

$$D\left(h_{ij,g},\widetilde{h}_{ij,g}^{(t)}\right)\le\max_k D\left(\overline{h}_{ij,k},\widetilde{h}_{ij,k}^{(t)}\right); \tag{58}$$

thus by (30)

$$\mathrm{CI}\left(H_g\right)\le\max_k\left\{\mathrm{CI}\left(H_k\right)\right\}. \tag{59}$$

□

Now, the consensus is another very important aspect of group decision making. To deal with the consensus issue among all the individual decision makers, the following consensus index is defined.

Definition 27 (consensus index). Let $H_k=(h_{ij,k})_{n\times n}$ $(k=1,2,3,\ldots,m)$ be m HPFPRs provided by decision makers and their NHPFPRs $\overline{H}_k$ are obtained with optimized parameter ζ_k $(k=1,2,\ldots,m)$. The group HPFPR H_g is computed by (48). Then the group consensus index (GCI) of HPFPR H_k is defined to be the distance measured between $\overline{H}_k$ and H_g; that is,

$$\mathrm{GCI}\left(H_k\right)=D\left(\overline{H}_k,H_g\right). \tag{60}$$

The agreement between individual decision maker with group decision is measured by the distance between individual HPFPR H_k and group HPFPR H_g. Therefore, $\mathrm{GCI}(H_k) = 0$ means k^{th} decision maker has full agreement with group decision; otherwise, the smaller the value of $\mathrm{GCI}(H_k)$ is, the better the consensus will be. In many real life scenarios, it is important to have consensus among all decision makers, although we have to live with difference of opinion and it is hard to reach complete consensus; for this reason a threshold value can be decided based on practical nature of the problem to allow the difference of opinion to some extent.

Definition 28 (acceptably consensus HPFPRs). Let $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ be m HPFPRs provided by decision makers and their NHPFPRs $\overline{H}_k$ are obtained with optimized ζ_k $(k = 1, 2, \ldots, m)$. Furthermore compute group HPFPR H_g and $\mathrm{GCI}(H_k)$ $(k = 1, 2, \ldots, m)$. Consider GCI_r a tolerance value of consensus measure; then HPFPR H_k is said to be acceptably consensus with group HPFPR H_g if

$$\mathrm{GCI}(H_k) \leq \mathrm{GCI}_r. \tag{61}$$

But it is possible that a decision maker has an unacceptable difference of opinion with group decision; in this regard a novel algorithm is proposed to modify his HPFPR to reach acceptable consensus.

Algorithm 29 (consensus improving algorithm).

Input. The tolerance value GCI_r of consensus measure, HPFPRs $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$, and parameter $\beta \in (0, 1)$.

Output. The acceptably consensus HPFPRs $H_k^{(t)}$, group consensus index $\mathrm{GCI}(H_k^{(t)})$ $(k = 1, 2, \ldots, m)$, the group HPFPR $H_g^{(t)}$, and number of iterations value t.

Step 1. Compute $\overline{H}_k = (\overline{h}_{ij,k})_{n\times n}$ with optimized parameter ζ_k. Let $t = 0$, $H_k^{(0)} = (h_{ij,k}^{(0)})_{n\times n} = \overline{H}_k = (\overline{h}_{ij,k})_{n\times n}$.

Step 2. Compute group HPFPR $H_g^{(t)} = (h_{ij,g}^{(t)})_{n\times n}$ by using all individual HPFPRs $H_k^{(t)}$ $(k = 1, 2, \ldots, m)$ according to (48), where

$$h_{ij,g}^{(t)} = \bigoplus_{k=1}^{m} \omega_k h_{ij,k}^{(t)}. \tag{62}$$

Step 3. Calculate $\mathrm{GCI}(H_k^{(t)}) = D(H_k^{(t)}, H_g^{(t)})$ $(k = 1, 2, \ldots, m)$. If $\mathrm{GCI}(H_k^{(t)}) \leq \mathrm{GCI}_r$ for all $k = 1, 2, \ldots, m$, then go to Step 5; otherwise go to Step 4.

Step 4. Put $H_k^{(t+1)} = (h_{ij,k}^{(t+1)})_{n\times n}$, where

$$h_k^{(t+1)} = \beta h_k^{(t)} \oplus (1-\beta)\, h_{ij,g}^{(t)}. \tag{63}$$

Also let $t = t + 1$ and go to Step 2.

Step 5. Output the adjusted HPFPRs $H_k^{(t)} = (h_{ij,k}^{(t)})_{n\times n}$, the group consistency index $\mathrm{GCI}(H_k^{(t)})$ $(k = 1, 2, \ldots, m)$, the group HPFPR $H_g^{(t)}$, and number of iterations value t.

Step 6. End.

The convergence of the above algorithm follows from the following result.

Theorem 30. *Consider $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ HPFPRs given by decision makers. Let $\{H_k^{(t)}\}$ be the sequence obtained from Algorithm 29. Then*

$$\begin{gathered} GCI\left(H_k^{(t+1)}\right) < GCI\left(H_k^{(t)}\right), \\ \lim_{t\to\infty} GCI\left(H_k^{(t)}\right) = 0 \\ \forall k = 1, 2, \ldots, m. \end{gathered} \tag{64}$$

Proof. By (62), (3), and (7),

$$\begin{aligned} &\left(h_{ij,g}^{(t+1)}\right)^{\sigma(s)} \mid \left(p_{ij,g}^{(t+1)}\right)^{\sigma(s)} \\ &\quad = \sum_{l=1}^{m} \omega_l \left(h_{ij,l}^{(t+1)}\right)^{\sigma(s)} \mid \sum_{l=1}^{m} \omega_l \left(p_{ij,l}^{(t+1)}\right); \end{aligned} \tag{65}$$

thus,

$$\begin{aligned} &\left|\left(h_{ij,k}^{(t+1)}\right)^{\sigma(s)} - \left(h_{ij,g}^{(t+1)}\right)^{\sigma(s)}\right| = \left|\beta\left(h_{ij,k}^{(t)}\right)^{\sigma(s)}\right. \\ &\quad \left. + (1-\beta)\left(h_{ij,g}^{(t)}\right)^{\sigma(s)} - \sum_{l=1}^{m}\omega_l\left(h_{ij,l}^{(t+1)}\right)^{\sigma(s)}\right| \\ &\quad = \left|\beta\left(h_{ij,k}^{(t)}\right)^{\sigma(s)} + (1-\beta)\left(h_{ij,g}^{(t)}\right)^{\sigma(s)}\right. \\ &\quad \left. - \sum_{l=1}^{m}\omega_l\left(\beta\left(h_{ij,l}^{(t)}\right)^{\sigma(s)} + (1-\beta)\left(h_{ij,g}^{(t)}\right)^{\sigma(s)}\right)\right| \\ &\quad = \left|\beta\left(h_{ij,k}^{(t)}\right)^{\sigma(s)} - \sum_{l=1}^{m}\omega_l\beta\left(h_{ij,l}^{(t)}\right)^{\sigma(s)}\right| = \beta\left|\left(h_{ij,k}^{(t)}\right)^{\sigma(s)}\right. \\ &\quad \left. - \left(h_{ij,g}^{(t)}\right)^{\sigma(s)}\right|; \end{aligned} \tag{66}$$

similarly,

$$\begin{aligned} &\left|\left(p_{ij,k}^{(t+1)}\right)^{\sigma(s)} - \left(p_{ij,g}^{(t+1)}\right)^{\sigma(s)}\right| \\ &\quad = \beta\left|\left(p_{ij,k}^{(t)}\right)^{\sigma(s)} - \left(p_{ij,g}^{(t)}\right)^{\sigma(s)}\right|. \end{aligned} \tag{67}$$

Hence,

$$D\left(h_{ij,k}^{(t+1)}, h_{ij,g}^{(t+1)}\right) = \beta D\left(h_{ij,k}^{(t)}, h_{ij,g}^{(t)}\right); \tag{68}$$

now by definition of GCI (60)

$$\begin{aligned}\mathrm{GCI}\left(H_k^{(t+1)}, H_g^{(t+1)}\right) &= \frac{2}{n(n-1)}\sum_{i<j}^{n} D\left(h_{ij,k}^{(t+1)}, h_{ij,g}^{(t+1)}\right)\\ &= \beta\,\mathrm{GCI}\left(H_k^{(t)}, H_g^{(t)}\right)\\ &< \mathrm{GCI}\left(H_k^{(t)}, H_g^{(t)}\right);\end{aligned} \tag{69}$$

it also implies

$$\begin{aligned}\lim_{t\to\infty}\mathrm{GCI}\left(H_k^{(t)}\right) &= \lim_{t\to\infty}\beta\,\mathrm{GCI}\left(H_k^{(t-1)}\right)\\ &= \lim_{t\to\infty}\beta^2\mathrm{GCI}\left(H_k^{(t-2)}\right) = \cdots\\ &= \lim_{t\to\infty}\beta^t\mathrm{GCI}\left(H_k^{(0)}\right) = 0\\ &\qquad (k = 1, 2, \ldots, m).\end{aligned} \tag{70}$$

□

Now, 1000 random sets of m HPFPRs are generated and Algorithm 29 is applied to develop consensus. The average value of iterations of Algorithm 29 is presented in Table 3 for different values of parameters. The readings of Table 3 suggest that the increase in value of parameter β has adverse effects on the number of iterations of Algorithm 29. So, the value of parameter β must be small. The more iterations are needed to develop consensus when GCI_r is nearer to 1. Furthermore, Algorithm 29 does not disturb the consistency of HPFPRs; that is, if the individual HPFPRs $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ are consistent, then output of Algorithm 29 and the adjusted HPFPRs $H_k^{(t)} = (h_{ij,k}^{(t)})_{n\times n}$ are also consistent.

Proposition 31. *Consider $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ HPFPRs given by decision makers. Let $\{H_k^{(t)}\}$ and $\{H_g^{(t)}\}$ be the sequences obtained from Algorithm 29. If $\max_k\{CI(H_k^{(t)})\} \le CI_r$ then*

$$\max_k\left\{CI\left(H_k^{(t+1)}\right)\right\} \le \max_k\left\{CI\left(H_k^{(t)}\right)\right\} \le CI_r. \tag{71}$$

Proof. It follows from Theorem 20 and (63). □

5. Decision Support Model for Group Decision Making with HPFPRs

Now, the issues of consistency and consensus are addressed. Algorithms 11–29 will provide the consistent individual HPF-PRs and the group HPFPR with agreement among decision makers. To comprehend final standing of the alternatives, first for the alternative x_i the i^{th} row of group HPFPR is aggregated by HPFA operator (44) and secondly the aggregated HPFEs are ordered according to their score and deviation [35]. In form of the following algorithm a complete decision model is presented.

Algorithm 32 (decision support model).

Input. The HPFPRs $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, \ldots, m)$ made by decision makers, the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ of

Table 3: Average values of iterations in Algorithm 29.

n	m	d	GCI_r	β = 0.2	0.4	0.7	0.9
4	4	4	0.01	2.002	3.233	3.233	25.301
			0.05	1	1.604	3.185	9.672
			0.1	0.949	0.956	1.346	3.384
	3	3	0.01	2	3.221	7.766	25.102
			0.05	1.003	1.701	3.47	10.452
			0.1	0.961	0.968	1.542	4.051
	2	2	0.01	2.03	3.476	8.166	26.48
			0.05	0.991	1.297	2.582	7.754
			0.1	0.872	0.911	1.517	3.975
5	4	5	0.01	2	3.137	7.631	24.652
			0.05	1	1.294	2.833	8.521
			0.1	0.868	0.864	0.974	2.186
	3	4	0.01	2	3.234	7.845	25.342
			0.05	1	1.223	2.681	7.775
			0.1	0.879	0.859	1.114	2.715
	3	3	0.01	2.001	3.167	7.816	25.32
			0.05	1	1.62	3.197	9.641
			0.1	0.802	0.817	0.901	2.024
6	5	6	0.01	2	3.006	7.154	22.888
			0.05	1	1.081	2.495	7.288
			0.1	0.623	0.606	0.643	1.128
	5	5	0.01	2	3.026	7.368	23.74
			0.05	1	1.128	2.638	7.784
			0.1	0.705	0.706	0.741	1.358
	4	4	0.01	2	3.069	7.544	24.335
			0.05	1	1.345	2.897	8.539
			0.1	0.849	0.838	0.955	1.99
7	7	7	0.01	2	3	6.908	21.984
			0.05	1	1.022	2.417	6.912
			0.1	0.511	0.485	0.479	0.71
	7	6	0.01	2	3.004	7.208	23.336
			0.05	1	1.135	2.72	7.903
			0.1	0.768	0.777	0.8	1.351
	8	4	0.01	2	3.025	7.454	24.147
			0.05	1	1.29	2.954	8.655
			0.1	0.991	0.992	1.163	2.9
8	9	8	0.01	2	3	6.81	21.664
			0.05	1	1.006	2.249	6.615
			0.1	0.263	0.256	0.257	0.339
	8	7	0.01	2	3	6.907	21.885
			0.05	1	1.006	2.232	6.611
			0.1	0.329	0.341	0.333	0.435
	8	6	0.01	2	3.001	7.022	22.476
			0.05	1	1.036	2.453	7.215
			0.1	0.309	0.305	0.342	0.356
9	10	9	0.01	2	3	6.633	21.16
			0.05	1	1.001	2.052	5.883
			0.1	0.083	0.065	0.072	0.1
	7	8	0.01	2	3	6.468	20.746
			0.05	1	1.003	2.094	6.072
			0.1	0.087	0.126	0.105	0.127
	9	7	0.01	2	3	6.812	21.546
			0.05	1	1.004	2.134	6.298
			0.1	0.161	0.162	0.157	0.205

Table 3: Continued.

n	m	d	GCI_r	β 0.2	0.4	0.7	0.9
			0.01	2	3	6.619	21.049
	9	10	0.05	1	1	1.955	5.152
			0.1	0.052	0.05	0.062	0.062
			0.01	2	3	6.334	20.466
10	8	9	0.05	1	1	2.038	5.746
			0.1	0.036	0.038	0.036	0.048
			0.01	2	3	6.454	20.795
	8	8	0.05	1	1	2.018	5.545
			0.1	0.059	0.061	0.047	0.05

decision makers in normalized form, consistency tolerance value CI_r, group consensus tolerance value GCI_r, the maximum number of iterations allowed t_{max}, and the parameters $0 < \alpha, \beta < 1$ for modification.

Output. The final standing of all the alternatives.

Step 1. Compute NHPFPRs $\overline{H}_k = (\overline{h}_{ij,k})_{n\times n}$ with optimized parameters ζ_k. Put $t = 0$ and $H_k^{(0)} = (h_{ij,k}^{(0)})_{n\times n} = \overline{H}_k = (\overline{h}_{ij,k})_{n\times n}$.

Step 2. Calculate the consistent HPFPRs $\widetilde{H}_k^{(t)} = (\widetilde{h}_{ij,k}^{(t)})_{n\times n}$ by Algorithm 11 and consistency indexes $CI(H_k^{(t)})$, where

$$CI\left(H_k^{(t)}\right) = \frac{2}{n(n-1)}\sum_{i<j}^{n} D\left(h_{ij,k}^{(t)}, \widetilde{h}_{ij,k}^{(t)}\right). \quad (72)$$

Step 3. If $CI(H_k^{(t)}) \leq CI_r$ for all $k = 1, 2, \ldots, m$ then go to Step 5; otherwise, go to Step 4.

Step 4. Make the adjusted HPFPRs $H_k^{(t+1)} = (h_{ij,k}^{(t+1)})$, where

$$h_{ij,k}^{(t+1)} = \begin{cases} h_{ij,k}^{(t)}, & CI\left(H_k^{(t)}\right) \leq CI_r \\ \alpha h_{ij,k}^{(t)} \oplus (1-\alpha)\,\widetilde{h}_{ij,k}^{(t)}, & CI\left(H_k^{(t)}\right) > CI_r. \end{cases} \quad (73)$$

Let $t = t + 1$; now return to Step 2.

Step 5. Apply HPFWA operator (43) to individual HPFPRs $H_k^{(t)} = (h_{ij,k}^{(t)})_{n\times n}$ $(k = 1, 2, \ldots, m)$, to get group HPFPR $H_g^{(t)} = (h_{ij,g}^{(t)})_{n\times n}$, where

$$h_{ij,g}^{(t)} = \bigoplus_{k=1}^{m} \omega_k h_{ij,k}^{(t)}. \quad (74)$$

Step 6. Calculate $GCI(H_k^{(t)}) = D(H_k^{(t)}, H_g^{(t)})$ $(k = 1, 2, \ldots, m)$. If $GCI(H_k^{(t)}) \leq GCI_r$ for all $k = 1, 2, \ldots, m$ or $t > t_{max}$, then go to Step 8; otherwise, go to Step 7.

Step 7. Let $H_k^{(t+1)} = (h_{ij,k}^{(t+1)})_{n\times n}$, where

$$h_k^{(t+1)} = \beta h_k^{(t)} \oplus (1-\beta)\, h_{ij,g}^{(t)}. \quad (75)$$

Also let $t = t + 1$ and go to Step 5.

Step 8. Aggregate each i^{th}-row of HPFPR $H_g^{(t)}$ by HPFA operator (44),

$$h_{i,g} = \text{HPFA}\left(h_{i1,g}^{(t)}, h_{i2,g}^{(t)}, \ldots, h_{in,g}^{(t)}\right) = \bigoplus_{j=1}^{n}\left(\frac{1}{n} h_{ij,g}^{(t)}\right), \quad (76)$$

to get collective preference degrees of alternative x_i over all other alternatives $(i = 1, 2, \ldots, n)$.

Step 9. Compute scores $s(h_{i,g})$ and deviations $d(h_{i,g})$ [35] as follows:

$$\begin{aligned} s\left(h_{i,g}\right) &= \sum_{s=1}^{d} h_{i,g}^{\sigma(s)} p_{i,g}^{\sigma(s)}, \\ d\left(h_{i,g}\right) &= \sum_{s=1}^{d}\left(h_{i,g}^{\sigma(s)} - s\left(h_{i,g}\right)\right)^2 p_{i,g}^{\sigma(s)}, \\ & \qquad i = (1, 2, \ldots, n). \end{aligned} \quad (77)$$

Step 10. Determine the final ranking of alternatives x_i, $i = 1, 2, 3, \ldots, n$ by comparing the score and deviation values of respective $h_{i,g}$ and output the ranking vector.

Step 11. End

Flowchart of the decision making model is presented in Figure 7.

6. Case Study

The proposed decision making model will be applied to a practical problem of investment in Forex.

Example 33. The Flagship Investment Company (FIC) is known for investment plans in different commodities in Forex. The investors hire FIC for making profitable plans of investments. An investor is interested to invest in four commodities: oil, gold, wheat, and copper denoted as x_1, x_2, x_3, and x_4, respectively. He approached FIC to indicate which commodity is more profitable to invest in. FIC makes a committee of four economic experts e_1, e_2, e_3, and e_4. Each expert will provide its preferences in terms of HPFPR depending upon many factors like previous market rates, market trends, possible future decisions of different regulatory bodies, economic stability, global peace situation, and so on. The complex nature of the Forex required tools that will model the vagueness and produce results closer to real life. HPFPR will allow the economic experts to express their

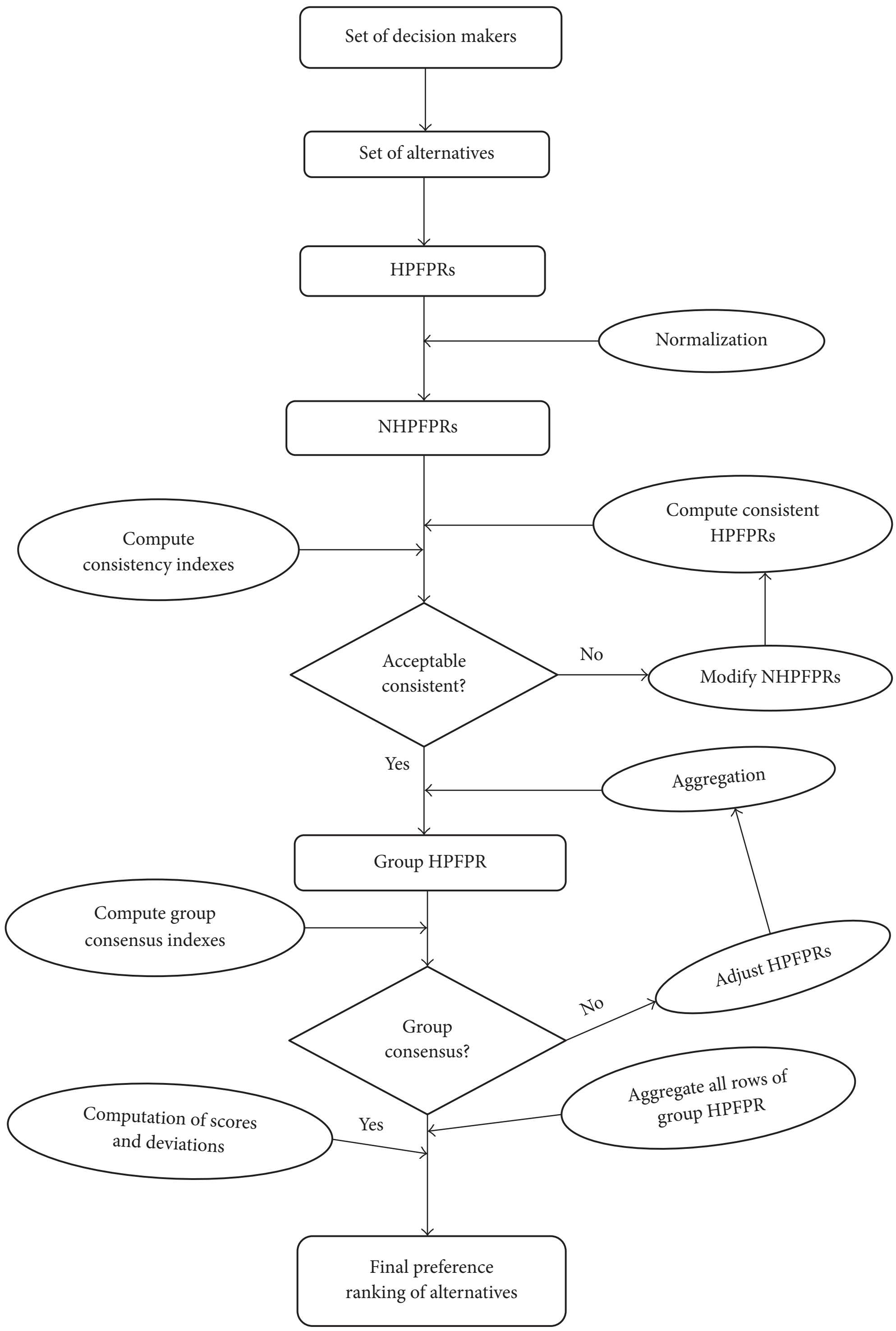


Figure 7: Flowchart of the proposed decision model.

hesitancy and provide the probabilities of different preference degrees, which are bound to be different because of the factors discussed above. Based on the experience and economic knowledge, weight vector of experts is $\omega = (0.1, 0.4, 0.2, 0.3)^T$. The economic expert e_k provides his HPFPR H_k ($k = 1, 2, 3, 4$) as follows.

$$H_1 = \begin{pmatrix} \{.5 \mid 1\} & \{0.1174 \mid 0.2979, 0.910 \mid .7021\} & \{0.4242 \mid 0.6622, 0.8341 \mid 0.3378\} & \{0.2625 \mid 0.0636, 0.7956 \mid 0.9364\} \\ \{0.8826 \mid 0.2979, 0.0900 \mid 0.7021\} & \{.5 \mid 1\} & \{0.1921 \mid 0.2236, 0.9289 \mid 0.7764\} & \{0.2373 \mid 0.4313, 0.5785 \mid 0.5687\} \\ \{0.5758 \mid 0.6622, 0.1659 \mid 0.3378\} & \{0.8079 \mid 0.2236, 0.0711 \mid 0.2236\} & \{.5 \mid 1\} & \{0.2316 \mid 0.7145, 0.6211 \mid 0.2855\} \\ \{0.7375 \mid 0.0636, 0.2044 \mid 0.9364\} & \{0.7627 \mid 0.4313, 0.4214 \mid 0.5687\} & \{0.7684 \mid 0.7145, 0.3789 \mid 0.2855\} & \{.5 \mid 1\} \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \{.5 \mid 1\} & \{0.3674 \mid 0.5018, 0.3955 \mid 0.3181, 0.9880 \mid 0.1800\} & \{0.0987 \mid 0.3339, 0.2619 \mid 0.6322, 0.5913 \mid 0.0338\} & \{0.3197 \mid 0.2412, 0.8691 \mid 0.7588\} \\ \{0.6326 \mid 0.5018, 0.6044 \mid 0.3181, 0.0120 \mid 0.1800\} & \{.5 \mid 1\} & \{0.1340 \mid 0.3627, 0.8909 \mid 0.1892, 0.9037 \mid 0.4481\} & \{0.7440 \mid 1\} \\ \{0.9013 \mid 0.3339, 0.7381 \mid 0.3339, 0.4087 \mid 0.0338\} & \{0.8660 \mid 0.3627, 0.1091 \mid 0.1892, 0.0963 \mid 0.4481\} & \{.5 \mid 1\} & \{0.4111 \mid 0.7060, 0.9047 \mid 0.2940\} \\ \{0.6803 \mid 0.2412, 0.1309 \mid 0.7588\} & \{0.2559 \mid 1\} & \{0.5889 \mid 0.7060, 0.0953 \mid 0.2940\} & \{.5 \mid 1\} \end{pmatrix},$$

$$H_3 = \begin{pmatrix} \{.5 \mid 1\} & \{0.1829 \mid 0.1317, 0.2399 \mid 0.4745, 0.5767 \mid 0.3938\} & \{0.3817 \mid 0.7453, 0.9787 \mid 0.2547\} & \{0.0596 \mid 0.2951, 0.7219 \mid 0.7049\} \\ \{0.8171 \mid 0.1317, 0.7601 \mid 0.4745, 0.4232 \mid 0.3938\} & \{.5 \mid 1\} & \{0.5 \mid 0.1232, 0.5961 \mid 0.8767\} & \{0.1499 \mid 0.2398, 0.7224 \mid 0.5387, 0.8175 \mid 0.2214\} \\ \{0.6183 \mid 0.7453, 0.0213 \mid 0.2547\} & \{0.5 \mid 0.1233, 0.4039 \mid 0.8767\} & \{.5 \mid 1\} & \{0.4538 \mid 0.5612, 0.8003 \mid 0.4388\} \\ \{0.9404 \mid 0.2951, 0.2781 \mid 0.7049\} & \{0.8501 \mid 0.2398, 0.2775 \mid 0.5387, 0.1825 \mid 0.2215\} & \{0.5462 \mid 0.5612, 0.1997 \mid 0.4388\} & \{.5 \mid 1\} \end{pmatrix},$$

$$H_4 = \begin{pmatrix} \{.5 \mid 1\} & \{0.0991 \mid 0.629, 0.7317 \mid 0.371\} & \{0.3498 \mid 0.8924, 0.9447 \mid 0.1076\} & \{0.1835 \mid 0.2908, 0.3685 \mid 0.5157, 0.7256 \mid 0.1936\} \\ \{0.9009 \mid 0.629, 0.2683 \mid 0.371\} & \{.5 \mid 1\} & \{0.2481 \mid 0.7128, 0.5 \mid 0.2855, 0.9199 \mid 0.0017\} & \{0.3786 \mid 0.6610, 0.6443 \mid 0.0808, 0.7948 \mid 0.2582\} \\ \{0.6502 \mid 0.8924, 0.0553 \mid 0.1076\} & \{0.7519 \mid 0.7128, 0.5 \mid 0.2855, 0.0801 \mid 0.0017\} & \{.5 \mid 1\} & \{0.1349 \mid 0.4378, 0.6225 \mid 0.163, 0.8759 \mid 0.3992\} \\ \{0.8165 \mid 0.2908, 0.6315 \mid 0.5157, 0.2744 \mid 0.1936\} & \{0.6214 \mid 0.6610, 0.3557 \mid 0.0808, 0.2052 \mid 0.2582\} & \{0.8651 \mid 0.4378, 0.3775 \mid 0.163, 0.1241 \mid 0.3992\} & \{.5 \mid 1\} \end{pmatrix}. \quad (78)$$

Step 1. NHPFPRs $H_k^{(0)} = \overline{H}_k$ $(k = 1, 2, 3, 4)$ are calculated by (10) as follows.

$$H_1^{(0)} = \begin{pmatrix} \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.1174 \mid 0.149, 0.1174 \mid 0.149, 0.91 \mid 0.7021\} & \{0.4242 \mid 0.3311, 0.4242 \mid 0.3311, 0.8341 \mid 0.3378\} & \{0.2625 \mid 0.0318, 0.2625 \mid 0.0318, 0.7956 \mid 0.9364\} \\ \{0.8826 \mid 0.149, 0.8826 \mid 0.149, 0.09 \mid 0.7020\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.1921 \mid 0.1118, 0.1921 \mid 0.1118, 0.9289 \mid 0.7764\} & \{0.2373 \mid 0.2156, 0.2373 \mid 0.2156, 0.5785 \mid 0.5687\} \\ \{0.5758 \mid 0.3311, 0.5758 \mid 0.3311, 0.1659 \mid 0.3378\} & \{0.8079 \mid 0.1118, 0.8079 \mid 0.1118, 0.0711 \mid 0.7764\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.2316 \mid 0.3573, 0.2316 \mid 0.3573, 0.6211 \mid 0.2855\} \\ \{0.7375 \mid 0.03180, 0.7375 \mid 0.03180, 0.2044 \mid 0.9364\} & \{0.7627 \mid 0.2156, 0.7627 \mid 0.2156, 0.4215 \mid 0.5687\} & \{0.7684 \mid 0.3573, 0.7684 \mid 0.3573, 0.3789 \mid 0.2855\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} \end{pmatrix},$$

$$H_2^{(0)} = \begin{pmatrix} \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.3674 \mid 0.5018, 0.3955 \mid 0.3181, 0.988 \mid 0.18\} & \{0.0987 \mid 0.3339, 0.2619 \mid 0.6322, 0.5913 \mid 0.0338\} & \{0.3197 \mid 0.1206, 0.3197 \mid 0.1206, 0.8691 \mid 0.7588\} \\ \{0.6326 \mid 0.5018, 0.6045 \mid 0.3181, 0.012 \mid 0.18\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.134 \mid 0.3627, 0.8909 \mid 0.1892, 0.9037 \mid 0.4481\} & \{0.7441 \mid 0.3333, 0.7441 \mid 0.3333, 0.7441 \mid 0.3333\} \\ \{0.9013 \mid 0.3339, 0.7381 \mid 0.6322, 0.4087 \mid 0.03383\} & \{0.866 \mid 0.3626, 0.1091 \mid 0.1892, 0.0963 \mid 0.4481\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.4111 \mid 0.353, 0.4111 \mid 0.353, 0.9047 \mid 0.294\} \\ \{0.6803 \mid 0.1206, 0.6803 \mid 0.1206, 0.1309 \mid 0.7588\} & \{0.2559 \mid 0.3333, 0.2559 \mid 0.3333, 0.2559 \mid 0.3333\} & \{0.5889 \mid 0.353, 0.5889 \mid 0.353, 0.0953 \mid 0.294\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} \end{pmatrix},$$

$$H_3^{(0)} = \begin{pmatrix} \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.1829 \mid 0.1317, 0.2399 \mid 0.4744, 0.5767 \mid 0.3938\} & \{0.3817 \mid 0.3727, 0.3817 \mid 0.3727, 0.9787 \mid 0.2547\} & \{0.0596 \mid 0.1476, 0.0596 \mid 0.1476, 0.7219 \mid 0.7049\} \\ \{0.8171 \mid 0.1317, 0.4745 \mid 0.7601, 0.4233 \mid 0.3938\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.5 \mid 0.0616, 0.5 \mid 0.0616, 0.5961 \mid 0.8767\} & \{0.1499 \mid 0.2398, 0.7224 \mid 0.5387, 0.8175 \mid 0.2215\} \\ \{0.6183 \mid 0.3727, 0.6183 \mid 0.3727, 0.0213 \mid 0.2547\} & \{0.5 \mid 0.0616, 0.5 \mid 0.0616, 0.4039 \mid 0.8767\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.4538 \mid 0.2806, 0.4538 \mid 0.2806, 0.8003 \mid 0.4388\} \\ \{0.9404 \mid 0.1476, 0.9404 \mid 0.1476, 0.2781 \mid 0.7049\} & \{0.8501 \mid 0.239, 0.2776 \mid 0.5387, 0.1825 \mid 0.2214\} & \{0.5462 \mid 0.2807, 0.5462 \mid 0.2807, 0.1997 \mid 0.4388\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} \end{pmatrix},$$

$$H_4^{(0)} = \begin{pmatrix} \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.0991 \mid 0.3145, 0.0991 \mid 0.3145, 0.7317 \mid 0.371\} & \{0.3498 \mid 0.4462, 0.3498 \mid 0.4462, 0.9447 \mid 0.1076\} & \{0.1835 \mid 0.2908, 0.3685 \mid 0.5157, 0.7256 \mid 0.1936\} \\ \{0.9009 \mid 0.3145, 0.9009 \mid 0.3145, 0.2683 \mid 0.371\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.2481 \mid 0.7128, 0.5 \mid 0.2855, 0.9199 \mid 0.0017\} & \{0.3786 \mid 0.661, 0.6443 \mid 0.0808, 0.7948 \mid 0.2581\} \\ \{0.6502 \mid 0.4462, 0.6502 \mid 0.4462, 0.0553 \mid 0.1076\} & \{0.7519 \mid 0.7128, 0.5 \mid 0.2855, 0.0801 \mid 0.0017\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} & \{0.1349 \mid 0.4378, 0.6225 \mid 0.163, 0.8759 \mid 0.3992\} \\ \{0.8165 \mid 0.2908, 0.6315 \mid 0.5157, 0.2744 \mid 0.1936\} & \{0.6214 \mid 0.661, 0.3557 \mid 0.0808, 0.2052 \mid 0.2581\} & \{0.8651 \mid 0.4378, 0.3775 \mid 0.163, 0.1241 \mid 0.3992\} & \{.5 \mid 0.3333, .5 \mid 0.3333, .5 \mid 0.3333\} \end{pmatrix}. \quad (79)$$

Step 2. After calculating consistent HPFPRs $\widetilde{H}_k^{(0)} = (\widetilde{h}_{ij,k}^{(0)})_{n\times n}$ by Algorithm 11 consistency indexes $\mathrm{CI}(H_k^{(t)})$ are computed by (72) as follows:

$$\mathrm{CI}\left(H_1^{(0)}\right) = .1366,$$
$$\mathrm{CI}\left(H_2^{(0)}\right) = .2533,$$
$$\mathrm{CI}\left(H_3^{(0)}\right) = .2159,$$
$$\mathrm{CI}\left(H_4^{(0)}\right) = .2116. \quad (80)$$

Step 3. The consistency tolerance value CI_r is decided to be 0.01; therefore, all the HPFPRs $H_k^{(0)}$ $(k = 1, 2, 3, 4)$ are needed to be adjusted.

Repeating Steps 2 to 4 in Section 5, we get the acceptably consistent HPFPR,

$$H_1^{(2)} = \begin{pmatrix} \{0.5 \mid 0.2293, 0.5 \mid 0.2293, 0.5 \mid 0.5414\} & \{0.3705 \mid 0.2265, 0.3705 \mid 0.2265, 0.7373 \mid 0.547\} & \{0.2984 \mid 0.237, 0.2984 \mid 0.237, 0.9195 \mid 0.5261\} & \{0.1352 \mid 0.2288, 0.1352 \mid 0.2288, 0.8828 \mid 0.5425\} \\ \{0.6295 \mid 0.2265, 0.6295 \mid 0.2265, 0.2627 \mid 0.547\} & \{0.5 \mid 0.2274, 0.5 \mid 0.2274, 0.5 \mid 0.5452\} & \{0.4219 \mid 0.2338, 0.4219 \mid 0.2338, 0.6873 \mid 0.5323\} & \{0.2606 \mid 0.2296, 0.2606 \mid 0.2296, 0.6475 \mid 0.5407\} \\ \{0.7015 \mid 0.237, 0.7015 \mid 0.237, 0.0804 \mid 0.5261\} & \{0.5781 \mid 0.2338, 0.5781 \mid 0.2338, 0.3127 \mid 0.5323\} & \{0.5 \mid 0.2447, 0.5 \mid 0.2447, 0.5 \mid 0.5106\} & \{0.3356 \mid 0.2397, 0.3356 \mid 0.2397, 0.4649 \mid 0.5206\} \\ \{0.8648 \mid 0.2288, 0.8648 \mid 0.2288, 0.1171 \mid 0.5425\} & \{0.7394 \mid 0.2296, 0.7394 \mid 0.2296, 0.3525 \mid 0.5407\} & \{0.6644 \mid 0.2397, 0.6644 \mid 0.2397, 0.5351 \mid 0.5206\} & \{0.5 \mid 0.2343, 0.5 \mid 0.2343, 0.5 \mid 0.5315\} \end{pmatrix},$$

$$
\begin{gathered}
H_2^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.3272, 0.5 \mid 0.341, 0.5 \mid 0.3318\} & \{0.3193 \mid 0.3466, 0.1865 \mid 0.324, 0.7001 \mid 0.3294\} & \{0.1513 \mid 0.3341, 0.428 \mid 0.3516, 0.758 \mid 0.3144\} & \{0.3152 \mid 0.3142, 0.3626 \mid 0.3196, 0.9903 \mid 0.3663\} \\
\{0.6807 \mid 0.3466, 0.8135 \mid 0.324, 0.2999 \mid 0.3294\} & \{0.5 \mid 0.3625, 0.5 \mid 0.3073, 0.5 \mid 0.3301\} & \{0.3311 \mid 0.352, 0.7467 \mid 0.3303, 0.5659 \mid 0.3177\} & \{0.4989 \mid 0.334, 0.6792 \mid 0.3048, 0.7939 \mid 0.3612\} \\
\{0.8487 \mid 0.3341, 0.5718 \mid 0.3516, 0.242 \mid 0.3144\} & \{0.6689 \mid 0.352, 0.2533 \mid 0.3303, 0.4341 \mid 0.3177\} & \{0.5 \mid 0.3409, 0.5 \mid 0.3561, 0.5 \mid 0.303\} & \{0.6608 \mid 0.3233, 0.4331 \mid 0.3295, 0.7336 \mid 0.3472\} \\
\{0.6848 \mid 0.3142, 0.6374 \mid 0.3196, 0.0097 \mid 0.3663\} & \{0.5011 \mid 0.334, 0.3201 \mid 0.3049, 0.2061 \mid 0.3612\} & \{0.3392 \mid 0.3233, 0.5669 \mid 0.3295, 0.2663 \mid 0.3472\} & \{0.5 \mid 0.3054, 0.5 \mid 0.3024, 0.5 \mid 0.3922\}
\end{pmatrix}, \\
H_3^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.2437, 0.5 \mid 0.3263, 0.5 \mid 0.4301\} & \{0.2883 \mid 0.2256, 0.1753 \mid 0.3335, 0.6098 \mid 0.4408\} & \{0.2642 \mid 0.2486, 0.2783 \mid 0.3062, 0.7651 \mid 0.4452\} & \{0.07175 \mid 0.243, 0.2276 \mid 0.3224, 0.9025 \mid 0.4347\} \\
\{0.7117 \mid 0.2256, 0.8247 \mid 0.3335, 0.3902 \mid 0.4408\} & \{0.5 \mid 0.2116, 0.5 \mid 0.338, 0.5 \mid 0.4504\} & \{0.474 \mid 0.2295, 0.6016 \mid 0.3089, 0.6523 \mid 0.4615\} & \{0.2812 \mid 0.2279, 0.5563 \mid 0.3321, 0.7944 \mid 0.44\} \\
\{0.7358 \mid 0.2486, 0.7217 \mid 0.3062, 0.2349 \mid 0.4452\} & \{0.526 \mid 0.2295, 0.3984 \mid 0.3089, 0.3477 \mid 0.4615\} & \{0.5 \mid 0.2529, 0.5 \mid 0.2853, 0.5 \mid 0.4618\} & \{0.3103 \mid 0.2489, 0.452 \mid 0.3032, 0.6429 \mid 0.4479\} \\
\{0.9283 \mid 0.243, 0.7724 \mid 0.3224, 0.0975 \mid 0.4347\} & \{0.7188 \mid 0.227, 0.44378 \mid 0.3321, 0.2056 \mid 0.44\} & \{0.6897 \mid 0.2489, 0.548 \mid 0.3032, 0.3571 \mid 0.4479\} & \{0.5 \mid 0.246, 0.5 \mid 0.3222, 0.5 \mid 0.4318\}
\end{pmatrix}, \\
H_4^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.2437, 0.5 \mid 0.3263, 0.5 \mid 0.4301\} & \{0.2883 \mid 0.2256, 0.1753 \mid 0.3335, 0.6098 \mid 0.4408\} & \{0.2642 \mid 0.2486, 0.2783 \mid 0.3062, 0.7651 \mid 0.4452\} & \{0.0717 \mid 0.243, 0.2276 \mid 0.3224, 0.9025 \mid 0.4347\} \\
\{0.7117 \mid 0.2256, 0.8247 \mid 0.3335, 0.3902 \mid 0.4408\} & \{0.5 \mid 0.2116, 0.5 \mid 0.338, 0.5 \mid 0.4504\} & \{0.474 \mid 0.2295, 0.60158 \mid 0.3089, 0.6523 \mid 0.4615\} & \{0.2812 \mid 0.2279, 0.5562 \mid 0.3321, 0.7944 \mid 0.44\} \\
\{0.7358 \mid 0.2486, 0.7217 \mid 0.3062, 0.2349 \mid 0.4452\} & \{0.5260 \mid 0.2295, 0.3984 \mid 0.3089, 0.3477 \mid 0.4615\} & \{0.5 \mid 0.2529, 0.5 \mid 0.2853, 0.5 \mid 0.4618\} & \{0.3103 \mid 0.2489, 0.452 \mid 0.3032, 0.6429 \mid 0.4479\} \\
\{0.9283 \mid 0.243, 0.7724 \mid 0.3224, 0.0975 \mid 0.4347\} & \{0.7188 \mid 0.2279, 0.4437 \mid 0.3321, 0.2056 \mid 0.44\} & \{0.6897 \mid 0.2489, 0.548 \mid 0.3032, 0.3571 \mid 0.4479\} & \{0.5 \mid 0.246, 0.5 \mid 0.3222, 0.5 \mid 0.4318\}
\end{pmatrix},
\end{gathered}
\tag{81}
$$

and consistent HPFPRs,

$$
\begin{gathered}
\widetilde{H}_1^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.2304, 0.5 \mid 0.2304, 0.5 \mid 0.5392\} & \{0.373 \mid 0.2299, 0.373 \mid 0.229, 0.7356 \mid 0.5403\} & \{0.2972 \mid 0.2346, 0.2972 \mid 0.2346, 0.9204 \mid 0.5308\} & \{0.1339 \mid 0.2317, 0.1339 \mid 0.2317, 0.8837 \mid 0.5365\} \\
\{0.627 \mid 0.2299, 0.627 \mid 0.2299, 0.2644 \mid 0.5403\} & \{0.5 \mid 0.2293, 0.5 \mid 0.2293, 0.5 \mid 0.5413\} & \{0.4242 \mid 0.2341, 0.4242 \mid 0.2341, 0.6848 \mid 0.5319\} & \{0.2608 \mid 0.2312, 0.2608 \mid 0.2312, 0.6482 \mid 0.5376\} \\
\{0.7028 \mid 0.2346, 0.7028 \mid 0.2346, 0.0796 \mid 0.5308\} & \{0.5758 \mid 0.2341, 0.5758 \mid 0.2341, 0.3152 \mid 0.5318\} & \{0.5 \mid 0.2388, 0.5 \mid 0.2388, 0.5 \mid 0.5224\} & \{0.3367 \mid 0.2359, 0.3367 \mid 0.2359, 0.46337 \mid 0.5281\} \\
\{0.8661 \mid 0.2317, 0.8661 \mid 0.2317, 0.1163 \mid 0.5365\} & \{0.7392 \mid 0.2312, 0.7392 \mid 0.2312, 0.3518 \mid 0.5376\} & \{0.6633 \mid 0.2359, 0.6633 \mid 0.2359, 0.5367 \mid 0.5281\} & \{0.5 \mid 0.2331, 0.5 \mid 0.2331, 0.5 \mid 0.5338\}
\end{pmatrix}, \\
\widetilde{H}_2^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.3305, 0.5 \mid 0.3340, 0.5 \mid 0.3355\} & \{0.3188 \mid 0.3396, 0.18448 \mid 0.3253, 0.6971 \mid 0.335\} & \{0.1519 \mid 0.334, 0.4297 \mid 0.338, 0.7597 \mid 0.328\} & \{0.3152 \mid 0.3249, 0.363 \mid 0.3241, 0.9916 \mid 0.3511\} \\
\{0.6812 \mid 0.3396, 0.8156 \mid 0.3253, 0.3029 \mid 0.335\} & \{0.5 \mid 0.3488, 0.5 \mid 0.3166, 0.5 \mid 0.3346\} & \{0.3331 \mid 0.3432, 0.7453 \mid 0.3292, 0.5625 \mid 0.3276\} & \{0.4964 \mid 0.334, 0.6786 \mid 0.3153, 0.7944 \mid 0.3507\} \\
\{0.8481 \mid 0.334, 0.5703 \mid 0.338, 0.2403 \mid 0.328\} & \{0.6669 \mid 0.3432, 0.2547 \mid 0.3292, 0.4375 \mid 0.3276\} & \{0.5 \mid 0.3376, 0.5 \mid 0.3417, 0.5 \mid 0.3206\} & \{0.6633 \mid 0.3284, 0.4333 \mid 0.328, 0.7319 \mid 0.3436\} \\
\{0.6848 \mid 0.3249, 0.637 \mid 0.3241, 0.0084 \mid 0.3511\} & \{0.5036 \mid 0.33, 0.3214 \mid 0.3153, 0.2056 \mid 0.3507\} & \{0.3367 \mid 0.328, 0.5667 \mid 0.328, 0.2681 \mid 0.3436\} & \{0.5 \mid 0.3192, 0.5 \mid 0.3141, 0.5 \mid 0.3667\}
\end{pmatrix}, \\
\widetilde{H}_3^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.2402, 0.5 \mid 0.3221, 0.5 \mid 0.4377\} & \{0.2893 \mid 0.2319, 0.1747 \mid 0.3251, 0.6101 \mid 0.4429\} & \{0.263 \mid 0.2426, 0.2771 \mid 0.3115, 0.7629 \mid 0.4459\} & \{0.07187 \mid 0.2408, 0.2293 \mid 0.321, 0.9043 \mid 0.4381\} \\
\{0.7107 \mid 0.2319, 0.8253 \mid 0.3251, 0.3899 \mid 0.4429\} & \{0.5 \mid 0.2237, 0.5 \mid 0.3281, 0.5 \mid 0.4481\} & \{0.4737 \mid 0.2343, 0.6026 \mid 0.3145, 0.6528 \mid 0.4511\} & \{0.2826 \mid 0.2326, 0.5546 \mid 0.3241, 0.7942 \mid 0.4434\} \\
\{0.737 \mid 0.2426, 0.7227 \mid 0.3115, 0.2371 \mid 0.4459\} & \{0.5263 \mid 0.2343, 0.3974 \mid 0.3145, 0.3472 \mid 0.4511\} & \{0.5 \mid 0.245, 0.5 \mid 0.3009, 0.5 \mid 0.4541\} & \{0.3088 \mid 0.2432, 0.452 \mid 0.3105, 0.6413 \mid 0.4463\} \\
\{0.9281 \mid 0.2408, 0.7707 \mid 0.321, 0.0957 \mid 0.4381\} & \{0.7174 \mid 0.2326, 0.4454 \mid 0.3241, 0.2058 \mid 0.4434\} & \{0.6912 \mid 0.2432, 0.548 \mid 0.3105, 0.3587 \mid 0.4463\} & \{0.5 \mid 0.2414, 0.5 \mid 0.32, 0.5 \mid 0.4386\}
\end{pmatrix}, \\
\widetilde{H}_4^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.4125, 0.5 \mid 0.3373, 0.5 \mid 0.2502\} & \{0.2762 \mid 0.4365, 0.193 \mid 0.3148, 0.6047 \mid 0.2487\} & \{0.2739 \mid 0.4331, 0.2612 \mid 0.3229, 0.8477 \mid 0.2441\} & \{0.0824 \mid 0.4253, 0.3632 \mid 0.3178, 0.9496 \mid 0.257\} \\
\{0.7238 \mid 0.4365, 0.806 \mid 0.3148, 0.3953 \mid 0.2487\} & \{0.5 \mid 0.4606, 0.5 \mid 0.2922, 0.5 \mid 0.2471\} & \{0.4977 \mid 0.4572, 0.5681 \mid 0.3003, 0.7429 \mid 0.2425\} & \{0.3062 \mid 0.4493, 0.6701 \mid 0.2952, 0.8448 \mid 0.2554\} \\
\{0.7262 \mid 0.4331, 0.7388 \mid 0.3229, 0.1523 \mid 0.2441\} & \{0.5023 \mid 0.4572, 0.4319 \mid 0.3003, 0.2571 \mid 0.2425\} & \{0.5 \mid 0.4537, 0.5 \mid 0.3084, 0.5 \mid 0.237\} & \{0.3085 \mid 0.4459, 0.602 \mid 0.3033, 0.6019 \mid 0.2508\} \\
\{0.9176 \mid 0.4253, 0.6368 \mid 0.3177, 0.0504 \mid 0.257\} & \{0.6938 \mid 0.4493, 0.3299 \mid 0.2952, 0.1552 \mid 0.2554\} & \{0.6915 \mid 0.4459, 0.398 \mid 0.3033, 0.3981 \mid 0.2508\} & \{0.5 \mid 0.438, 0.5 \mid 0.2982, 0.5 \mid 0.2638\}
\end{pmatrix}.
\end{gathered}
\tag{82}
$$

Also, Figures 8, 9, 10, and 11 present the comparison between the area graphs of score values for normalized, acceptably consistent, and consistent HPFPRs made from H_1, H_2, H_3, and H_4, respectively, by the end of Step 4 in Algorithm 32.

Step 4. By Applying HPFWA operator (43) to individual HPFPRs $H_k^{(t)} = (h_{ij,k}^{(t)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ we get group HPFPR $H_g^{(t)} = (h_{ij,g}^{(t)})_{n\times n}$ as follows:

$$
\begin{gathered}
H_g^{(2)} \\
= \begin{pmatrix}
\{0.5 \mid 0.3081, 0.5 \mid 0.3226, 0.5 \mid 0.3693\} & \{0.3099 \mid 0.3237, 0.2228 \mid 0.3053, 0.6612 \mid 0.3711\} & \{0.2403 \mid 0.325, 0.3224 \mid 0.3152, 0.8189 \mid 0.3598\} & \{0.161 \mid 0.3113, 0.2903 \mid 0.3045, 0.9384 \mid 0.3843\} \\
\{0.6901 \mid 0.3238, 0.7772 \mid 0.3053, 0.3388 \mid 0.3711\} & \{0.5 \mid 0.3399, 0.5 \mid 0.2883, 0.5 \mid 0.3718\} & \{0.427 \mid 0.3407, 0.5989 \mid 0.2952, 0.6611 \mid 0.3641\} & \{0.3501 \mid 0.3294, 0.5681 \mid 0.2878, 0.7799 \mid 0.3827\} \\
\{0.7597 \mid 0.325, 0.6776 \mid 0.3152, 0.1811 \mid 0.3598\} & \{0.573 \mid .3407, 0.4011 \mid 0.2952, 0.3389 \mid 0.3641\} & \{0.5 \mid 0.3411, 0.5 \mid 0.3053, 0.5 \mid 0.3537\} & \{0.4195 \mid 0.3298, 0.4681 \mid 0.2964, 0.623 \mid 0.3738\} \\
\{0.839 \mid 0.3114, 0.7097 \mid 0.3045, 0.0616 \mid 0.3843\} & \{0.6499 \mid 0.3294, 0.4319 \mid 0.2878, 0.2201 \mid 0.3827\} & \{0.5805 \mid 0.3298, 0.5319 \mid 0.2964, 0.3777 \mid 0.3738\} & \{0.5 \mid 0.3179, 0.5 \mid 0.2885, 0.5 \mid 0.3936\}
\end{pmatrix}.
\end{gathered}
\tag{83}
$$

Step 5. The group consensus indexes are

$$
\begin{gathered}
\mathrm{GCI}\left(H_1^{(t)}\right) = 0.0823, \\
\mathrm{GCI}\left(H_2^{(t)}\right) = 0.0709, \\
\mathrm{GCI}\left(H_3^{(t)}\right) = 0.0899, \\
\mathrm{GCI}\left(H_4^{(t)}\right) = 0.0743.
\end{gathered}
\tag{84}
$$

Now GCI_r is decided to be 0.1; therefore all the HPFPRs are acceptably consensus with group HPFPR.

Step 6. Now, all the rows of HPFPR H_g are aggregated by HPFA operator (44) as follows:

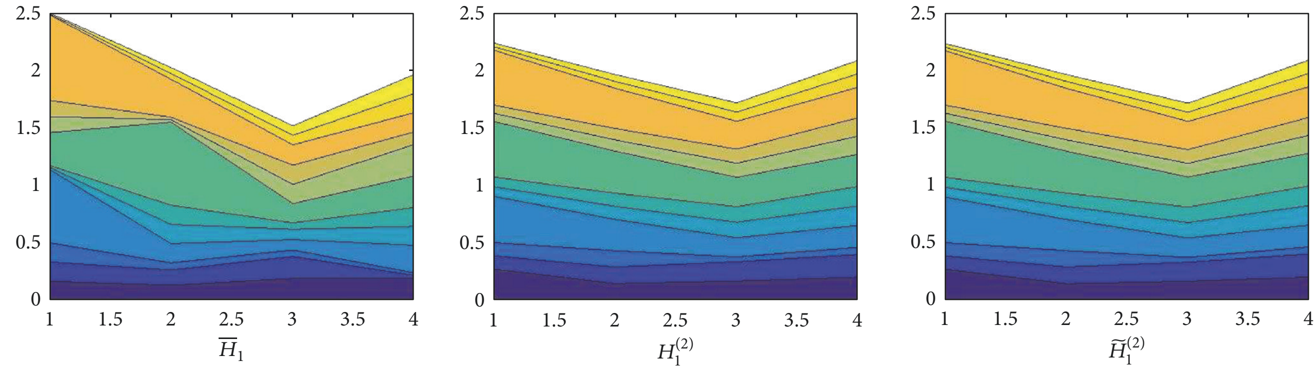


FIGURE 8: Area graphs of score values of H_1.

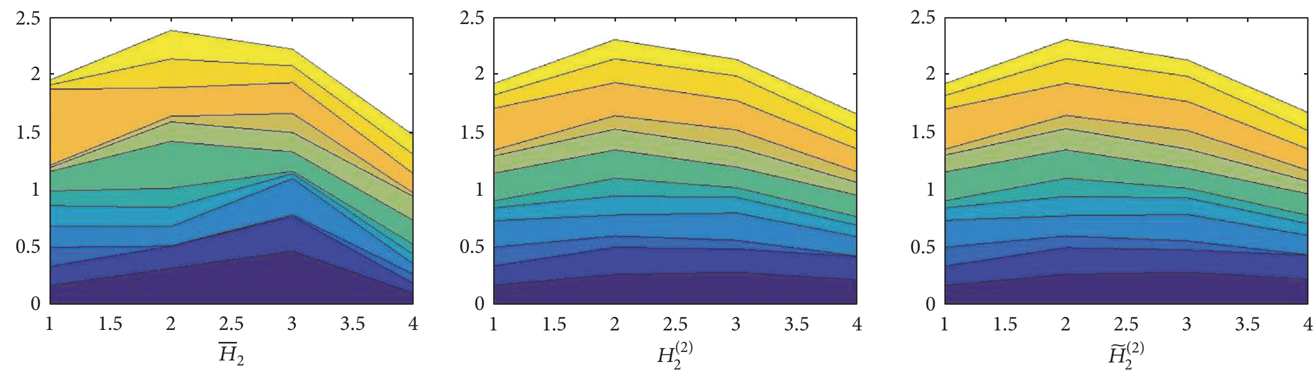


FIGURE 9: Area graphs of score values of H_2.

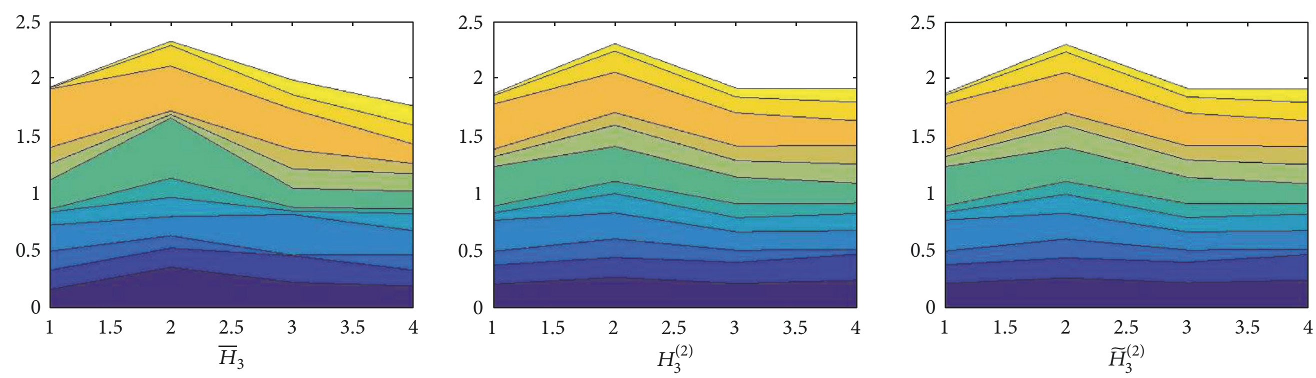


FIGURE 10: Area graphs of score values of H_3.

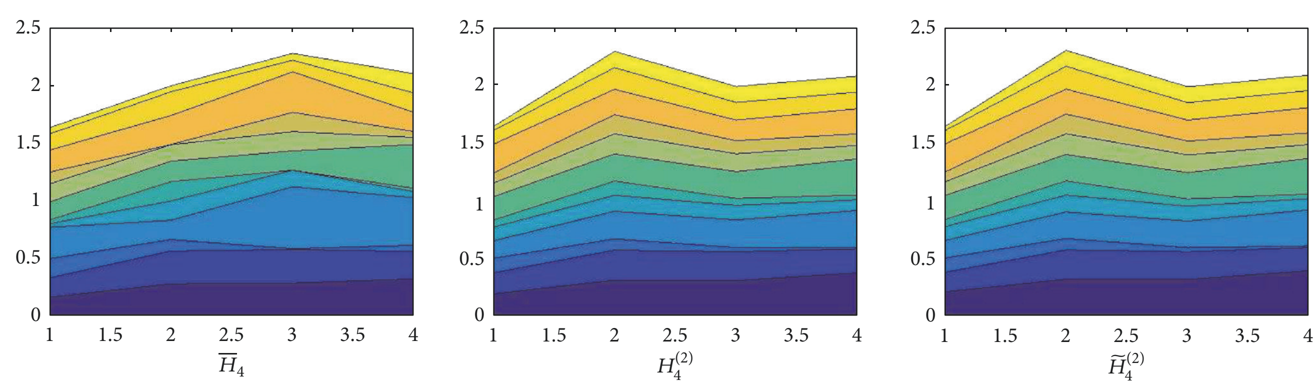


FIGURE 11: Area graphs of score values of H_4.

$$\begin{aligned}
h_{1,g} &= \{0.3028 \mid 0.317, 0.3339 \mid 0.3119, 0.7296 \mid 0.3711\} \\
h_{2,g} &= \{0.4918 \mid 0.3334, 0.6111 \mid 0.2942, 0.57 \mid 0.3724\} \\
h_{3,g} &= \{0.563 \mid 0.3341, 0.5117 \mid 0.303, 0.4108 \mid 0.3628\} \\
h_{4,g} &= \{0.6424 \mid 0.3221, 0.5434 \mid 0.2943, 0.2897 \mid 0.3836\}.
\end{aligned} \tag{85}$$

Step 7. By (77), scores and deviations of $h_{i,g}$ $(i = 1, 2, 3, 4)$ are calculated as follows.

$$\begin{aligned}
s(h_{1,g}) &= .4709, \\
s(h_{2,g}) &= .556, \\
s(h_{3,g}) &= .4922, \\
s(h_{4,g}) &= .4779, \\
d(h_{1,g}) &= .0397, \\
d(h_{2,g}) &= .0023, \\
s(h_{3,g}) &= .0042, \\
s(h_{4,g}) &= .0236.
\end{aligned} \tag{86}$$

Step 8. Comparing scores and deviations computed in Step 7 provides us the following final preference ranking:

$$x_2 > x_3 > x_4 > x_1. \tag{87}$$

Step 9. End.

Remark 34. Example 33 provides the complete illustration of the proposed model dealing with consistency and creating enough consensus among all economic experts. Based on the results it is found that gold is the most profitable commodity, whereas oil is the least preferable for investment.

6.1. Effects of Probability on Decision Making. In order to show how important is the role played by the probabilities of preference degrees two new versions are generated from the HPFPRs $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, 3, 4)$ by changing the probabilities only and preserving the preference degrees as follows.

$$\begin{aligned}
H_1^* &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.1174 \mid 0.2, 0.91 \mid 0.8\} & \{0.4242 \mid 0.1, 0.8341 \mid 0.9\} & \{0.2625 \mid 0.231, 0.7956 \mid 0.769\} \\
\{0.8826 \mid 0.2, 0.0900 \mid 0.8\} & \{.5 \mid 1\} & \{0.1921 \mid 0.85, 0.9289 \mid 0.15\} & \{0.2373 \mid 0.925, 0.5785 \mid 0.075\} \\
\{0.5758 \mid 0.1, 0.1659 \mid 0.9\} & \{0.8079 \mid 0.85, 0.0711 \mid 0.15\} & \{.5 \mid 1\} & \{0.2316 \mid 0.1, 0.6211 \mid 0.9\} \\
\{0.7375 \mid 0.231, 0.2044 \mid 0.769\} & \{0.7627 \mid 0.925, 0.4214 \mid 0.075\} & \{0.7684 \mid 0.1, 0.3789 \mid 0.9\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_2^* &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.3674 \mid 0.1, 0.3955 \mid 0.1, 0.988 \mid 0.8\} & \{0.0987 \mid 0.2, 0.2619 \mid 0.1, 0.5913 \mid 0.7\} & \{0.3197 \mid 0.35, 0.8691 \mid 0.65\} \\
\{0.6326 \mid 0.1, 0.6044 \mid 0.1, 0.0120 \mid 0.8\} & \{.5 \mid 1\} & \{0.1340 \mid 0.8, 0.8909 \mid 0.1, 0.9037 \mid 0.1\} & \{0.7440 \mid 1\} \\
\{0.9013 \mid 0.2, 0.7381 \mid 0.1, 0.4087 \mid 0.7\} & \{0.8660 \mid 0.8, 0.1091 \mid 0.1, 0.0963 \mid 0.1\} & \{.5 \mid 1\} & \{0.4111 \mid 0.11, 0.9047 \mid 0.89\} \\
\{0.6803 \mid 0.35, 0.1309 \mid 0.65\} & \{0.2559 \mid 1\} & \{0.5889 \mid 0.11, 0.0953 \mid 0.89\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_3^* &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.1829 \mid 0.01, 0.2399 \mid 0.23, 0.5767 \mid 0.76\} & \{0.3817 \mid 0.289, 0.9787 \mid 0.711\} & \{0.0596 \mid 0.0586, 0.7219 \mid 0.9414\} \\
\{0.8171 \mid 0.01, 0.7601 \mid 0.23, 0.4232 \mid 0.76\} & \{.5 \mid 1\} & \{0.5 \mid 0.7, 0.5961 \mid 0.3\} & \{0.1499 \mid 0.8, 0.7224 \mid 0.1, 0.8175 \mid 0.1\} \\
\{0.6183 \mid 0.289, 0.0213 \mid 0.711\} & \{0.5 \mid 0.7, 0.4039 \mid 0.3\} & \{.5 \mid 1\} & \{0.4538 \mid 0.15, 0.8003 \mid 0.85\} \\
\{0.9404 \mid 0.0586, 0.2781 \mid 0.9414\} & \{0.8501 \mid 0.8, 0.2775 \mid 0.1, 0.1825 \mid 0.1\} & \{0.5462 \mid 0.15, 0.1997 \mid 0.85\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_4^* &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.0991 \mid 0.05, 0.7317 \mid 0.95\} & \{0.3498 \mid 0.15, 0.9447 \mid 0.85\} & \{0.1835 \mid 0.1, 0.3685 \mid 0.05, 0.7256 \mid 0.85\} \\
\{0.9009 \mid 0.05, 0.2683 \mid 0.95\} & \{.5 \mid 1\} & \{0.2481 \mid 0.75, 0.5 \mid 0.1, 0.9199 \mid 0.15\} & \{0.3786 \mid 0.75, 0.6443 \mid 0.1, 0.7948 \mid 0.15\} \\
\{0.6502 \mid 0.15, 0.0553 \mid 0.85\} & \{0.7519 \mid 0.75, 0.5 \mid 0.1, 0.0801 \mid 0.15\} & \{.5 \mid 1\} & \{0.1349 \mid 0.15, 0.6225 \mid 0.1, 0.8759 \mid 0.75\} \\
\{0.8165 \mid 0.1, 0.6315 \mid 0.05, 0.2744 \mid 0.85\} & \{0.6214 \mid 0.75, 0.3557 \mid 0.1, 0.2052 \mid 0.15\} & \{0.8651 \mid 0.15, 0.3775 \mid 0.1, 0.1241 \mid 0.75\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_1^{**} &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.1174 \mid 0.2979, 0.91 \mid 0.7021\} & \{0.4242 \mid 0.9, 0.8341 \mid 0.1\} & \{0.2625 \mid 0.9, 0.7956 \mid 0.1\} \\
\{0.8826 \mid 0.2979, 0.0900 \mid 0.7021\} & \{.5 \mid 1\} & \{0.1921 \mid 0.85, 0.9289 \mid 0.15\} & \{0.2373 \mid 0.925, 0.5785 \mid 0.075\} \\
\{0.5758 \mid 0.9, 0.1659 \mid 0.1\} & \{0.8079 \mid 0.85, 0.0711 \mid 0.15\} & \{.5 \mid 1\} & \{0.2316 \mid 0.1, 0.6211 \mid 0.9\} \\
\{0.7375 \mid 0.9, 0.2044 \mid 0.1\} & \{0.7627 \mid 0.925, 0.4214 \mid 0.075\} & \{0.7684 \mid 0.1, 0.3789 \mid 0.9\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_2^{**} &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.3674 \mid 0.5018, 0.3955 \mid 0.3181, 0.9880 \mid 0.18\} & \{0.0987 \mid 0.85, 0.2619 \mid 0.1, 0.5913 \mid 0.05\} & \{0.3197 \mid 0.75, 0.8691 \mid 0.25\} \\
\{0.6326 \mid 0.5018, 0.6044 \mid 0.3181, 0.0120 \mid 0.18\} & \{.5 \mid 1\} & \{0.1340 \mid 0.8, 0.8909 \mid 0.1, 0.9037 \mid 0.1\} & \{0.7440 \mid 1\} \\
\{0.9013 \mid 0.85, 0.7381 \mid 0.1, 0.4087 \mid 0.05\} & \{0.8660 \mid 0.8, 0.1091 \mid 0.1, 0.0963 \mid 0.1\} & \{.5 \mid 1\} & \{0.4111 \mid 0.95, 0.9047 \mid 0.05\} \\
\{0.6803 \mid 0.75, 0.1309 \mid 0.25\} & \{0.2559 \mid 1\} & \{0.5889 \mid 0.95, 0.0953 \mid 0.05\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_3^{**} &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.1829 \mid 0.1317, 0.2399 \mid 0.4745, 0.5767 \mid 0.3938\} & \{0.3817 \mid 0.89, 0.9787 \mid 0.11\} & \{0.0596 \mid 0.9414, 0.7219 \mid 0.0586\} \\
\{0.8171 \mid 0.1317, 0.7601 \mid 0.4745, 0.4232 \mid 0.3938\} & \{.5 \mid 1\} & \{0.5 \mid 0.9, 0.5961 \mid 0.1\} & \{0.1499 \mid 0.8, 0.7224 \mid 0.1, 0.8175 \mid 0.1\} \\
\{0.6183 \mid 0.89, 0.0213 \mid 0.11\} & \{0.5 \mid 0.9, 0.4039 \mid 0.1\} & \{.5 \mid 1\} & \{0.4538 \mid 0.888, 0.8003 \mid 0.112\} \\
\{0.9404 \mid 0.9414, 0.2781 \mid 0.0586\} & \{0.8501 \mid 0.8, 0.2775 \mid 0.1, 0.1825 \mid 0.1\} & \{0.5462 \mid 0.888, 0.1997 \mid 0.112\} & \{.5 \mid 1\}
\end{pmatrix} \\
H_4^{**} &= \begin{pmatrix}
\{.5 \mid 1\} & \{0.0991 \mid 0.629, 0.7317 \mid 0.371\} & \{0.3498 \mid 0.9, 0.9447 \mid 0.1\} & \{0.1835 \mid 0.8, 0.3685 \mid 0.05, 0.7256 \mid 0.15\} \\
\{0.9009 \mid 0.629, 0.2683 \mid 0.371\} & \{.5 \mid 1\} & \{0.2481 \mid 0.8, 0.5 \mid 0.1, 0.9199 \mid 0.1\} & \{0.3786 \mid 0.8, 0.6443 \mid 0.1, 0.7948 \mid 0.1\} \\
\{0.6502 \mid 0.9, 0.0553 \mid 0.1\} & \{0.7519 \mid 0.8, 0.5 \mid 0.1, 0.0801 \mid 0.1\} & \{.5 \mid 1\} & \{0.1349 \mid 0.9, 0.6225 \mid 0.05, 0.8759 \mid 0.05\} \\
\{0.8165 \mid 0.8, 0.6315 \mid 0.05, 0.2744 \mid 0.15\} & \{0.6214 \mid 0.8, 0.3557 \mid 0.1, 0.2052 \mid 0.1\} & \{0.8651 \mid 0.9, 0.3775 \mid 0.05, 0.1241 \mid 0.05\} & \{.5 \mid 1\}
\end{pmatrix}.
\end{aligned} \tag{88}$$

Table 4: Effects of probability on decision making.

HPFPRs	Ranking of alternatives	Best commodity
H_1, H_2, H_3, H_4	$x_2 > x_3 > x_4 > x_1$	Gold
$H_1^*, H_2^*, H_3^*, H_4^*$	$x_1 > x_2 > x_3 > x_4$	Oil
$H_1^{**}, H_2^{**}, H_3^{**}, H_4^{**}$	$x_4 > x_2 > x_3 > x_1$	Copper

Remark 35. Inspection of Table 4 will reveal why it is important to consider the probabilities of different preference degrees. Many a time, decision makers are taking decisions related to future scenarios like our case study. The future is a mystery that cannot be predicted completely; probability is a useful tool to deal with uncertainties. In real life scenarios probability of different events can alter the decisions we make today.

7. Comparison with Existing Models

Preference relations are one of the important tools that are used for practical decision making in different complex scenarios. A wide number of researchers developed different kinds of preference relations models as fuzzy [10, 11, 32, 40, 43], intuitionistic fuzzy [23, 26, 27], and hesitant fuzzy [6, 24, 28, 33, 38]. Of course fuzzy preference relations are subcase of hesitant fuzzy preference relations, as we have discussed the occurrence probability is considered to be the same which is impractical as Table 4 reflects the impact of probabilities on decision making. So, for this reason the proposed model has provided better modelling of real world problems. In hesitant probabilistic fuzzy environment Zhou and Xu [41] proposed and studied probabilistic fuzzy preference relations. The drawback and difference between their definition of preference relation and proposed Definition 6 have already been discussed in Remark 7. First, they proposed expected consistency based upon multiplicative consistency of fuzzy preference relation [44]; then based on expected consistency multiobjective goal programming models are developed to compute occurrence probability and priority weight vector. Also, an iterative optimization algorithm to improve consistency of nonconsistent preference relations was designed. The comparison is summarized as follows.

(i) Zhou and Xu's definition [41, Definition 4] is not suitable for decision making based on β-normalization principle, whereas proposed Definition 6 is consistent with β-normalization; in fact HPFPRs are normalized following β-normalization principle.

(ii) They proposed expected consistency, taking a unifying approach to handle both preference degrees and their occurrence probabilities, whereas additive consistency [44] and weak stochastic transitivity [45, Page 27] are adopted for preference degrees and probabilities, respectively, in our proposal.

(iii) They defined expected consistency index based on priority weights computed from multiobjective goal programming model made from expected consistency, whereas for HPFPR H Algorithm 11 (based on convex combination technique) will provide its corresponding NHPFPR $\overline{H}$ and consistent HPFPR $\widetilde{H}$ and consistency index is defined to be the distance between $\overline{H}$ and $\widetilde{H}$, in our proposal.

(iv) They developed an algorithm which will replace the preference degrees by optimal ones iteratively and occurrence probabilities are computed accordingly to improve consistency, whereas Algorithm 19 (based on convex combination technique) will improve consistency, in our proposal.

(v) Both proposals use different techniques and ideas to define and deal with consistency as the above points reflect. However, Zhou and Xu make no effort for group decision making; the main advantage of our proposal is that it is a complete model, not only dealing with group decision making but also providing solution to consensus problem among decision maker, which is a key issue in group decision making.

The HFPR is a special case of HPFPR to provide experimental comparison with existing models of group decision making models with HFPRs [33, 38]; we solved their case studies by our proposed model, in a way that preference degrees are the same and their occurrence probabilities are considered to be equal. The comparison is provided in Table 5; clearly our proposed model is quite efficient as it provides the correct results.

8. Conclusion

In this paper, we have developed a complete group decision support model based on HPFPRs; it consists of three parts: a consistency improving process, a consensus reaching process, and the selection process. The consistency measure of a HPFPR has been defined in the consistency improving process. For HPFPRs that have unacceptable consistency, an optimization method is proposed to improve the consistency until the HPFPRs have an acceptable consistency. In the consensus reaching process, a consensus index is defined to measure the consensus level. For HPFPRs that have an unacceptable consensus, an optimization method is designed to assist them in achieving a predefined consensus level. The proposed model can be used to address GDM problems with HPFPRs. Optimization methods are also developed to help the individual HPFPRs to achieve a predefined consistency level and consensus level with fewer interactions of the DMs. As a consequence, our model is time saving, efficient, and convenient for practical applications. The consistency improving process is performed to ensure that the DMs are neither random nor illogical in their pairwise comparisons. Procedure to reach a consensus level ensures that the adjusted HPFPRs not only achieve the predefined level of consensus but also maintain acceptable consistency. Proposed model also ensures that the consistent HPFPRs do not change at each iteration. This property can retain the DMs' original decision making information to the greatest extent possible. Here we did not discuss the effects of the application of different controlling parameters and distance functions in the developed model. This model is not useful for the incomplete

Table 5: Experimental comparison.

Reference	Existing models solution	Proposed model solution
[38, Case study 5.1]	$x_3 > x_2 > x_1 > x_4$	$x_3 > x_2 > x_1 > x_4$
[33, Example 5.1]	$x_3 > x_2 > x_1 > x_4$	$x_3 > x_2 > x_1 > x_4$

preference relation. In the future, we will pay attention to addressing these problems. Future research can also focus on the extension of the proposed model to other types of preference relations.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, pp. B141–B164, 1970.

[3] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.

[4] I. Beg and T. Rashid, "Ideal solutions for hesitant fuzzy soft sets," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 30, no. 1, pp. 143–150, 2016.

[5] I. Beg and T. Rashid, "Hesitant 2-tuple linguistic information in multiple attributes group decision making," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 30, no. 1, pp. 109–116, 2016.

[6] N. Chen, Z. S. Xu, and M. M. Xia, "Interval-valued hesitant preference relations and their applications to group decision making," *Knowledge-Based Systems*, vol. 37, pp. 528–540, 2013.

[7] G. Qian, H. Wang, and X. Q. Feng, "Generalized hesitant fuzzy sets and their application in decision support system," *Knowledge-Based Systems*, vol. 37, pp. 357–365, 2013.

[8] B. Zhu, Z. Xu, and M. Xia, "Dual hesitant fuzzy sets," *Journal of Applied Mathematics*, vol. 2012, Article ID 879629, 13 pages, 2012.

[9] T. Rashid and I. Beg, "Convex hesitant fuzzy sets," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 30, no. 5, pp. 2791–2796, 2016.

[10] F. Chiclana, F. Herrera, and E. Herrera-Viedma, "Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations," *Fuzzy Sets and Systems*, vol. 122, no. 2, pp. 277–291, 2001.

[11] E. Herrera-Viedma, S. Alonso, F. Chiclana, and F. Herrera, "A consensus model for group decision making with incomplete fuzzy preference relations," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 5, pp. 863–877, 2007.

[12] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, "A model of consensus in group decision making under linguistic assessments," *Fuzzy Sets and Systems*, vol. 78, no. 1, pp. 73–87, 1996.

[13] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, "Direct approach processes in group decision making using linguistic OWA operators," *Fuzzy Sets and Systems*, vol. 79, no. 2, pp. 175–190, 1996.

[14] J. Kacprzyk, "Group decision making with a fuzzy linguistic majority," *Fuzzy Sets and Systems*, vol. 18, no. 2, pp. 105–118, 1986.

[15] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Springer, Dordrecht, Netherlands, 1994.

[16] T. L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, NY, USA, 1980.

[17] T. L. Saaty and L. G. Vargas, "Uncertainty and rank order in the analytic hierarchy process," *European Journal of Operational Research*, vol. 32, no. 1, pp. 107–117, 1987.

[18] S. A. Orlovsky, "Decision-making with a fuzzy preference relation," *Fuzzy Sets and Systems*, vol. 1, no. 3, pp. 155–167, 1978.

[19] Z. Xu and J. Chen, "Group decision-making procedure based on incomplete reciprocal relations," *Soft Computing*, vol. 12, no. 6, pp. 515–521, 2008.

[20] Y. Dong, Y. Xu, and H. Li, "On consistency measures of linguistic preference relations," *European Journal of Operational Research*, vol. 189, no. 2, pp. 430–444, 2008.

[21] F. Herrera, E. Herrera-Viedma, and J. L. Verdegay, "A sequential selection process in group decision making with a linguistic assessment approach," *Information Sciences*, vol. 85, no. 4, pp. 223–239, 1995.

[22] Z. S. Xu, "Generalized fuzzy, consistency matrix and its priority method," *Journal of PLA University of Science and Technology*, vol. 1, no. 6, pp. 97–99, 2000.

[23] Z. Xu, "Intuitionistic preference relations and their application in group decision making," *Information Sciences*, vol. 177, no. 11, pp. 2363–2379, 2007.

[24] B. Zhu and Z. Xu, "Regression methods for hesitant fuzzy preference relations," *Technological and Economic Development of Economy*, vol. 19, pp. S214–S227, 2014.

[25] M. Xia and Z. Xu, "Hesitant fuzzy information aggregation in decision making," *International Journal of Approximate Reasoning*, vol. 52, no. 3, pp. 395–407, 2011.

[26] M. Xia, Z. Xu, and H. Liao, "Preference relations based on intuitionistic multiplicative information," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 1, pp. 113–133, 2013.

[27] Z. Xu, "Priority weight intervals derived from intuitionistic multiplicative preference relations," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 4, pp. 642–654, 2013.

[28] B. Zhu, Z. Xu, and J. Xu, "Deriving a ranking from hesitant fuzzy preference relations under group decision making," *IEEE Transactions on Cybernetics*, vol. 44, no. 8, pp. 1328–1337, 2014.

[29] M. Xia, Z. Xu, and J. Chen, "Algorithms for improving consistency or consensus of reciprocal [0,1]-valued preference relations," *Fuzzy Sets and Systems*, vol. 216, pp. 108–133, 2013.

[30] Y. Dong, G. Zhang, W.-C. Hong, and Y. Xu, "Consensus models for AHP group decision making under row geometric mean prioritization method," *Decision Support Systems*, vol. 49, no. 3, pp. 281–289, 2010.

[31] E. Herrera-Viedma, F. Herrera, and F. Chiclana, "A consensus model for multiperson decision making with different preference structures," *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 32, no. 3, pp. 394–402, 2002.

[32] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, "Some issues on consistency of fuzzy preference relations," *European Journal of Operational Research*, vol. 154, no. 1, pp. 98–109, 2004.

[33] Z. Zhang, C. Wang, and X. Tian, "A decision support model for group decision making with hesitant fuzzy preference relations," *Knowledge-Based Systems*, vol. 86, pp. 77–101, 2015.

[34] Q. Pang, H. Wang, and Z. Xu, "Probabilistic linguistic term sets in multi-attribute group decision making," *Information Sciences*, vol. 369, pp. 128–143, 2016.

[35] Z. Xu and W. Zhou, "Consensus building with a group of decision makers under the hesitant probabilistic fuzzy environment," *Fuzzy Optimization and Decision Making. A Journal of Modeling and Computation Under Uncertainty*, vol. 16, no. 4, pp. 481–503, 2017.

[36] Y. Zhai, Z. Xu, and H. Liao, "Probabilistic linguistic vector-term set and its application in group decision making with multi-granular linguistic information," *Applied Soft Computing*, vol. 49, pp. 801–816, 2016.

[37] S. Zhang, Z. Xu, and Y. He, "Operations and integrations of probabilistic hesitant fuzzy information in decision making," *Information Fusion*, vol. 38, pp. 1–11, 2017.

[38] Y. Xu, F. J. Cabrerizo, and E. Herrera-Viedma, "A consensus model for hesitant fuzzy preference relations and its application in water allocation management," *Applied Soft Computing*, vol. 58, pp. 265–284, 2017.

[39] A. M. Feldman and R. Serrano, *Welfare Economics and Social Choice Theory*, Springer, Boston, MA, USA, 1980.

[40] M. Xia and Z. Xu, "Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 21, no. 6, pp. 865–897, 2013.

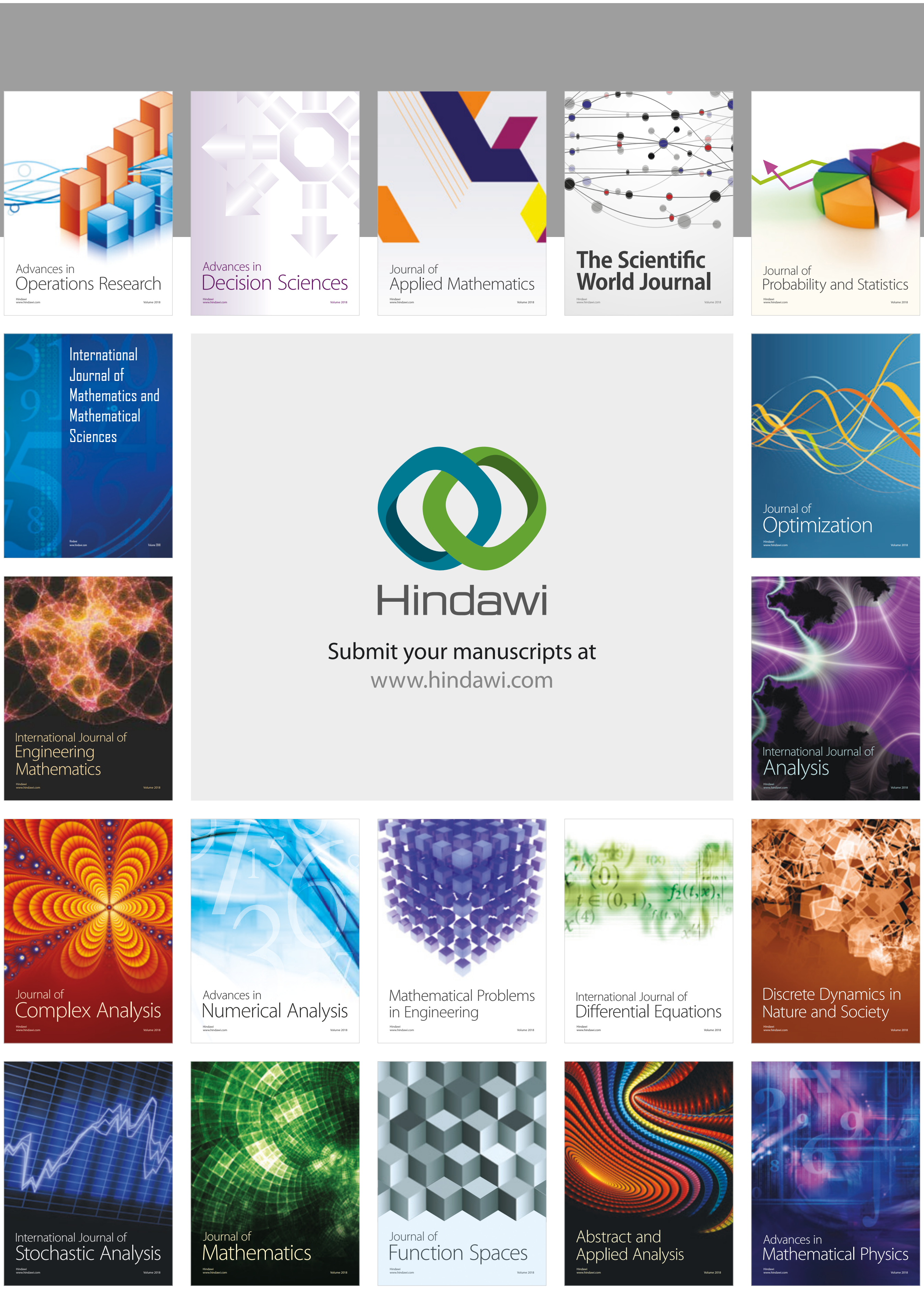
[41] W. Zhou and Z. S. Xu, "Probability calculation and element optimization of probabilistic hesitant fuzzy preference relations based on expected consistency," *IEEE Transactions on Fuzzy Systems*, 2017.

[42] Z. Xu and M. Xia, "Distance and similarity measures for hesitant fuzzy sets," *Information Sciences*, vol. 181, no. 11, pp. 2128–2138, 2011.

[43] Z. Wu and J. Xu, "A consistency and consensus based decision support model for group decision making with multiplicative preference relations," *Decision Support Systems*, vol. 52, no. 3, pp. 757–767, 2012.

[44] T. Tanino, "Fuzzy preference orderings in group decision making," *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 117–131, 1984.

[45] P. Bossuyt, *A Comparison of Probabilistic Unfolding Theories for Paired Comparisons Data*, Springer-Verlag, 1990.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