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Research Article

Newtonian and Non-Newtonian Fluids through Permeable Boundaries

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We considered the situation where a container with a permeable boundary is immersed in a larger body of fluid of the same kind. In this paper, we found mathematical expressions at the permeable interface Γ of a domain Ω, where $\Omega \subset \mathfrak{R}^3$. Γ is defined as a smooth two-dimensional (at least class C^2) manifold in Ω. The Sennet-Frenet formulas for curves without torsion were employed to find the expressions on the interface Γ. We modelled the flow of Newtonian as well as non-Newtonian fluids through permeable boundaries which results in nonhomogeneous dynamic and kinematic boundary conditions. The flow is assumed to flow through the boundary only in the direction of the outer normal $\mathbf{n}$, where the tangential components are assumed to be zero. These conditions take into account certain assumptions made on the curvature of the boundary regarding the surface density and the shape of Ω; namely, that the curvature is constrained in a certain way. Stability of the rest state and uniqueness are proved for a special case where a "shear flow" is assumed.

1. Introduction

The flow of incompressible Navier-Stokes fluids and fluids of second grade through permeable boundaries and past porous walls has been studied under various conditions. The equation of motion for incompressible flows in Newtonian fluids (Navier-Stokes equations) under no-slip boundary conditions has been studied extensively from many perspectives. Since the pioneering papers of Leray [1–3] and Hopf [4] questions of the existence, stability [5, 6], and uniqueness of both classical and weak solutions have received more than their fair share of attention.

Recently the same issues have been studied for non-Newtonian fluids of second grade. The studies cover both weak solutions [7–12] and classical solutions for homogeneous Dirichlet boundary data [13] and nonhomogeneous boundary data [6, 14, 15].

Unlike Newtonian fluids, fluids of second grade (and other non-Newtonian species) have the property of developing "normal stresses differences" at boundaries. It was shown, for example, by Berker [16] that if an incompressible flow of a fluid of grade two satisfies the homogeneous Dirichlet boundary condition. The stress at the boundary is given by $\mathbf{t} = (-p + \alpha|\boldsymbol{\omega}|^2)\mathbf{n} + [\mu\boldsymbol{\omega} + 2\alpha\partial_t\boldsymbol{\omega}] \wedge \mathbf{n}$, where $\mathbf{n}$ is the unit exterior normal to the boundary and $\boldsymbol{\omega} = \nabla\wedge\mathbf{v}$ is the vorticity. The wedge denotes a vector product. Thus there is a normal component of stress at the boundary in addition to the pressure. The question becomes *what governs the flow across the boundary*? Possible ways of circumventing this question may be to "prescribe" the normal component of the velocity field at the boundary or to prescribe mass or momentum flux. The prescription of shear stress has also been suggested. ([16, 17]). *Nonlinear* or *non-Newtonian* fluids are fluids like molten metals, multigrade oils, printing inks, paints, suspensions, polymer solutions, molten plastics, blood, protein solutions, and ice [18]. These fluids cannot be described by the above model. The study of these interesting substances has proved to be very important with the growth of the polymer and plastics industry over the last four decades. Consequently, an interest has arisen to study the flow of these nonlinear fluids and, in the case of this model, second-grade fluids, through permeable boundaries. The boundary conditions alone in such circumstances are an interesting topic for study. Works by Berker [16] and Rajagopal and Gupta [19] can be mentioned in this regard.

In this study we shall provide an alternative approach through the formulation of “dynamics at the boundary,” the idea being that the normal component of velocity at the boundary is viewed as an unknown function which satisfies a differential equation intricately coupled to the flow in the region “enclosed” by the boundary.

A glimpse of the history of the research on non-Newtonian and Newtonian fluids around porous boundaries is given in Section 2. Notation and definitions precede Sections 4 and 5 which deal with the constitutive equations and the modelling of permeability. In Section 6 the expressions on the interface Γ are given. The alternative model is studied and the stability and uniqueness are proved in Section 7. Section 8 concludes the study and further explorations are discussed.

2. Backgroud

Berker [16] studied the two-dimensional creeping flow of a second-order fluid with nonparallel porous walls. An additional velocity boundary condition was needed. The other conditions they used were due to the usual no-slip conditions. This additional velocity boundary condition was to prescribe the rate of shear at the wall. The problem was then solved numerically by a standard routine.

In 1989 Rajagopal and Kaloni [20] wrote remarks on boundary conditions for flows of fluids of the differential type. Rajagopal [21] discusses a lot of related issues. Rajagopal and Gupta [19] studied the flow of an incompressible fluid of second grade past an infinite porous plate subjected to either suction or blowing at the plate. They studied fluids modelled by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A} + \alpha_1\left[D_t\mathbf{A} + \mathbf{A}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}\right] + \alpha_2\mathbf{A}^2. \quad (1)$$

No assumptions were made about the material moduli α_1 and α_2. For the boundary value problem they considered, it was found that the velocity distributions do not depend on the normal stress modulus α_2, but the pressure does. They found that it was possible to produce an exact solution which is asymptotic in nature for both “suction” and “blowing” at the plate if the material modulus $\alpha_1 > 0$. For $\alpha_1 < 0$, they found that such solutions could not exist in the case of blowing, a result which was in keeping with the classical incompressible fluid. Fosdick and Rajagopal [22] have shown that the model (1) whose material modulus $\alpha_1 < 0$ exhibits anomalous behaviour was not to be expected of any fluid of rheological interest (also see [23]). Proudman studied an example of steady laminar flow at a large Reynolds number [24].

Beavers and Joseph [25] studied the flow of a Newtonian fluid over a porous surface in 1967. They found that if the governing differential system was not to be underdetermined, it was necessary to specify some condition on the tangential component of the velocity of the free fluid at the porous interface. It is usual in these analyses to approximate the fluid motion near the true boundary with an adherence condition for the tangential component of velocity of the free fluid at some boundary. Because of a certain ambiguity which is implied by the notion of a “true” boundary for a permeable material, it was found useful to define a nominal boundary. They fixed a nominal boundary by first defining a smooth geometric surface and then assuming that the outermost perimeters of all the surface pores of the permeable material are in this surface. Thus, if the surface pores were filled with solid material to the level of their respective perimeters, a smooth impermeable boundary of the assumed shape would result. This definition is precise when the geometry is simple (planes, spheres, cylinders, etc.) but may not be fully adequate in more complex situations. Beavers and Joseph’s [25] experiment was designed to examine the tangential flow in the boundary region of a permeable interface. The results of this experiment indicate that the effects of viscous shear appear to penetrate into the permeable material in a boundary layer region, producing a velocity distribution similar to that depicted in the following figure. The tangential component of the velocity of the free fluid at the porous boundary can be considerably greater than the mean filter velocity within the body of the porous material.

In Figure 1 the plane $y = 0$ defines a nominal surface for the permeable material. The flow through the body of the permeable material, which is homogeneous and isotropic, is assumed to be governed by Darcy’s Law. Read more of the status on Darcy’s Law in [26]. In the absence of body forces Darcy’s Law may be written as $Q = -(k/\mu)(dP/dx)$, where k is the “permeability” of the material and Q is the volume flow rate per unit of the cross-sectional area. As such, Q represents the filter velocity rather than the true velocity of the fluid in the pores. The measured pressure gradient is denoted by dP/dx.

3. Basic Notation

We work in Euclidean 3 space. The following notation will be used throughout:

$$|\mathbf{x}| := \sqrt{\sum_1^3 x_i^2} \text{ denotes the Euclidean norm.}$$

$$\partial_i := \frac{\partial}{\partial x_i}; \quad i = 1, 2, 3.$$

$$\partial_t := \frac{\partial}{\partial t}.$$

$$[\nabla p]_i := \partial_i p \quad \text{if } p \text{ is a scalar field.}$$

$$[\nabla\mathbf{v}]_{ij} := \partial_j v_i; \quad i, j = 1, 2, 3, \text{ if } \mathbf{v} \text{ is a vector field.}$$

$$[\nabla\mathbf{v}]_{ij}^T := \partial_i v_j; \quad i, j = 1, 2, 3, \text{ if } \mathbf{v} \text{ is a vector field.}$$

$$\nabla\cdot\mathbf{v} := \sum_{i=1}^3 \partial_i v_i \quad \text{if } \mathbf{v} \text{ is a vector field.}$$

$$\mathbf{v}\cdot\nabla := \sum_{i=1}^3 v_i\partial_i \quad \text{if } \mathbf{v} \text{ is a vector field.}$$

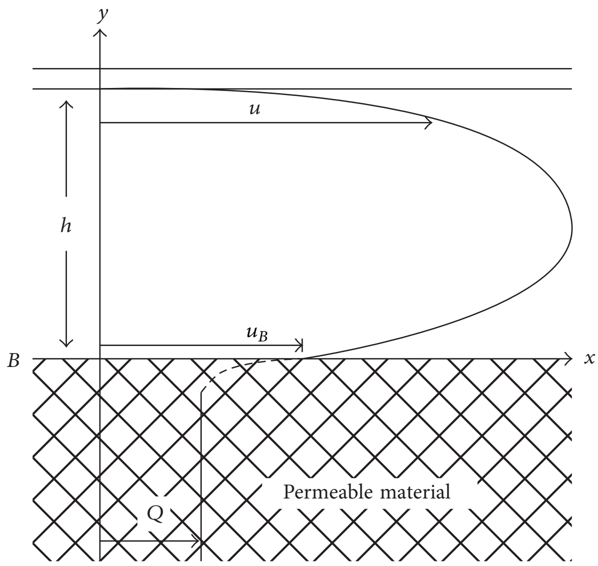

FIGURE 1: Velocity profile for the rectilinear flow in a horizontal channel formed by a permeable lower wall ($y = 0$) and an impermeable upper wall ($y = h$).

$$\begin{aligned}
&[\nabla \cdot \mathbf{T}]_j := \sum_{i=1}^{3} \partial_i T_{ij}; \quad j = 1,2,3, \\
&\qquad \text{if } \mathbf{T} \text{ is a matrix (tensor) with} \\
&\qquad \text{Euclidean components } T_{ij}. \\
&[\mathbf{v} \otimes \mathbf{v}]_{ij} := v_i v_j; \quad i, j = 1,2,3, \text{ if } \mathbf{v} \text{ is a vector.} \\
&D_t := \partial_t + \mathbf{v} \cdot \nabla; \quad D_t \text{ is the material time derivative.} \\
&\mathbf{v} \wedge \mathbf{u} := \text{denotes the usual vector} \\
&\qquad \text{product of the vectors } \mathbf{v} \text{ and } \mathbf{u} \\
&\nabla \wedge \mathbf{v} := \text{curl } \mathbf{v}.
\end{aligned} \tag{2}$$

If $\mathbf{A}$ and $\mathbf{B}$ are second order tensors we shall use the notations $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij}$ and $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a smooth (at least $\mathbf{C}^2$) boundary Γ. Let $\mathbf{n} = \mathbf{n}(x)$ denote the unit exterior normal to Γ at x. We shall be concerned with smooth vector fields $\mathbf{v} = \mathbf{v}(x)$ defined in Ω such that on Γ it has the form $\gamma_o \mathbf{v}(x) = -\eta(x)\mathbf{n}(x)$, where γ_o is the trace operator denoting boundary values and η is a smooth scalar field defined on Γ. Associated with $\nabla \mathbf{v}$ we define the symmetric and skew-symmetric tensors $\mathbf{A}$ and $\mathbf{W}$ as $\mathbf{A} = \mathbf{A}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^T$ and $\mathbf{W} = \mathbf{W}(\mathbf{v}) = \nabla \mathbf{v} - (\nabla \mathbf{v})^T$, where $(\nabla \mathbf{v})^T$ denotes the transpose of the gradient of $\mathbf{v}$. The rate of deformation tensor is related to $\mathbf{A}$ by $\mathbf{D}(\mathbf{v}) = (1/2)\mathbf{A}(\mathbf{v})$. We note that if $\mathbf{v}$ is solenoidal ($\nabla \cdot \mathbf{v} = 0$) then trace $\mathbf{A}(\mathbf{v}) = 2\nabla \cdot \mathbf{v} = 0$ and, for any vector $\mathbf{a}$, $\mathbf{W}(\mathbf{v})\mathbf{a} = \boldsymbol{\omega} \wedge \mathbf{a}$, where $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ denotes the vorticity associated with $\mathbf{v}$.

4. The Constitutive Equations

The stress tensor for the linear viscous Newtonian model is $\mathbf{T} = -p\mathbf{I} + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$, with p as the pressure, μ as the coefficient of viscosity, and $\mathbf{v}$ as the velocity of the fluid. This model describes the flow of fluids like water and other similar fluids. Lamb [27] and Ladyzhenskaya [28] wrote mathematical theories on viscous incompressible flow.

Fluids of a *differential type* [29–31], of which Rivlin-Ericksen fluids are a subclass, are depicted by a popular nonlinear model. Fluids of *complexity* n form an important subclass of the fluids of a differential type. For incompressible fluids of complexity n the Cauchy stress tensor is of the form $\mathbf{T} = -p\mathbf{I} + \mathbf{F}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. The pressure p is not a thermodynamic variable and the term $-p\mathbf{I}$ reflects Pascal's law, which is inherent to all fluids. $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are the first n Rivlin-Ericksen tensors [21] defined recursively by

$$\begin{aligned}
\mathbf{A}_1 &= \nabla \mathbf{v} + (\nabla \mathbf{v})^T = \mathbf{A}, \\
\mathbf{A}_n &= D_t \mathbf{A}_{n-1} + \mathbf{A}_{n-1} (\nabla \mathbf{v}) + (\nabla \mathbf{v})^T \mathbf{A}_{n-1}, \quad n \geq 2.
\end{aligned} \tag{3}$$

Fluids of *grade n* are examples of fluids of complexity n. The stress tensors for fluids of grades 1 and 2 respectively, are assumed to be of the form

$$\begin{aligned}
\mathbf{T}^{[1]} &= -p\mathbf{I} + \mu \mathbf{A}_1, \\
\mathbf{T}^{[2]} &= \mathbf{T}^{[1]} + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2,
\end{aligned} \tag{4}$$

where μ and α_i are material coefficients (possibly temperature-dependent).

For incompressible fluids of second grade, the stress-deformation relation then becomes

$$\begin{aligned}
\mathbf{T} = \mathbf{T}^{[2]} &= -p\mathbf{I} + \mu \mathbf{A} + \alpha_1 D_t \mathbf{A} \\
&\quad + \alpha_1 \left(\mathbf{A} \nabla \mathbf{v} + (\nabla \mathbf{v})^T \mathbf{A} \right) + \alpha_2 \mathbf{A}^2,
\end{aligned} \tag{5}$$

where p and $\mathbf{v}$ are the pressure and the velocity fields. Here μ is the coefficient of viscosity and α_1 and α_2 are material coefficients or "normal stress moduli." In this case $\mathbf{A} = \mathbf{A}_1$.

To use the relation (5) for the modelling of a fluid, the fluid has to be compatible with thermodynamics in the sense that all flows of the fluid must satisfy the Clausius-Duhem inequality, and the assumption must be made that the specific Helmholtz free energy is at a minimum when the fluid is in equilibrium. Under these assumptions, α_1 and α_2 [32] must satisfy

$$\alpha_1 + \alpha_2 = 0. \tag{6}$$

Considerations of stability of the rest state require the assumptions μ and α_1 to be nonnegative; that is, $\mu > 0$, $\alpha_1 > 0$. See [32]. Under assumption (6), which we shall follow throughout, the form of the stress tensor $\mathbf{T}$ given in (5) reduces to a more compact expression. To obtain this we note that $\nabla \mathbf{v} = (1/2)(\mathbf{A} + \mathbf{W})$ and $(\nabla \mathbf{v})^T = (1/2)(\mathbf{A} - \mathbf{W})$, so that

$$\begin{aligned}
\alpha_1 \left(\mathbf{A} \nabla \mathbf{v} + (\nabla \mathbf{v})^T \mathbf{A} \right) &= \frac{\alpha_1}{2} \left[\mathbf{A} (\mathbf{A} + \mathbf{W}) + (\mathbf{A} - \mathbf{W}) \mathbf{A} \right] \\
&= \alpha_1 \mathbf{A}^2 + \frac{\alpha_1}{2} (\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}).
\end{aligned} \tag{7}$$

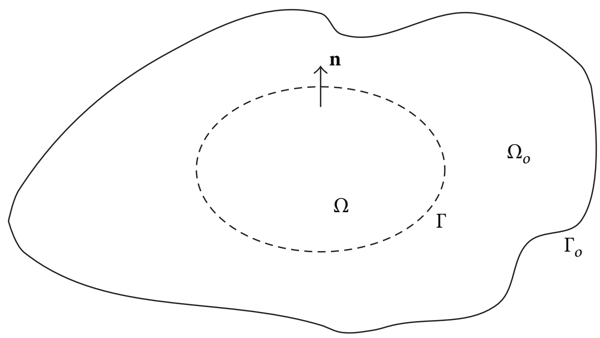

FIGURE 2: Profile for normal flow through the permeable wall Γ.

Therefore, by (5) and (7)

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A} + \alpha D_t\mathbf{A} + \frac{\alpha}{2}(\mathbf{AW} - \mathbf{WA}), \quad (8)$$

where we have set $\alpha_1 = \alpha$.

Remark 1. Please note that for the Navier-Stokes equations we take $\alpha = 0$ [33].

5. Modelling of Permeability

We study the motion of fluids around and through a fixed porous container filled with the same fluid. The interior of the porous container is an open bounded set $\Omega \subset \mathbf{R}^3$ and the porous boundary, Γ, is smooth. The surrounding fluid domain, Ω_o, is bounded and its outer boundary is denoted by Γ_o. The exterior normal to Ω on Γ is denoted by **n**. Figure 2 illustrates the situation where the curvature of the boundary Γ of Ω is nonnegative.

Permeability of the walls of the container is described by assuming that at the boundary Γ the flow **v** has the direction of the normal:

$$\gamma_o\mathbf{v}(x,t) = -\eta(x,t)\,\mathbf{n}(x). \quad (9)$$

The velocity component η is treated as an unknown and an evolution equation has to be found for it. We model the surface Γ as having an *effective area measure da* which has a density function $\zeta(x)$ with respect to the area measure ds. Thus $da = \zeta(x)ds$. The effective area through which fluid can permeate is not more than the surface area and therefore $0 \le \zeta(x) \le 1$ for any $x \in \Gamma$. If $\zeta(x) \equiv 0$, the wall is impermeable and if $\zeta(x) \equiv 1$, there is no wall.

In order to obtain expressions for mass and momentum in a boundary patch Γ', we let the patch be heuristically represented by a volume G built from copies of Γ' (Figure 3). This is in line with the Beavers-Joseph thinking which was discussed before. For this volume we set up a coordinate system consisting of a "radial part" r, which has the direction of the normal vector **n**, and a "surface part" made up by vectors tangential to Γ'. For the mass of G we obtain

$$\int_G \rho\,dx = \int_{\Gamma'}\int_0^\delta \rho\,dr\,da = \int_{\Gamma'}\int_0^\delta \rho\,dr\,\zeta\,ds = \int_{\Gamma'} \rho\zeta\delta\,ds, \quad (10)$$

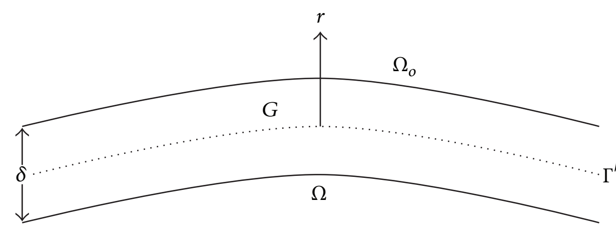

FIGURE 3: Heuristics of the permeable boundary.

where δ is some measure of thickness. With the aid of these concepts we introduce the *surface density* of the fluid at $x \in \Gamma$ as

$$\sigma(x) = \delta(x)\,\zeta(x)\,\rho, \quad (11)$$

where ρ is the volume density of the fluid.

To obtain the equation of motion for fluid in the boundary, we assume that the rate of change of linear momentum in the boundary is explained by stress forces at both sides of the boundary.

Let **T** and $\mathbf{T}'$ denote the stress tensors at the sides of the boundary facing Ω and Ω_o, respectively, and let **P** and $\mathbf{P}'$ denote the transfer-of-momentum tensors on the two sides. On an arbitrary boundary patch $\Gamma' \subset \Gamma$ the law of conservation of linear momentum is stated in the following way:

$$\partial_t \int_{\Gamma'} \sigma(x)\,\gamma_o\mathbf{v}\,ds = \int_{\Gamma'} \left[\mathbf{Pn} - \mathbf{P}'\mathbf{n}\right] da + \int_{\Gamma'} \left[-\mathbf{T}'(-\mathbf{n}) - \mathbf{Tn}\right] ds \quad (12)$$

with σ as defined in (11), and it follows that

$$\sigma(x)\,\partial_t\gamma_o\mathbf{v} - \zeta\left[\mathbf{P} - \mathbf{P}'\right]\mathbf{n} = \left[\mathbf{T}' - \mathbf{T}\right]\mathbf{n}. \quad (13)$$

From (13) we have $\sigma(x)\partial_t\eta(x,t) + \zeta\mathbf{n}\cdot[\mathbf{P} - \mathbf{P}']\mathbf{n} = \mathbf{n}\cdot\mathbf{Tn} - \mathbf{n}\cdot\mathbf{T}'\mathbf{n}$. In the domain Ω the momentum flux tensor is given by $\mathbf{P} = \rho\mathbf{v}\otimes\mathbf{v}$. In accordance with this, we shall take $\mathbf{P} = \rho\eta^2\mathbf{n}\otimes\mathbf{n}$ at the boundary. The tensor $\mathbf{P}'$ will be taken as zero.

We take $\mathbf{T}' = \ell\mathbf{I}$ to obtain from (13)

$$\sigma(x)\,\partial_t\eta + \zeta\rho\eta^2 = \mathbf{n}\cdot\mathbf{Tn} - \ell(t). \quad (14)$$

From the incompressibility of the flow in Ω it follows that

$$-\int_\Gamma \eta\,ds = 0. \quad (15)$$

6. Expressions at the Interface

In order to obtain expressions for the stress tensors **T** and $\mathbf{T}'$ as well as the acceleration at the boundary through which only normal flow occurs, we obtain a formal expression for the symmetric tensor **A** on a surface which is immersed in fluid. We shall eventually use these expressions in postulating the form of $\mathbf{T}$ and $\mathbf{T}'$ and in formulating a boundary condition which expresses zero tangential acceleration at a wall.

We consider a smooth vector field $\mathbf{v}(\mathbf{x})$ defined on a domain $\Omega \subset \mathbf{R}^3$ and a smooth two-dimensional (at least class C^2) manifold $\Gamma \subset \Omega$ so that $\mathbf{v}$ and $\nabla\mathbf{v}$ are defined on Γ. Let $\mathbf{n}(\mathbf{x})$ be the unit normal to Γ at the point $\mathbf{x} \in \Gamma$.

At any point $\mathbf{x}$ on Γ we consider two orthogonal curves c_1 and c_2 in a neighbourhood of x parametrised by arc lengths s_1 and s_2, respectively. Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be the unit tangents to the principal normal curves at a point on the surface. For local coordinates we use the orthogonal system formed by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\mathbf{n}$. Under the convention that $\boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_2 = \mathbf{n}$ we have $\mathbf{n} \wedge \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$ and $\mathbf{n} \wedge \boldsymbol{\tau}_2 = -\boldsymbol{\tau}_1$. Let κ_1 and κ_2 represent the principal curvatures at a point on the surface and let $K = \kappa_1 + \kappa_2$ denote the *mean curvature*.

Assumptions

(1) We shall assume throughout that the surface density is bounded and bounded away from zero; that is, there exist constants s and S such that

$$0 < s \le \sigma(x) \le S \quad \forall x \in \Gamma. \tag{16}$$

Also, we assume $\sigma \in C^\infty(\Gamma)$.

(2) Apart from the smoothness of Γ we make two additional assumptions regarding the shape of Ω; namely, that the curvatures κ_1, κ_2, and K are constrained in the following way:

(a) There exist constants g and G such that

$$0 < g \le K(x) \le G \quad \forall x \in \Gamma. \tag{17}$$

(b) There exists a constant H such that

$$0 \le \kappa_1^2 + \kappa_2^2 \le H^2 \quad \text{on } \Gamma. \tag{18}$$

Note that these assumptions allow cases where κ_1 and κ_2 can be of opposite signs.

The Frenet-Serret [34–36] formulae in this case, providing that there is no torsion, are then

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial s_1} &= -\kappa_1 \boldsymbol{\tau}_1 & \frac{\partial \mathbf{n}}{\partial s_2} &= -\kappa_2 \boldsymbol{\tau}_2 \\ \frac{\partial \boldsymbol{\tau}_1}{\partial s_1} &= \kappa_1 \mathbf{n} & \frac{\partial \boldsymbol{\tau}_1}{\partial s_2} &= 0 \\ \frac{\partial \boldsymbol{\tau}_2}{\partial s_1} &= 0 & \frac{\partial \boldsymbol{\tau}_2}{\partial s_2} &= \kappa_2 \mathbf{n}. \end{aligned} \tag{19}$$

The surface gradient ∇_s of a scalar function f may be written as

$$\gamma_o(\nabla f) = \nabla_s(\gamma_o f) + \mathbf{n}\gamma_1 f, \tag{20}$$

where the trace operator γ_1 denotes the normal derivative. Also consider

$$\begin{aligned} \nabla_s f &= \frac{\partial f}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial f}{\partial s_2}\boldsymbol{\tau}_2, \\ \Delta_s f &= \nabla_s \cdot (\nabla_s f) = \frac{\partial^2 f}{\partial^2 s_1^2} + \frac{\partial^2 f}{\partial^2 s_2^2}. \end{aligned} \tag{21}$$

If $\mathbf{f}$ is a vector field defined on Γ, the surface gradient ∇_s is defined as the tensor

$$\nabla_s \mathbf{f} = \frac{\partial \mathbf{f}}{\partial s_1} \otimes \boldsymbol{\tau}_1 + \frac{\partial \mathbf{f}}{\partial s_2} \otimes \boldsymbol{\tau}_2. \tag{22}$$

Surface divergence and surface curl are defined as

$$\begin{aligned} \nabla_s \cdot \mathbf{f} &= \boldsymbol{\tau}_1 \cdot \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau}_2 \cdot \frac{\partial \mathbf{f}}{\partial s_2}, \\ \nabla_s \wedge \mathbf{f} &= \boldsymbol{\tau}_1 \wedge \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau}_2 \wedge \frac{\partial \mathbf{f}}{\partial s_2}. \end{aligned} \tag{23}$$

The relationship between the surface operators and the volume operators for a function defined in Ω is given by

$$\gamma_o(\nabla \mathbf{f}) = \nabla_s \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{f}] \otimes \mathbf{n}, \tag{24}$$

$$\gamma_o(\nabla \cdot \mathbf{f}) = \nabla_s \cdot \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{f}] \cdot \mathbf{n}, \tag{25}$$

$$\gamma_o(\nabla \wedge \mathbf{f}) = \nabla_s \wedge \gamma_o \mathbf{f} + \mathbf{n} \wedge \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{f}]. \tag{26}$$

We use (20)–(25) to prove more important results to make the calculations easier.

Lemma 2. *Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be two orthogonal unit tangential vectors and let $\mathbf{n}$ be the exterior unit normal vector to Γ. Let α, β, and γ be scalar functions; then*

$$\begin{aligned} &(a)\ \nabla_s \cdot (\alpha \boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1} \\ &(b)\ \nabla_s \cdot (\beta \boldsymbol{\tau}_2) = \frac{\partial \beta}{\partial s_2} \\ &(c)\ \nabla_s \cdot (\gamma \mathbf{n}) = -\gamma K. \end{aligned} \tag{27}$$

Proof. Consider the following:

(a) $\nabla_s \cdot (\alpha \boldsymbol{\tau}_1) = \boldsymbol{\tau}_1 \cdot \frac{\partial}{\partial s_1}(\alpha \boldsymbol{\tau}_1) + \boldsymbol{\tau}_2 \cdot \frac{\partial}{\partial s_2}(\alpha \boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1}$.

(b) Similar to (a).

$$\begin{aligned} (c)\ \nabla_s \cdot (\gamma \mathbf{n}) &= \boldsymbol{\tau}_1 \cdot \left[\frac{\partial \gamma}{\partial s_1}\mathbf{n} - \kappa_1 \gamma \boldsymbol{\tau}_1\right] \\ &\quad + \boldsymbol{\tau}_2 \cdot \left[\frac{\partial \gamma}{\partial s_2}\mathbf{n} - \gamma \kappa_2 \boldsymbol{\tau}_2\right] \\ &= -\gamma[\kappa_1 + \kappa_2] = -\gamma K. \end{aligned} \tag{28}$$

□

We shall apply the expressions above to $\mathbf{v}$. By the Frenet-Serret formulae (torsion is zero) $(\partial/\partial s_1)(\gamma_o \mathbf{v}) = -(\partial\eta/\partial s_1)\mathbf{n} - \eta(\partial \mathbf{n}/\partial s_1) = -(\partial \eta/\partial s_1)\mathbf{n} + \kappa_1 \boldsymbol{\tau}_1 \eta$, and, similarly, $\partial(\gamma_0 \mathbf{v})/\partial s_2 = -(\partial\eta/\partial s_2)\mathbf{n} + \kappa_2 \boldsymbol{\tau}_2 \eta$. Hence

$$\begin{aligned}\nabla_s\gamma_o\mathbf{v} &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad -\left[\frac{\partial\eta}{\partial s_1}\mathbf{n}\otimes\boldsymbol{\tau_1}+\frac{\partial\eta}{\partial s_2}\mathbf{n}\otimes\boldsymbol{\tau_2}\right] \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad -\mathbf{n}\otimes\left[\frac{\partial\eta}{\partial s_1}\boldsymbol{\tau_1}+\frac{\partial\eta}{\partial s_2}\boldsymbol{\tau_2}\right] \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right]-\mathbf{n}\otimes\nabla_s\eta.\end{aligned} \tag{29}$$

The transpose is given by

$$\left(\nabla_s\left(\gamma_o\mathbf{v}\right)\right)^T = \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right]-\nabla_s\eta\otimes\mathbf{n}. \tag{30}$$

To find an expression for $\mathbf{A}$ at Γ, we need an expression for $\nabla\mathbf{v}$ on the boundary:

$$\gamma_o\left(\nabla\mathbf{v}\right) = \nabla_s\left(\gamma_o\mathbf{v}\right)+\gamma_o\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\otimes\mathbf{n}. \tag{31}$$

Although we know that the divergence of $\mathbf{v}$ will be zero, it is helpful to observe that $\theta = \gamma_o(\nabla\cdot\mathbf{v}) = \nabla_s\cdot(\gamma_o\mathbf{v})+\gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n}$, where

$$\begin{aligned}\nabla_s\cdot\left(\gamma_o\mathbf{v}\right) &= -\boldsymbol{\tau_1}\cdot\frac{\partial}{\partial s_1}\left(\eta\mathbf{n}\right)-\boldsymbol{\tau_2}\cdot\frac{\partial}{\partial s_2}\left(\eta\mathbf{n}\right) \\ &= -\boldsymbol{\tau_1}\cdot\left[\frac{\partial\eta}{\partial s_1}\mathbf{n}-\eta\kappa_1\boldsymbol{\tau_1}\right]-\boldsymbol{\tau_2}\cdot\left(\frac{\partial\eta}{\partial s_2}\mathbf{n}-\eta\kappa_2\boldsymbol{\tau_2}\right) \\ &= \eta\left(\kappa_1+\kappa_2\right) \\ &= \eta K.\end{aligned} \tag{32}$$

Hence $\theta = \eta K+\gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n}$.

We proceed to find expressions for $\gamma_o(\mathbf{v}\cdot\nabla)\mathbf{v}$, $\gamma_o(\nabla\mathbf{v})$, and $\gamma_o(\nabla\mathbf{v})^T$.

We know that $\boldsymbol{\omega}\wedge\mathbf{n} = \mathbf{W}(\mathbf{v})\mathbf{n} = (\mathbf{n}\cdot\nabla)\mathbf{v}-(\nabla\mathbf{v})^T\mathbf{n}$, and

$$\begin{aligned}\gamma_o(\nabla\mathbf{v})^T\mathbf{n} &= \left(\nabla_s\left(\gamma_o\mathbf{v}\right)\right)^T\mathbf{n}+\gamma_o\left(\left[\mathbf{n}\cdot\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\mathbf{n}\right) \\ &= -\left(\nabla_s\eta\right)+\left(\theta-K\eta\right)\mathbf{n}.\end{aligned} \tag{33}$$

Therefore,

$$\left(\mathbf{n}\cdot\nabla\right)\gamma_o\mathbf{v} = \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)-\nabla_s\eta+\left(\theta-K\eta\right)\mathbf{n}. \tag{34}$$

Multiply (34) with $-\eta$ to obtain

$$\gamma_o\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = -\eta\left(\mathbf{n}\cdot\nabla\right)\gamma_o\mathbf{v} = K\eta^2\mathbf{n}+\eta\left[\nabla_s\eta-\boldsymbol{\gamma_0\omega}\wedge\mathbf{n}\right]. \tag{35}$$

From (31) we now obtain

$$\begin{aligned}\gamma_o\left(\nabla\mathbf{v}\right) &= \nabla_s\mathbf{v}+\left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}-\nabla_s\eta+\left(\theta-K\eta\right)\mathbf{n}\right]\otimes\mathbf{n} \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad -\mathbf{n}\otimes\nabla_s\eta-\nabla_s\eta\otimes\mathbf{n}+\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n} \\ &\quad +\left(\theta-K\eta\right)\mathbf{n}\otimes\mathbf{n} \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}-K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad -\left[\mathbf{n}\otimes\nabla_s\eta+\nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad +\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n}+\theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{36}$$

The transpose is

$$\begin{aligned}\gamma_o(\nabla\mathbf{v})^T &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}-K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad -\left[\mathbf{n}\otimes\nabla_s\eta+\nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad +\mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)+\theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{37}$$

Thus we have

$$\begin{aligned}\gamma_o\left(\mathbf{A}\right) &= \gamma_o\left(\nabla\mathbf{v}\right)+\gamma_o(\nabla\mathbf{v})^T \\ &= \nabla_s\gamma_o\mathbf{v}+\left(\nabla_s\gamma_o\mathbf{v}\right)^T+\gamma_o\left(\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\otimes\mathbf{n}\right) \\ &\quad +\gamma_o\left(\mathbf{n}\otimes\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\right) \\ &= 2\eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}-K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad -2\left[\mathbf{n}\otimes\nabla_s\eta+\nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad +\mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)+\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n}.+2\theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{38}$$

Let us define the symmetrical tensors $\mathbf{M}$ and $\mathbf{N}$ by

$$\begin{aligned}\mathbf{M} &= \left[K\mathbf{n}\otimes\mathbf{n}-\left(\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1}+\kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right)\right], \\ \mathbf{N} &= \mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)+\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n} \\ &\quad +2\theta\mathbf{n}\otimes\mathbf{n}-2\left(\mathbf{n}\otimes\nabla_s\eta+\nabla_s\eta\otimes\mathbf{n}\right) \\ &= \mathbf{n}\otimes\left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}-2\nabla_s\eta\right] \\ &\quad +\left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}-2\nabla_s\eta\right]\otimes\mathbf{n}+2\theta\mathbf{n}\otimes\mathbf{n} \\ &= \mathbf{n}\otimes\boldsymbol{\psi}+\boldsymbol{\psi}\otimes\mathbf{n}-2\theta\mathbf{n}\otimes\mathbf{n},\end{aligned} \tag{39}$$

with

$$\boldsymbol{\psi} = \boldsymbol{\gamma_o\omega}\wedge\mathbf{n}-2\nabla_s\eta \tag{40}$$

a tangential vector. Then, for a vector field of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ, from (38) we have

$$\gamma_o\mathbf{A} = -2\eta\mathbf{M}+\mathbf{N} \quad \text{on } \Gamma. \tag{41}$$

In local coordinates we have the representations

$$\mathbf{M} = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & K \end{pmatrix},$$
$$\mathbf{N} = \begin{pmatrix} 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau_1} \\ 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau_2} \\ \boldsymbol{\psi}\cdot\boldsymbol{\tau_1} & \boldsymbol{\psi}\cdot\boldsymbol{\tau_2} & -2\theta \end{pmatrix}. \tag{42}$$

If $\nabla\cdot\mathbf{v} = 0$, it follows that $\mathrm{tr}\mathbf{A} = 0$, which is in line with incompressibility.

We would further like to obtain expressions for the terms $\mathbf{n}\cdot\Delta\mathbf{v}$, $\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}$, and $\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}$ on the boundary Γ.

Lemma 3. *Let* $\mathbf{n}$ *be the exterior normal to the boundary* Γ, $\mathbf{v}\in\mathscr{D}$, *and* $\mathbf{A} = -2\eta\mathbf{M}+\mathbf{N}$ *with* $\mathbf{M}$ *and* $\mathbf{N}$ *as defined in* (39). *We assume that* $\nabla\cdot\mathbf{v}=0$ *and* $\boldsymbol{\gamma_o\omega}\wedge\mathbf{n} = 2\nabla_s\eta$, *which implies that* $\mathbf{N}=0$. *Then*

$$(a)\ \gamma_o(-\mathbf{n}\cdot\Delta\mathbf{v}) = \Delta_s\eta$$
$$(b)\ \gamma_o[\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}] = -4\eta^2K_G - 2\eta\Delta_s\eta \tag{43}$$
$$(c)\ \gamma_o[\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}] = 0,$$

where K_G *denotes the Gauss-curvature.*

Proof. (a) We have chosen $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$, and $\mathbf{n}$ so that $\boldsymbol{\tau_1}\wedge\boldsymbol{\tau_2} = \mathbf{n}$. In view of the incompressibility and the fact that there is zero tangential velocity

$$\begin{aligned}\gamma_o(-\Delta\mathbf{v}) &= \nabla\wedge\boldsymbol{\gamma_o\omega}\\ &= \boldsymbol{\tau_1}\wedge\partial_{s_1}[\eta_1\boldsymbol{\tau_2}-\eta_2\boldsymbol{\tau_1}] + \boldsymbol{\tau_2}\wedge\partial_{s_2}[\eta_1\boldsymbol{\tau_2}-\eta_2\boldsymbol{\tau_1}]\\ &\quad + \text{a tangential term}\\ &= \boldsymbol{\tau_1}\wedge[\partial_{s_1}\eta_1\boldsymbol{\tau_2} - \eta_2\kappa_1\mathbf{n}]\\ &\quad + \boldsymbol{\tau_2}\wedge[\eta_1\kappa_2\mathbf{n} - \partial_{s_1}\eta_2\boldsymbol{\tau_1}] + \cdots\\ &= (\partial_{s_1}\eta_1 + \partial_{s_2}\eta_2)\mathbf{n} + \cdots\\ &= \Delta_s\eta\mathbf{n}.\end{aligned} \tag{44}$$

(b) Consider the tensor $\nabla\cdot\mathbf{A}$ built from "row vectors" with $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ a basis for $\mathbf{R}^3$. Then

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \begin{pmatrix}\nabla\cdot\mathbf{Ae}_1\\ \nabla\cdot\mathbf{Ae}_2\\ \nabla\cdot\mathbf{Ae}_3\end{pmatrix}\\ &= \begin{pmatrix}\nabla_s\cdot\mathbf{Ae}_1\\ \nabla_s\cdot\mathbf{Ae}_2\\ \nabla_s\cdot\mathbf{Ae}_3\end{pmatrix} + \begin{pmatrix}\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{Ae}_1]\\ \mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{Ae}_2]\\ \mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{Ae}_3]\end{pmatrix}\\ &= \nabla_s\cdot\mathbf{A} + [(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}.\end{aligned} \tag{45}$$

Hence, $\gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n} = \gamma_o[\nabla\cdot\mathbf{A}] - \nabla_s\cdot\gamma_o(\mathbf{A})$.

Furthermore,

$$\nabla_s\cdot\gamma_o[\mathbf{A}] = -2(M\nabla_s\eta + \eta\nabla_s\cdot M) \tag{46}$$

and now

$$\begin{aligned}\mathbf{n}\cdot[\nabla_s\cdot[\gamma_o\mathbf{A}]] &= -2(\mathbf{n}\cdot M\nabla_s\eta + \eta\mathbf{n}\cdot(\nabla_s\cdot M))\\ &= -2\eta\mathbf{n}\cdot(\nabla_s\cdot M).\end{aligned} \tag{47}$$

Here we used the fact that $M\mathbf{n} = -K\mathbf{n}$. Determine $\mathbf{n}\cdot[\nabla_s\cdot M]$ term by term to obtain

$$\begin{aligned}\mathbf{n}\cdot[\nabla_s\cdot[\gamma_o\mathbf{A}]] &= -2\eta(\kappa_1^2+\kappa_2^2-K^2)\\ &= -2\eta(2\kappa_1\kappa_2) = -4K_G\eta.\end{aligned} \tag{48}$$

K_G denotes the Gauss-curvature and is bounded by assumptions (17) and (18). Hence

$$\gamma_o\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n} = \gamma_o[\mathbf{n}\cdot\Delta\mathbf{v}] - 4\eta K_G = -4\eta K_G - \Delta_s\eta. \tag{49}$$

The term we use in the proof of (57) is therefore

$$-\eta\gamma_o[\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}] = +4\eta^2K_G + \eta\Delta_s\eta. \tag{50}$$

(c) $\mathbf{n}\cdot(\mathbf{AW}-\mathbf{WA})\mathbf{n} = \mathbf{An}\cdot\mathbf{Wn} + \mathbf{Wn}\cdot\mathbf{An} = 2\mathbf{An}\cdot\mathbf{Wn}$.

Here we make use of the additional boundary conditions (52) and (55) and the fact that $\mathbf{Wn} = \boldsymbol{\gamma_o\omega}\wedge\mathbf{n}$ to obtain that

$$\begin{aligned}\mathbf{An}\cdot\mathbf{Wn} &= \mathbf{An}\cdot(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n})\\ &= (\boldsymbol{\gamma_o\omega}\wedge\mathbf{n})\cdot[-2\eta K\mathbf{n}]\\ &= 0.\end{aligned} \tag{51}$$

□

6.1. Explicit Form of the Dynamic Boundary Condition. It is shown that for a smooth two-dimensional manifold Γ contained in a domain $\Omega\subset\mathbf{R}^3$ the following is true for a vector field $\mathbf{v}$ which is of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ:

$$\gamma_o[\mathbf{A}] = -2\eta\mathbf{M} + \mathbf{N}, \tag{52}$$

where $\mathbf{M}$ and $\mathbf{N}$ are defined in (39).

If $\mathbf{v}$ is solenoidal, as in the case under consideration, $\theta = 0$. A straightforward application of Stokes' theorem shows that $\boldsymbol{\omega}$ is tangential to the boundary, which implies that $\boldsymbol{\psi}$ is tangential to the boundary. Indeed, let Γ' be any patch of the surface Γ; then

$$\int_{\Gamma'}(\nabla\wedge\mathbf{f})\cdot\mathbf{n}\,ds = \int_{\partial\Gamma'}\mathbf{f}\cdot d\boldsymbol{\tau}, \tag{53}$$

where $d\boldsymbol{\tau}$ is a vector tangential to the boundary. Now if $\mathbf{f} = \gamma_o\mathbf{v} = -\eta\mathbf{n}$, then $\int_{\partial\Gamma'}\mathbf{f}\cdot d\boldsymbol{\tau} = 0$, and that implies that $\int_{\Gamma'}(\nabla\wedge\mathbf{v})\cdot\mathbf{n}ds = 0$ for all $\Gamma'\subset\Gamma$, which in turn implies that $(\nabla\wedge\mathbf{v})\cdot\mathbf{n} = 0$.

In the problem under consideration we shall assume that the "rate of deformation" tensor $\mathbf{A}$ has precisely the form (52) on the boundary Γ with $\mathbf{n}$ the unit exterior normal (the traditional rate of deformation is defined as $\mathbf{D} = (1/2)\mathbf{A}$).

We shall consider a kinematic boundary condition, which has a physical meaning in that there are no tangential components of deformation at the interface boundary. This concerns the form of the tensor $\mathbf{N}$.

Towards this, we observe from (52) that

$$\gamma_0 [\mathbf{A}\mathbf{n}] = -2K\eta\mathbf{n} + \boldsymbol{\psi}. \tag{54}$$

It follows from (54) that there are no tangential components of deformation at Γ if and only if $\boldsymbol{\psi} = 0$; that is,

$$\boldsymbol{\gamma}_{\mathbf{o}}\boldsymbol{\omega} \wedge \mathbf{n} = 2\nabla_s \eta. \tag{55}$$

This is the kinematic boundary condition.

The various terms in $\mathbf{n}\cdot\mathbf{T}\mathbf{n}$, with $\mathbf{T}$ on a surface Γ, given by (8), may be expressed as follows (see Lemma 3):

$$\begin{gathered}\mathbf{n}\cdot\gamma_o\mathbf{A}\mathbf{n} = -2K\eta \\ \mathbf{n}\cdot\gamma_o\partial_t[\mathbf{A}\mathbf{n}] = \partial_t[\mathbf{n}\cdot\mathbf{A}\mathbf{n}] = -2K\eta_t\end{gathered} \tag{56}$$

$$\gamma_o [\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}] = +4K_G\eta^2 + \eta\Delta_s\eta \tag{57}$$

$$\mathbf{n}\cdot[\mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A}]\mathbf{n} = 0. \tag{58}$$

Guided by these expressions and (8), we assume that, at Γ,

$$\mathbf{n}\cdot\gamma_o\mathbf{T}\mathbf{n} = -\left[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta\right]. \tag{59}$$

For the stress tensor $\mathbf{T}'$ in the fluid exterior to Ω we assume that $\mathbf{n}\cdot\mathbf{T}'\mathbf{n} = \ell(t)$. This amounts to the situation where the fluid in Ω_o is at rest. As a result we have

$$\begin{aligned}\mathbf{n}\cdot(\delta\mathbf{T})\mathbf{n} &= \mathbf{n}\cdot\left[\mathbf{T} - \mathbf{T}'\right]\mathbf{n} \\ &= -\left[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta\right] \\ &\quad - \ell(t).\end{aligned} \tag{60}$$

From (13), (14), and (60) we obtain

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta) \tag{61}$$

with $s(\eta) = \sigma^{-1/2}[(-k + 4\alpha K_G)\eta^2 - 2\mu\eta K + \alpha\eta\Delta_s\eta - \ell(t)]$, and $k = \zeta\rho$.

Equation (61) is the explicit form of the dynamic boundary condition.

7. An Alternative Model: *Problem $\mathscr{A}$*

Although it was possible to prove stability and uniqueness for the original model (see [33, 37]), we could not find a way to a possible proof of existence for a classical solution. In this chapter we describe an alternative model which displays all the properties of the original problem with respect to stability and uniqueness.

In the alternative model the dynamics at the boundary are formulated by assuming a "shear flow" of the form

$$\mathbf{v}^*(y,t) = -\eta(s_1, s_2, t)\,\mathbf{n}(y) \tag{62}$$

with s_1 and s_2 as the surface parameters (like arc length). It is assumed that the "body force" acting on the shearing fluid at the boundary is proportional to the difference between the pressures $\gamma_o p$ and $\ell(t)$. Under these assumptions the equation governing the evolution of η is

$$\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta + \delta^{-1}\ell(t), \tag{63}$$

where $\gamma_o\mathbf{v} = -\eta\mathbf{n}$, and p is the resulting pressure through the boundary with thickness δ. Δ_s is the Laplace-Beltrami operator ($\Delta_s = \nabla_s\cdot\nabla_s$) and ∇_s denotes the surface gradient. The parameter δ has the physical dimension of length and may be thought of as the "thickness" of the "shear layer" (see [38], Sect 123, p. 506). Equation (63) is derived by calculating the stress tensor for a shear flow and noticing that terms of the form $\mathbf{v}^*\cdot\nabla_s$ vanish. The term $\delta^{-1}\ell(t)$ may be left out since, as before, it disappears when projections are taken. The kinematic boundary condition is still imposed.

7.1. Definitions. All spaces of vector fields are denoted by boldface letters.

(1) Let Ω be a bounded domain in $\mathbf{R}^3$ with a smooth boundary Γ (of class C^∞), $\Omega_T = \Omega\times(0,T)$, and $\Gamma_T = \Gamma\times(0,T)$.

(2) $H^{m,q}(\Omega)$, for m a nonnegative integer and $1 < q < \infty$, is the usual Sobolev space (of real-valued functions) embedded in $L^q(\Omega)$ with norm $\|\cdot\|_{m,q}$. $H^m(\Omega)$, for m a nonnegative integer, denotes the Sobolev space $H^{m,2}(\Omega)$ of order m. By this agreement $\mathbf{H}^m(\Omega)$ is the Sobolev space of three vector fields and the components are elements of $H^m(\Omega)$. In particular, the norm and scalar products in $\mathbf{H}^1(\Omega)$ are defined by $\|\mathbf{u}\|_1^2 = \|\mathbf{u}\|_\Omega^2 + \|\nabla\mathbf{u}\|_\Omega^2$ and $(\mathbf{u},\mathbf{v})_1 = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}dx$.

(3) With the above notation $\mathbf{H}^0(\Omega)$ denotes the Hilbert space $\mathbf{L}^2(\Omega)$ of vector functions $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))$, with $x\in\Omega$, for which $|\mathbf{u}|^2$ is integrable on Ω. The norm and scalar products for $\mathbf{u},\mathbf{v}\in\mathbf{L}^2(\Omega)$ are defined as $\|\mathbf{u}\|_\Omega^2 = \int_\Omega |\mathbf{u}|^2dx$ and $(\mathbf{u},\mathbf{v})_\Omega = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx$.

(4) There exists a linear continuous operator $\gamma_o \in \mathscr{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\partial\Omega))$, called the *trace operator*, such that $\gamma_o\mathbf{u}$ = the "restriction" of $\mathbf{u}$ to $\partial\Omega$ for every function $\mathbf{u}\in\mathbf{H}^1(\Omega)$ which is continuous in $\overline{\Omega}$. The space $\mathbf{H}_o^1(\Omega)$ is the kernel of γ_o. The image space $\gamma_o(\mathbf{H}^1(\Omega))$ is a dense subspace of $\mathbf{L}^2(\Gamma)$ denoted by $\mathbf{H}^{1/2}(\Gamma)$. The trace operator is bounded. Indeed, there exists a constant $C_1 > 0$ such that

$$\|\gamma_o\mathbf{u}\|_\Gamma \le C_1\|\mathbf{u}\|_1 \quad \forall\mathbf{u}\in\mathbf{H}^1(\Omega). \tag{64}$$

Reference [39, Theorem 9.4, page 41]. We shall refer to this result (64) as the *Trace theorem*.

(5) For the deformation we use the following notation for the norm and scalar products: $(\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v}))_\Omega = \int_\Omega \mathbf{A}(\mathbf{u}) : \mathbf{A}(\mathbf{v}) dx$ and $\|\mathbf{A}\|_\Omega^2 = \int_\Omega |\mathbf{A}|^2 dx$.

(6) We define the domain $\mathscr{D}$ by

$$\mathscr{D} = \{\mathbf{v} \in \mathbf{H}^2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \gamma_o \mathbf{v} = -\eta \mathbf{n} \in \mathbf{L}^2(\Gamma), \mathbf{A}(\mathbf{v}) = -2\eta \mathbf{M} \text{ on } \Gamma\}. \quad (65)$$

$\mathscr{D}$ is a closed subspace of $\mathbf{H}^2(\Omega)$. Elements of $\mathscr{D}$ satisfy the kinematical boundary conditions (55).

(7) $\mathbf{H}_\varsigma^1(\Omega)$ denotes the closure of $\mathscr{D}$ in $\mathbf{H}^1(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_1$.

(8) The norm of $\gamma_o \mathbf{v} \in \mathbf{L}^2(\Gamma)$ on the boundary Γ is chosen as

$$\|\gamma_o \mathbf{v}\|_\Gamma^2 = \|\eta\|_\Gamma^2 = \int_\Gamma \sigma(x) |\gamma_o \mathbf{v}|^2 ds. \quad (66)$$

The associated scalar product is

$$(\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_\Gamma = \int_\Gamma \sigma(x) \gamma_o \mathbf{u} \cdot \gamma_o \mathbf{v}\, ds. \quad (67)$$

According to assumption (16) this is equivalent to the usual $\mathbf{L}^2$ metric. It is assumed that the function $\sigma \in C^\infty(\Gamma)$.

(9) For the purpose of stability and uniqueness we define the following norm:

$$\int_\Gamma \eta\, ds = \|\eta\|_{0,\Gamma}. \quad (68)$$

(10) We shall deal extensively with the energy associated with fluids of second grade defined for the purpose of this study by

$$\widetilde{E}_v = \frac{\alpha}{2} \|\mathbf{A}(\mathbf{v})\|_\Omega^2 + \rho \|\mathbf{v}\|_\Omega^2 + C_1 \|\eta\|_{0,\Gamma}^2 + C_2 \|\nabla_s \eta\|_{0,\Gamma}^2, \quad (69)$$

with $C_1 = (\delta\rho - 2\alpha K)$ and $C_2 = \alpha\delta$. $\widetilde{E}_v^{1/2}$ is evidently a norm on $\mathbf{H}_\varsigma^1(\Omega)$. We shall refer to the quantity $\widetilde{E}_v^{1/2}$ as the energy norm of $\mathbf{v}$.

(11) The constant C, which appears in inequalities, denotes a generic positive constant. Sometimes it is necessary to indicate the quantities on which a constant depends in brackets or by a subscript.

7.2. Important Identities

Identity 1. For any symmetric tensor $\mathbf{A}$ and any arbitrary tensor $\mathbf{B}$, we have $\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}_s$, with $\mathbf{B}_s = (1/2)(\mathbf{B} + \mathbf{B}^T)$.

Proof. Consider the following:

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^T : \mathbf{B}^T = \mathbf{A} : \mathbf{B}^T = \frac{1}{2}\mathbf{A}\left(\mathbf{B} + \mathbf{B}^T\right) = \mathbf{A} : \mathbf{B}_s. \quad (70)$$

□

Expressions are necessary for inner products of the form $(D_t \mathbf{F}, \mathbf{F})_\Omega$, where $\mathbf{F}$ is either a vector or a second order tensor. $D_t = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative. In order to obtain simple expressions for the scalar product, we notice that if $\circ$ denotes either the usual scalar product or the "colon" product, then

$$[\partial_t \mathbf{F} + (\mathbf{v} \cdot \nabla) \mathbf{F}] \circ \mathbf{F} = \frac{1}{2}\partial_t |\mathbf{F}|^2 + \frac{1}{2}\nabla \cdot \left(|\mathbf{F}|^2 \mathbf{v}\right), \quad (71)$$

provided $\nabla \cdot \mathbf{v} = 0$. Hence the following identity.

Identity 2. For any smooth vector or tensor quantity $\mathbf{F}(x, t)$ and any $\mathbf{v} \in \mathscr{D}$, we have

$$(D_t \mathbf{F}, \mathbf{F})_\Omega = \frac{1}{2}\partial_t \|\mathbf{F}\|_\Omega^2 - \frac{1}{2}\int_\Gamma |\mathbf{F}|^2 \eta\, ds. \quad (72)$$

Proof. By the divergence theorem

$$\begin{aligned}(\partial_t \mathbf{F} + (\mathbf{v} \cdot \nabla) \mathbf{F}, \mathbf{F}) &= \frac{1}{2}\partial_t \int_\Omega |\mathbf{F}|^2 dx + \frac{1}{2}\int_\Omega \nabla \cdot \left(|\mathbf{F}|^2 \mathbf{v}\right) dx \\ &= \frac{1}{2}\partial_t \|\mathbf{F}\|_\Omega^2 + \frac{1}{2}\int_\Gamma |\mathbf{F}|^2 \mathbf{v} \cdot \mathbf{n}\, ds \\ &= \frac{1}{2}\partial_t \|\mathbf{F}\|_\Omega^2 - \frac{1}{2}\int_\Gamma |\mathbf{F}|^2 \eta\, ds.\end{aligned} \quad (73)$$

□

Later in this study we shall employ the energy method. It will become necessary to use the various boundary conditions in order to prove stability. The following is important to obtain the required results.

Identity 3. If $f \in H^1(\Omega)$ is a scalar field and $\mathbf{v} \in \mathscr{D}$, then

$$\int_\Omega (\mathbf{v} \cdot \nabla) f\, dx = -\int_\Gamma \eta f\, ds. \quad (74)$$

Proof. Integrating by parts and using the fact that $\mathbf{v}$ is solenoidal

$$\int_\Omega (\mathbf{v} \cdot \nabla) f\, dx = \int_\Gamma f \mathbf{v} \cdot \mathbf{n}\, ds - \int_\Omega f \nabla \cdot \mathbf{v}\, dx = -\int_\Gamma \eta f\, ds. \quad (75)$$

□

We note that, in particular for $\mathbf{v} \in \mathscr{D}$, the imbedding of $\mathbf{H}^2(\Omega)$ in the space of bounded continuous functions makes the choice $f = |\mathbf{v}|^2$ possible, and it follows from Identity 3 that

$$\int_\Omega (\mathbf{v} \cdot \nabla) |\mathbf{v}|^2 ds = -\int_\Gamma |\eta|^3 ds. \quad (76)$$

For $\mathbf{v} \in \mathscr{D}$ we may also choose $f = |\mathbf{A}(\mathbf{v})|^2$, and it follows that

$$\int_\Omega (\mathbf{v}\cdot\nabla)|\mathbf{A}(\mathbf{v})|^2 dx = -\int_\Gamma |\mathbf{A}(\mathbf{v})|^2 \eta\, ds = -\int_\Gamma 4\eta^3 |\mathbf{M}|^2 ds \quad (77)$$

since $\mathbf{N} = 0$ on $\mathscr{D}$.

The following will be of immediate importance.

Identity 4. For any $\mathbf{v} \in \mathscr{D}$,

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 = 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma K(x)\eta^2 ds. \quad (78)$$

Proof. From the definition of $\mathbf{A}$ it is evident that $|\mathbf{A}(\mathbf{v})|^2 = 2|\nabla\mathbf{v}|^2 + 2\nabla\mathbf{v} : \nabla^T\mathbf{v}$. Now $\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}] = \nabla\mathbf{v} : \nabla^T\mathbf{v} + (\mathbf{v}\cdot\nabla)(\nabla\cdot\mathbf{v})$, and, since $\nabla\cdot\mathbf{v} = 0$, $\nabla\mathbf{v} : \nabla^T\mathbf{v} = \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]$, integration over Ω and Identity 1 yield

$$\begin{aligned}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Omega \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\,dx \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma \mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 - 2\int_\Gamma \eta\mathbf{n}\otimes\mathbf{n} : [\nabla\mathbf{v}]\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 - \int_\Gamma \eta\mathbf{n}\cdot\mathbf{A}(\mathbf{v})\,\mathbf{n}\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma K(x)\eta^2 ds.\end{aligned} \quad (79)$$

□

Thus, if the curvature K is positive everywhere on Γ, it becomes apparent that $\mathbf{A}(\mathbf{v}) = \mathbf{0}$ if and only if $\mathbf{v} = 0$.

Identity 5. For any bilinear form b on a Hilbert space H, we have, for any $\mathbf{v}, \mathbf{w} \in H$ and with $\mathbf{u} = \mathbf{v} - \mathbf{w}$ that $b(\mathbf{v},\mathbf{v}) - b(\mathbf{w},\mathbf{w}) = b(\mathbf{u},\mathbf{v}) + b(\mathbf{w},\mathbf{u})$.

Identity 6. Let $\mathbf{f}$ and $\mathbf{g}$ be tensor fields of the same order and let $\circ$ denote the "scalar product" in such tensor fields. For $\mathbf{v} \in \mathscr{D}$ it is true that

$$\int_\Omega [\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}]\,dx = -\int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}\,ds. \quad (80)$$

Proof. Consider the following:

$$\int_\Omega \mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g}\,dx = \int_\Gamma (\mathbf{v}\cdot\mathbf{n})\,\mathbf{f}\circ\mathbf{g}\,ds - \int_\Omega \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}\,dx, \quad (81)$$

thus

$$\int_\Omega [\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\mathbf{f}]\,dx = \int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}\,ds. \quad (82)$$

□

7.3. Inequalities

Lemma 4. *Under the assumptions* (17) *and* (16), *for* $\mathbf{v} \in \mathscr{D}$, $\mathbf{v} = 0$ *if and only if* $\mathbf{A}(\mathbf{v}) = 0$.

Proof. From (16), (78), and (17) we have

$$\frac{g}{S}\|\eta\|_\Gamma^2 + \|\nabla\mathbf{v}\|_\Omega^2 \le \frac{1}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \le \|\nabla\mathbf{v}\|_\Omega^2 + \frac{G}{s}\|\eta\|_\Gamma^2, \quad (83)$$

and the result follows. □

The following two lemmas are important in establishing a Poincaré inequality.

Lemma 5. *The bilinear forms* $a(\mathbf{u},\mathbf{v}) = (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))_\Omega$ *and* $b(\mathbf{u},\mathbf{v}) = \rho(\mathbf{u},\mathbf{v})_\Omega + C_1(\gamma_o\mathbf{u},\gamma_o\mathbf{v})_\Gamma + C_2(\nabla_s\eta_u,\nabla_s\eta_v)_\Gamma$ *are bounded in the space* $\mathbf{H}_\varsigma^1(\Omega)$. C_1 *and* C_2 *are positive constants.*

Proof. For $\mathbf{u}$ and $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ and by (16), (17), and the Schwartz inequality

$$\begin{aligned}|a(\mathbf{u},\mathbf{v})| &= |(\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))| \\ &= \left|2(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + 2\int_\Gamma K(x)\eta_u\eta_v ds\right| \\ &\le 2\|\mathbf{u}\|_1\|\mathbf{v}\|_1 + \frac{2G}{s^2}\|\gamma_o\mathbf{u}\|_\Gamma\|\gamma_o\mathbf{v}\|_\Gamma.\end{aligned} \quad (84)$$

Hence, by the Trace theorem

$$|a(\mathbf{u},\mathbf{v})| \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1. \quad (85)$$

Furthermore,

$$\begin{aligned}|b(\mathbf{u},\mathbf{v})| &= \left|\rho(\mathbf{u},\mathbf{v})_\Omega + C_1(\gamma_o\mathbf{u},\gamma_o\mathbf{v})_\Gamma + C_2(\nabla_s\eta_u,\nabla_s\eta_u)_\Gamma\right| \\ &\le \rho\|\mathbf{u}\|_1\|\mathbf{v}\|_1 + C_1\|\gamma_o\mathbf{u}\|_\Gamma\|\gamma_o\mathbf{v}\|_\Gamma \\ &\quad + C_2\|\nabla_s\eta_u\|_\Gamma\|\nabla_s\eta_v\|_\Gamma \\ &\le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1,\end{aligned} \quad (86)$$

by the Trace theorem and C_1 and C_2 as defined and C generic. □

Lemma 6. *The bilinear form* $|a(\mathbf{u},\mathbf{v})| = (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))_\Omega$ *is coercive in the sense that there exist constants* $c_1 > 0$ *and* $c_o \ge 0$ *such that*

$$|a(\mathbf{u},\mathbf{u})| \ge c_1\|\mathbf{u}\|_1^2 - c_o b(\mathbf{u},\mathbf{u}). \quad (87)$$

Proof. From (83) we have

$$\begin{aligned}a(\mathbf{u},\mathbf{u}) &= (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u}))_\Omega \\ &\ge 2\|\nabla\mathbf{u}\|_\Omega^2 + \frac{2g}{S}\|\eta\|_\Gamma^2 \\ &= 2\|\mathbf{u}\|_1^2 - 2\|\mathbf{u}\|_\Omega^2 + \frac{2g}{S}\|\eta\|_\Gamma^2\end{aligned}$$

$$\geq 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}\left(\rho\|\mathbf{u}\|_\Omega^2\right) - \frac{2}{\rho}C_1\|\eta\|_\Gamma^2 - \frac{2}{\rho}C_2\|\nabla_s\eta_u\|_\Gamma^2$$
$$= 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}b(\mathbf{u},\mathbf{u}). \tag{88}$$

□

We can now obtain a generalised Poincaré inequality.

Lemma 7. *There exists a smallest possible constant $\beta > 0$ such that, for every $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$,*

$$\frac{\beta}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \rho\|\mathbf{v}\|_\Omega^2 + C_1\|\gamma_o\mathbf{v}\|_\Gamma^2 + C_2\|\nabla_s\eta_u\|_\Gamma^2. \tag{89}$$

Proof. From the smoothness of Γ (which is always assumed), it follows that the embedding $J : \mathbf{u} \in \mathbf{H}^1(\Omega) \to \langle\mathbf{u}, \gamma_o\mathbf{u}\rangle \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Gamma)$ is compact [40]. From the boundedness and coerciveness proved above it follows that there exists a smallest eigenvalue λ and associated eigenfunction $\mathbf{u} \in \mathbf{H}_\varsigma^1(\Omega)$ for which $b(\mathbf{u},\mathbf{u}) = 1$ (see [34]):

$$\lambda = \inf\left\{a(\mathbf{v},\mathbf{v}) : \mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega); b(\mathbf{v},\mathbf{v}) = 1\right\} = a(\mathbf{u},\mathbf{u}) \tag{90}$$

$\lambda > 0$, for if it is zero, it follows that $\mathbf{u} = 0$, which cannot be. It follows from (90) that for any $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ the inequality

$$a(\mathbf{v},\mathbf{v}) \geq \lambda\left[\rho\|\mathbf{v}\|_\Omega^2 + C_1\|\gamma_o\mathbf{v}\|_\Gamma^2 + C_2\|\nabla_s\eta_u\|_\Gamma^2\right] \tag{91}$$

holds and that λ is the largest such constant. Finally, we set $\beta = 2/\lambda$. □

Remark 8. It is now easy to see that $\|\mathbf{A}(\cdot)\|_\Omega$ is a norm on $\mathbf{H}_\varsigma^1(\Omega)$.

In fact, we have the following lemma.

Lemma 9. *For all $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ we have*

$$\frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq \widetilde{E}_v \leq \frac{\alpha+\beta+2}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2. \tag{92}$$

Proof. Add $(\alpha/2)\|\mathbf{A}(\mathbf{v})\|_\Omega^2 + C_2\|\nabla_s\eta\|_{0\Gamma}^2$ to both sides of the inequality (89):

$$\widetilde{E}_v = \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 + \rho\|\mathbf{v}\|_\Omega^2 + C_1\|\eta\|_\Gamma^2 + C_2\|\nabla_s\eta\|_{0,\Gamma}$$
$$\leq \frac{\alpha+\beta+2}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2. \tag{93}$$

From the definition of the energy norm it is clear that

$$\frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq \widetilde{E}_v, \tag{94}$$

and the result follows. □

From Lemma 7 it is clear that these are the best estimates of this form.

Lemma 10. *The norms $\|\mathbf{A}(\mathbf{v})\|_\Omega$ and $\widetilde{E}_v^{1/2}$ are equivalent to the norm in the Sobolev space $\mathbf{H}^1(\Omega)$.*

Proof. From (83) and (89) it follows that

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq 2\|\nabla\mathbf{v}\|_\Omega^2,$$
$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \frac{2\rho}{\beta}\|\mathbf{v}\|_\Omega^2. \tag{95}$$

The addition of the two inequalities above yields

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \|\nabla\mathbf{v}\|_\Omega^2 + \frac{\rho}{\beta}\|\mathbf{v}\|_\Omega^2. \tag{96}$$

Let $k = \min(1, \rho/\beta)$; then

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq k\|\mathbf{v}\|_1^2. \tag{97}$$

Equation (83) yields

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq 2\|\nabla\mathbf{v}\|_\Omega^2 + \frac{2G}{s}\|\eta\|_\Gamma^2$$
$$\leq 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\|\mathbf{v}\|_\Omega^2 + \frac{2G}{s}\|\eta\|_\Gamma^2, \tag{98}$$

and from the Trace theorem it follows that

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq C\|\mathbf{v}\|_1^2. \tag{99}$$

From (92) it is evident that the energy norm is equivalent to the norm $\|\mathbf{A}(\mathbf{v})\|_\Omega$. □

Remark 11. From the above lemma we may claim from the embedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^3(\Gamma)$, [40], that there exists a constant $\tau > 0$ such that

$$\int_\Gamma |\gamma_o\mathbf{v}|_\Gamma^3 ds \leq \tau\|\mathbf{A}(\mathbf{v})\|_\Omega^3 \quad \text{for every } \mathbf{v} \in \mathbf{H}_\sigma^1(\Omega). \tag{100}$$

7.4. Stability and Uniqueness for the Model Problem $\mathscr{A}$. $\mathbf{v} \in \mathbf{H}^3(\Omega) \cap \mathscr{D}$ satisfies the system of evolution equations

$$D_t[\rho\mathbf{v}(x,t)] = \nabla\cdot\mathbf{T}(\rho,\mathbf{v}) \quad \text{in } \Omega\times(0,\infty)$$
$$\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta \quad \text{at } \Gamma\times(0,\infty) \tag{101}$$
$$\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta M \quad \text{at } \Gamma\times(0,\infty).$$

We now derive an energy identity for the solutions of (101). Take the $\mathbf{L}^2(\Omega)$, scalar product with $\mathbf{v}$ on both sides of $(101)_1$. This produces

$$(D_t(\rho\mathbf{v}),\mathbf{v}) = \frac{\rho}{2}\partial_t\|\mathbf{v}\|_\Omega^2 - \frac{\rho}{2}\int_\Omega \eta^3 ds$$
$$= (\nabla\cdot\mathbf{T},\mathbf{v})$$
$$= \int_\Gamma \gamma_o\mathbf{v}\cdot\mathbf{T}\mathbf{n}\,ds - (\mathbf{T},\nabla\mathbf{v})$$
$$= -\int_\Gamma \eta\mathbf{n}\cdot\mathbf{T}\mathbf{n}\,ds - \frac{1}{2}\int_\Omega \mathbf{T}:\mathbf{A}\,dx. \tag{102}$$

According to the formulation of the original problem on the boundary where $s(\eta) = \mathbf{n} \cdot \mathbf{Tn}$, we obtain

$$\begin{aligned} &-\int_\Gamma \eta \mathbf{n} \cdot \mathbf{Tn}\, ds \\ &= -\int_\Gamma \eta \Big(-\gamma_o p - 2\mu\eta K - 2\alpha K \eta_t \\ &\qquad + 4\alpha K_G \eta^2 - \alpha\eta\Delta_s\eta - \ell(t) \Big) ds. \end{aligned} \tag{103}$$

From $(101)_2$ we obtain

$$\gamma_o p = -\delta\partial_t \left[\rho\eta - \alpha\Delta_s\eta\right] + \delta\mu\Delta_s\eta. \tag{104}$$

Substitute (104) into (103) to obtain

$$\begin{aligned} -\int_\Gamma \eta\mathbf{n}\cdot\mathbf{Tn}\,ds &= -\partial_t \int_\Gamma \frac{\delta\rho}{2}|\eta|^2 ds - \delta\alpha\partial_t\|\nabla_s\eta\|^2_{0,\Gamma} \\ &\quad - \delta\mu\|\nabla_s\eta\|^2_{0,\Gamma} \\ &\quad + 2\mu\delta\int_\Gamma K\eta^2 ds + \partial_t\int_\Gamma \delta\alpha K|\eta|^2 ds \\ &\quad - 4\alpha\delta\int_\Gamma K_G\eta^3 ds - \alpha\delta\int_\Gamma \eta|\nabla_s\eta|^2 ds. \end{aligned} \tag{105}$$

Also

$$-\frac{1}{2}(\mathbf{T},\mathbf{A})_\Omega = -\frac{1}{2}\mu\|\mathbf{A}\|^2_\Omega - \frac{\alpha}{4}\partial_t\|\mathbf{A}\|^2_\Omega + \alpha\int_\Gamma |\mathbf{M}|^2\eta^3 ds. \tag{106}$$

Therefore the energy identity for *Problem $\mathscr{A}$* is

$$\begin{aligned} &\partial_t\left[\frac{\rho}{2}\|\mathbf{v}\|^2_\Omega + \frac{\alpha}{4}\|\mathbf{A}(\mathbf{v})\|^2_\Omega \right.\\ &\qquad \left. + \int_\Gamma\left(\frac{\rho\delta}{2} - \alpha K\right)|\eta|^2 ds + \delta\alpha\|\nabla_s\eta\|^2_{0,\Gamma}\right] \\ &= -\frac{1}{2}\mu\|\mathbf{A}(\mathbf{v})\|^2_\Omega + \alpha\int_\Gamma\left(|M|^2 - 4\delta K_G\right)|\eta|^3 ds \\ &\quad - \delta\mu\|\nabla_s\eta\|^2_\Gamma + 2\mu\delta\int_\Gamma K|\eta|^2 ds - \alpha\delta\int_\Gamma\eta|\nabla_s\eta|^2 ds. \end{aligned} \tag{107}$$

Now we can define an energy norm for *Problem $\mathscr{A}$* as follows:

$$\begin{aligned} \widetilde{E}_\nu(t) &= \rho\|\mathbf{v}\|^2_\Omega + \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|^2_\Omega \\ &\quad + \int_\Gamma(\delta\rho - 2\alpha K)|\eta|^2 ds + 2\delta\alpha\|\nabla_s\eta\|^2_{0,\Gamma}. \end{aligned} \tag{108}$$

Note that here we have to make the assumption that $\delta\rho - 2\alpha K > 0$, which gives us a restriction on K. We define a parameter

$$p_2 = \frac{\alpha K}{\delta\rho}. \tag{109}$$

It is now clear that stability can only be proved under the assumption that $p_2 \in (0, 1/2)$.

The Poincaré inequality (see [39]) states that there exists a smallest constant c such that $\|\eta\|^2_{0,\Gamma} \geq c\|\nabla_s\eta\|^2_{0,\Gamma}$. Applying the Schwartz inequality and the above Poincaré inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\widetilde{E}_\nu(t) &\leq -\mu\|\mathbf{A}(\mathbf{v})\|^2_\Omega + 2\alpha\left(G^2 + H^2 + 4\delta G^2\right)\|\eta\|^3_{0,\Gamma} \\ &\quad + 2\delta\|\nabla_s\eta\|^2_{0,\Gamma}\left(\mu + \alpha\|\eta\|_{0,\Gamma}\right) + 4\delta\mu G\|\eta\|^2_{0,\Gamma} \\ &\leq -\widetilde{E}_\nu\left[\mu - 2\alpha\left(G^2 + H^2 + 2\delta G^2 - \delta\right)\widetilde{E}_\nu^{1/2}\right. \\ &\qquad \left. - 2\delta\mu - 4\delta\mu G\right]. \end{aligned} \tag{110}$$

With $2\alpha(G^2+H^2+4\delta G^2-\delta) = \epsilon^*$ and $\mu(1-2\delta-4\delta G) = \epsilon^{**}$ we have

$$\frac{d}{dt}\widetilde{E}_\nu \leq -\widetilde{E}_\nu\left[\epsilon^{**}\mu - \epsilon^*\widetilde{E}_\nu^{1/2}\right]. \tag{111}$$

Theorem 12 (stability for problem $\mathscr{A}$). *If $p_2 \in (0, 1/2)$ and $\widetilde{E}_\nu(0) < (\epsilon^{**}\mu/\epsilon^*)^2$, then the energy $\widetilde{E}_\nu(t)$ decreases exponentially to zero as $t \to \infty$.*

The uniqueness of the solution of *Problem $\mathscr{A}$* is treated in the same way as the uniqueness of the solution of the original problem (see [33]).

8. Conclusion

An extensive study was conducted to find expressions for the stress tensors of Newtonian and non-Newtonian fluids at a permeable surface. We employed the Serret-Frenet formulae exactly for this reason. Stability of the rest state and uniqueness were proven for a special case where a shear flow was taken into account.

These results proved to be valuable in applications for the study of blood flow, where they were applied to model the permeability of special capillaries in the formation of cerebrospinal fluid [41, 42]. Here the authors have presented a mathematical model of the flow of blood through the permeable boundary of a blocked choroidal capillary in which the parameters could be controlled. The blood plasma was modelled as a Newtonian fluid and the nonlinear Stokes equations were supplemented with a boundary condition at the permeable interface of the specialized capillary. The existence of a unique weak solution, which depends on the viscosity and the nature of the curvature of the capillary, was proved. By incorporating in this model all the ultrafiltration parameters, which are presented in [41, 42], the authors have attempted (within the prescribed morphological and physiological properties of the microvascular environment) to adapt the model used by Maritz and Sauer [33] to real-life situations. Further applications could be found in the modelling of other permeable systems in the human body like the lymphatic glands and the urinary system.

With this research the authors have tried to prepare the ground for the applications of these results in the exploring of permeable surfaces in biosciences, engineering, and the natural sciences. The open question regarding the existence of a classical solution for the system (101) will be addressed in further research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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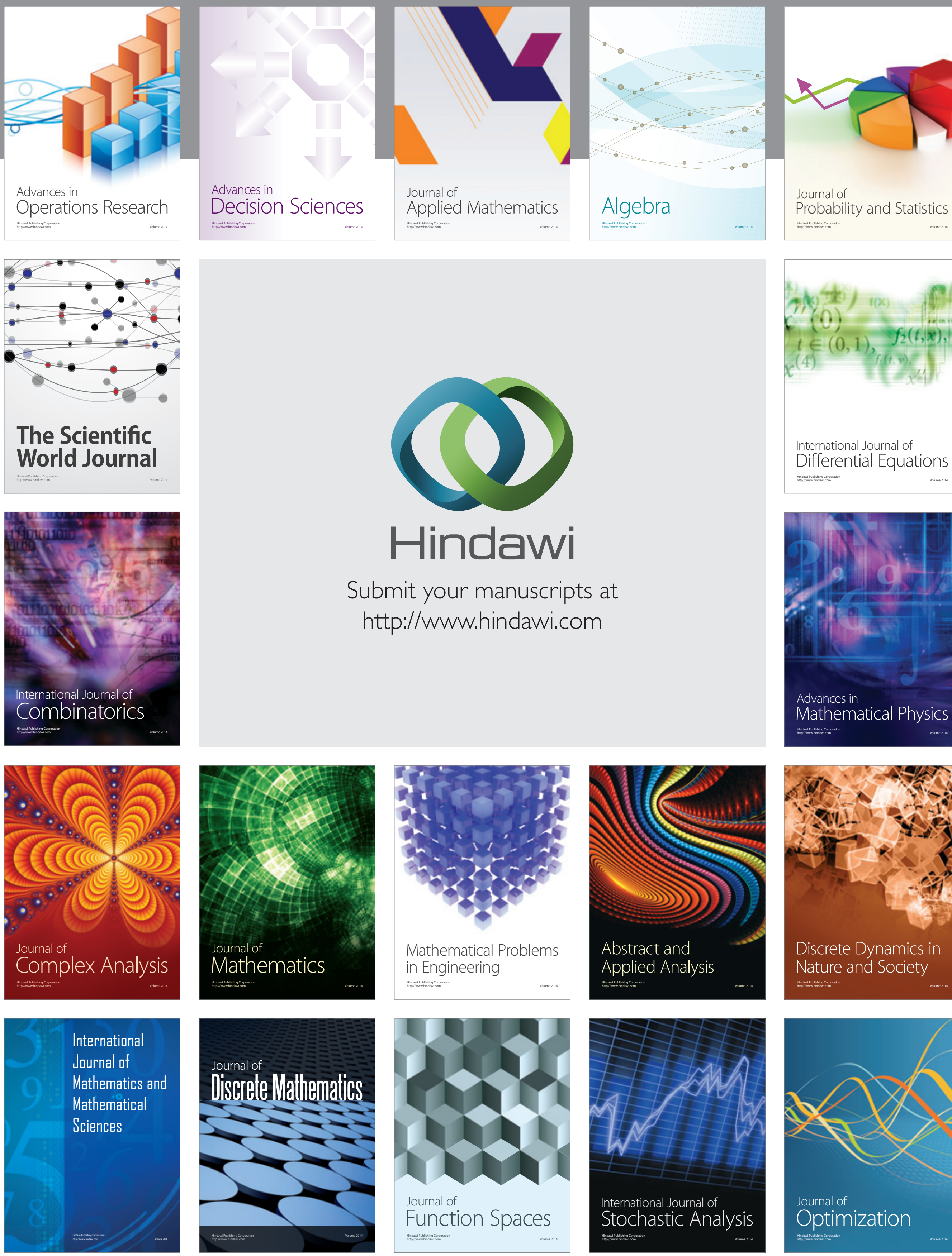
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