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CONVEX AND STARLIKE CRITERIA

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Abstract. We investigate an expression involving the quotient of the analytic representations of convex and starlike functions. Sufficient conditions are found for functions to be starlike of a positive order and convex.

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1. Introduction. Let S denote the class of functions f normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. A function f in S is said to be starlike of order $\alpha, 0 \le \alpha < 1$, and is denoted by $S^*(\alpha)$ if $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha, z \in \Delta$, and is said to be convex and is denoted by K if $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0, z \in \Delta$. Mocanu [9] studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. In [8], it was shown that if

$$\operatorname{Re}\left[\alpha(1 + zf''(z)/f'(z)) + (1-\alpha)zf'(z)/f(z)\right] > 0 \tag{1.1}$$

for $z \in \Delta$, then f is starlike for α real and convex for $\alpha \ge 1$.

In this note, we investigate the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class G_b consisting of normalized functions f defined by

$$G_b = \left\{ f : \left| \left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| < b,\ z \in \Delta \right\}. \tag{1.2}$$

We determine sharp values of b for which $G_b \subset S^*(\alpha), 1/2 \le \alpha < 1$, and also find values of b for which $G_b \subset K$. It is known ([7, 10]) that $K \subset S^*(1/2)$. We show that $G_1 \subset S^*(1/2) - K$. We also find values of b for which G_b is not starlike and not univalent.

We make use of the following lemma obtained by Jack in [4].

Lemma A. *Suppose* ω *is analytic for* $|z| \le r, \omega(0) = 0$ *and* $|\omega(z_0)| = \max_{|z|=r} |\omega(z)|$. *Then* $z_0\omega'(z_0) = k\omega(z_0), k \ge 1$.

2. Main results

Theorem 1. *If* $0 < b \le 1$ *and* G_b *is defined by (1.2), then* $G_b \subset S^*(2/(1+\sqrt{1+8b}))$. *The result is sharp for all* b.

We prove this theorem in an equivalent form, which we write as

Theorem 1a. *Set $b = (1-\alpha)/2\alpha^2, 1/2 \le \alpha < 1$. Then $G_b \subset S^*(\alpha)$, with extremal function $z/(1-z)^{2(1-\alpha)}$.*

Proof of Theorem 1a. It is well known that if $\omega(z)$ is analytic in Δ with $\omega(0) = 0$, then $\operatorname{Re}\left(\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\right) > \alpha, z \in \Delta$, if and only if $\omega(z)$ is a Schwarz function, i.e., $|\omega(z)| < 1$ for $z \in \Delta$ with $\omega(0) = 0$. Set

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)\omega(z)}{1-w(z)} \tag{2.1}$$

Then

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \tag{2.2}$$

and

$$\left|\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right) - 1\right| = \left|\frac{zp'(z)}{(p(z))^2}\right| = \left|\frac{2(1-\alpha)z\omega'(z)}{(1+(1-2\alpha)\omega(z))^2}\right|. \tag{2.3}$$

If $f \notin S^*(\alpha)$, then by Lemma A there is a $z_0 \in \Delta$ for which $|\omega(z_0)| = 1$ and $z_0\omega'(z_0) \ge \omega(z_0)$. It then follows from (2.3) that $\left|\frac{z_0p'(z_0)}{(p(z_0))^2}\right| \ge \frac{2(1-\alpha)}{(2\alpha)^2}$ which contradicts our hypothesis. This completes the proof. □

Corollary 1. $G_1 \subset S^*(1/2)$.

Proof. Set $b = 1$ in Theorem 1. □

Corollary 2. *If* $\operatorname{Re}\left(\frac{zf'(z)/f(z)}{1+zf''(z)/f'(z)}\right) > 1/2$ *for $z \in \Delta$, then $f \in S^*(1/2)$.*

Proof. This follows from Corollary 1 upon noting that for any complex value w, $|w-1| < 1 \Longleftrightarrow \operatorname{Re}(1/w) > 1/2$. □

We next give a partial converse to Corollary 1.

Theorem 2. *If $f \in S^*(1/2)$, then $\left|\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}\right) - 1\right| < 1$ for $|z| < (2\sqrt{3}-3)^{1/2} = 0.68\ldots$. The result is sharp.*

Proof. Set $p(z) = zf'(z)/f)(z) = 1/(1-\omega(z))$, where $\omega(z)$ is a Schwarz function. We need to find the largest disk $|z| < R$ for which $|zp'(z)/p(z))^2| = |z\omega'(z)| < 1$. Dieudonné [2] found the region of values for the derivative of Schwarz functions. This led to the sharp bound [3],

$$|\omega'(z)| \le \begin{cases} 1, & r = |z| \le \sqrt{2}-1 \\ \dfrac{(1+r^2)^2}{4r(1-r^2)}, & r \ge \sqrt{2}-1. \end{cases} \tag{2.4}$$

Since $|z\omega'(z)| \le (1+r^2)^2/4(1-r^2) = 1$ for $r = (2\sqrt{3}-3)^{1/2}$, the proof is complete. □

3. A counterexample.

The extreme points of the closed convex hull of convex functions and functions starlike of order $1/2$ are identical. See [1]. Since $G_1 \subset S^*(1/2)$, one might, also, expect to have $G_1 \subset K$. Surprisingly, this is not the case. We now construct a function $f \in G_1 - K$.

Theorem 3. $G_1 \not\subset K$.

Proof. $G_1 \subset S^*(1/2)$. Any of $f \in G_1$ satisfies $zf'(z)/f(z) = 1/(1-\omega(z))$ for some Schwarz function $\omega(z)$. Setting $\alpha = 1/2$ in (2.3), we see that $f \in G_1 \iff |z\omega'(z)| < 1$ for $z \in \Delta$, which means that $z\omega'(z)$ must, also, be a Schwarz function. Since $1 + zf''(z)/f(z) = (1+z\omega'(z))/(1-\omega(z))$, it suffices to construct a Schwarz function $\Omega(z) = z\omega'(z)$ for which

$$\operatorname{Re}\left\{\frac{1+\Omega(z)}{1-\omega(z)}\right\} < 0 \tag{3.1}$$

at some point $z \in \bar{\Delta}$. Let

$$A = \{z \in \Delta : |z-z_0| < 10^{-5}, z_0 = e^{\pi i/4} = e^{i\theta_0}\}, \tag{3.2}$$

and set

$$\phi(z) = (z_0+\bar{z}_0)\left[(1-\bar{z}_0 z)^{1/N} - 1\right], \tag{3.3}$$

where N is large enough so that $|\phi(z)/z| < 10^{-4}$ for $z \in \Delta - A$ and $|\operatorname{Im}\phi(z)| < 10^{-8}$ for $z \in A$. Define Ω by $\Omega(z) = 0.9999(z+\phi(z))$.

We first show that $\Omega(z)$ (and, consequently, $\omega(z)$) is a Schwarz function and then show that inequality (3.1) holds when $z = z_0$.

If

$$z \in \Delta - A, \tag{3.4}$$

then

$$|\Omega(z)| \le 0.9999(|z| + |\phi(z)|) \le 0.9999(1.0001) < 1. \tag{3.5}$$

If $z \in A$, set $z = z_0 - \epsilon e^{i\beta}$, $0 < \epsilon < 10^{-5}$, and note that $-2\cos\theta_0 \le \operatorname{Re}\phi(z) \le 0$. If $\operatorname{Re}(z+\phi(z)) \ge 0$, then $|z + \operatorname{Re}\phi(z)| \le |z| < 1$. If $\operatorname{Re}(z+\phi(z)) < 0$, then

$$|z+\operatorname{Re}\phi(z)| \le \sqrt{(\cos\theta_0+\epsilon)^2 + (\sin\theta_0+\epsilon)^2} < \sqrt{1+4\epsilon} < 1+2\epsilon < 1.0001. \tag{3.6}$$

Thus, if $z \in A$,

$$|\Omega(z)| \le 0.9999|z+\operatorname{Re}\phi(z)| + |\operatorname{Im}\phi(z)| < 0.9999(1.0001) + 10^{-8} = 1. \tag{3.7}$$

Therefore, $\Omega(z)$ is a Schwarz function.

We now show that (3.1) holds at $z = z_0$ for this choice of $\Omega(z)$. Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |\omega'(z) - 1| < 0.0002 \quad \text{for } z \in \Delta - A, \tag{3.8}$$

we may write $\omega(z) = z + \eta(z)$, where $|\eta(z)| < 0.0003$ for $z \in A$. Note that

$$\begin{aligned}(|1-\omega(z_0)|^2)\operatorname{Re}\left(\frac{1+\Omega(z_0)}{1-\Omega(z_0)}\right) &= \operatorname{Re}\left\{(1-\Omega(z_0))(1+\overline{\omega(z_0)})\right\} \\ &= \operatorname{Re}\left\{(1-0.9999\bar{z}_0)(1-\bar{z}_0-\overline{\eta(z_0)})\right\} \\ &\le 1 - 1.9999\cos\theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)| \\ &< 1 - 1.9999\cos(\pi/4) + 0.0006 < 0.\end{aligned} \tag{3.9}$$

Hence, the function f for which $1 + zf''(z)/f'(z) = (1+\Omega(z))/(1-\omega(z))$ must be in $G_1 - K$. □

4. Convexity. Since $G_1 \not\subset K$, we can ask if $G_b \subset K$ for some $b < 1$. In general, $S^*(\alpha) \not\subset K$ even for α arbitrary close to 1 (b close to 0). To see this, we note that $f_n(z) = z + a_n z^n$ is in $S^*(\alpha)$ if and only if $|a_n| \le (1-\alpha)/(n-\alpha)$ and $f_n(z) \in K$ if and only if $|a_n| \le 1/n^2$. Thus, $f(z) = z + (1-\alpha)/(n-\alpha) z^n \in S^*(\alpha) - K$ for $n > 2/(1-\alpha)$.

We next show that there are values of b for which the functions in G_b must be convex.

Theorem 4. *$G_b \subset K$ for $b \le \sqrt{2}/2$.*

Proof. Since $f \in G_b \subset G_1 \subset S^*(1/2)$, we may write $zf'(z)/f(z) = 1/(1-\omega(z))$, where ω is a Schwarz function. For $f \in G_b$, we take $\alpha = 1/2$ in (2.3) to obtain $|z\omega'(z)| < \sqrt{2}/2$ and, consequently, $|\omega(z)| < \sqrt{2}/2$, $z \in \Delta$. We need to show that

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} = \operatorname{Re}\left\{\frac{(1+z\omega'(z))}{(1-\omega(z))}\right\} > 0. \tag{4.1}$$

Since

$$\begin{aligned}\left|\arg\left(\frac{1+z\omega'(z)}{1-\omega(z)}\right)\right| &\le |\arg(1+z\omega'(z))| + |\arg(1-\omega(z))| \\ &\le \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},\end{aligned} \tag{4.2}$$

the result follows. □

In [6], MacGregor found the radius of convexity for $S^*(1/2)$ to be $(2\sqrt{3}-3)^{1/2} = 0.68\ldots$. Since $G_1 \subset S^*(1/2)$, we know that the radius of convexity is at least this large. The following consequence of Theorem 4 is that functions in G_1 are convex in the disk $|z| < \sqrt{2}/2$.

Corollary. *If $f \in G_b$, $\sqrt{2}/2 \le b \le 1$, then f is convex in the disk $|z| < \sqrt{2}/2b$.*

Proof. If $|z\omega'(z)| < 1$ for $z \in \Delta$, then $|z\omega'(z)| < t$ for $|z| < t < 1$. If $f \in G_b$, then $|z\omega'(z)| < b$ for $z \in \Delta$. Hence, $|z\omega'(z)| < \sqrt{2}/2$ when $|z| < \sqrt{2}/2b$. □

5. Examples. Theorem 1 gives a sharp order of starlikeness for G_b when $0 < b \le 1$, with $G_1 \subset S^*(1/2)$. Our methods do not extend to $b > 1$, but we expect the order of starlikeness to decrease from $1/2$ to 0 as b increases from 1 to some value b_0 after which functions in G_b need not be starlike. We do not have a sharp result for $b > 1$, but our next example shows that the univalent functions in G_b are not necessarily starlike for $b \ge 11.66$.

The function $h(z) = z(1-iz)^{i-1}$ is spiral-like [11] and, hence, in S because

$$\operatorname{Re}\left\{e^{\pi i/4}\frac{zh'(z)}{h(z)}\right\} = \frac{1}{\sqrt{2}}\left(\frac{1-|z|^2}{|1-iz|^2}\right) > 0, \quad z \in \Delta. \tag{5.1}$$

Since $zh'(z)/h(z) = (1+z)/(1-iz)$, we see that h is not starlike for $|z| < a$, $\sqrt{2}/2 < a < 1$. Thus, $f(z) = f_a(z) = h(az)/a$ is not starlike for $z \in \Delta$. Setting $p(z) = zf'(z)/f(z) = (1+az)/(1-aiz)$, we have

$$\left|\frac{zp'(z)}{(p(z))^2}\right| = \left|\frac{(1+i)az}{(1+az)^2}\right| \le \frac{\sqrt{2}a}{(1-a)^2} < 11.66 \tag{5.2}$$

for a sufficiently close to $\sqrt{2}/2$. Hence, $f \in G_b - S^*(0)$ for $b = 11.66$.

Finally, we show that the functions in G_b need not be univalent. In [5], it is shown for $h(z) = z(1-iz)^{i-1}$ that $g(z) = \int_0^z h(t)/t\,dt = (1-iz)^i - 1$ is not in S because $g(z_0) = g(-z_0)$ for $z_0 = i(e^{2\pi}-1)/(e^{2\pi}+1), |z_0| = 0.996\ldots$. We, thus, conclude that for $f(z) = g(cz)/c$, $c = 0.997$, $f \in G_b - S$ for b sufficiently large.

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