Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 495040, 11 pages
http://dx.doi.org/10.1155/2015/495040

Research Article

Existence and Multiplicity of Fast Homoclinic Solutions for a Class of Nonlinear Second-Order Nonautonomous Systems in a Weighted Sobolev Space

Qiongfen Zhang and Yuan Li

College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China

Correspondence should be addressed to Qiongfen Zhang; qfzhangcsu@163.com

Received 20 January 2015; Accepted 8 April 2015

Academic Editor: Yoshihiro Sawano

This paper is concerned with the following nonlinear second-order nonautonomous problem: $\ddot{u}(t)+q(t)\dot{u}(t)-\nabla K(t,u(t))+\nabla W(t,u(t))=0$, where $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, and $K, W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ are not periodic in t and $q:\mathbb{R}\to\mathbb{R}$ is a continuous function and $Q(t)=\int_0^t q(s)ds$ with $\lim_{|t|\to+\infty}Q(t)=+\infty$. The existence and multiplicity of fast homoclinic solutions are established by using Mountain Pass Theorem and Symmetric Mountain Pass Theorem in critical point theory.

1. Introduction

Consider fast homoclinic solutions of the following problem:

$$\ddot{u}(t)+q(t)\dot{u}(t)-\nabla K(t,u(t))+\nabla W(t,u(t))=0,\quad t\in\mathbb{R},\tag{1}$$

where $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, and $K, W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ are not periodic in t and $q:\mathbb{R}\to\mathbb{R}$ is a continuous function and $Q(t)=\int_0^t q(s)ds$ with

$$\lim_{|t|\to+\infty}Q(t)=+\infty.\tag{2}$$

When $q(t)\equiv 0$, problem (1) reduces to the following second-order Hamiltonian system:

$$\ddot{u}(t)-\nabla K(t,u(t))+\nabla W(t,u(t))=0,\quad t\in\mathbb{R}.\tag{3}$$

When $K(t,u(t))=(1/2)(L(t)u(t),u(t))$, where $L(t)$ is a positive definite symmetric matrix-valued function for all $t\in\mathbb{R}$, then problem (1) reduces to the following damped vibration problem:

$$\ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,\quad t\in\mathbb{R}.\tag{4}$$

If $K(t,u(t))=(1/2)(L(t)u(t),u(t))$ and $q(t)\equiv 0$, problem (1) reduces to the following second-order Hamiltonian system:

$$\ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,\quad t\in\mathbb{R}.\tag{5}$$

It is well known that the existence of homoclinic orbits plays a very important role in the study of the behavior of dynamical systems. The first work about homoclinic orbits was done by Poincaré [1].

In the past years, many researchers paid attention to the existence and multiplicity of homoclinic solutions for systems (3) and (5) by critical point theory. For example, see [2–13] and references cited therein. However, there is only a few researches about the existence of homoclinic solutions for damped vibration problems (4) when $q(t)\neq 0$. Wu and Zhou [14] established some results for a class of damped vibration problems with obstacles. Wu et al. [15] obtained some results for (4) with some boundary value conditions by variational methods. Zhang and Yuan [16] studied the existence of homoclinic solutions for (4) when $q(t)\equiv c$ is a constant. Later, Chen et al. [17] investigated fast homoclinic solutions for (4) and obtained the following Theorem A under more relaxed assumptions on $W(t,x)$, which resolved some open problems in [16]. Zhang et al. [18] considered a special nonautonomous problem and obtained some results of fast homoclinic orbits. Zhang [19] investigated a class of damped vibration problems with impulsive effects and established some existence and multiplicity results of fast homoclinic solutions. For the applications of weighted Sobolev space in elliptic equations, please see [20, 21] and references cited therein.

Theorem A (see [17]). *Assume that q, L, and W satisfy (2) and the following assumptions:*

(L) $L \in C(\mathbb{R}, \mathbb{R}^N \times \mathbb{R}^N)$ *is a $M^N(\mathbb{R})$-valued continuous function of $t \in \mathbb{R}$ and there exists constant $\beta > 0$ such that*

$$(L(t)x, x) \geq \beta |x|^2, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{6}$$

(W1) $W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and there exists constant $R > 0$ such that*

$$|\nabla W(t,x)| = o(|x|) \quad \text{as } x \longrightarrow 0 \tag{7}$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.

(W2) *There is constant $\mu > 2$ such that*

$$0 < \mu W_1(t,x) \leq (\nabla W_1(t,x), x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}. \tag{8}$$

(W3) $W_2(t,0) = 0$ *and there exists a constant $\varrho \in (2, \mu)$ such that*

$$\begin{gathered} W_2(t,x) \geq 0, \\ (\nabla W_2(t,x), x) \leq \varrho W_2(t,x), \\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \end{gathered} \tag{9}$$

Then problem (4) has at least one nontrivial fast homoclinic solution.

Otherwise, Chen et al. [17] obtained the multiplicity of fast homoclinic solutions for (4) if the following condition holds:

(W4) $W(t,-x) = W(t,x), \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Motivated by the abovementioned works, we will establish some results for (1). In order to introduce the concept of the fast homoclinic solutions for (1), we first state some properties of the weighted Sobolev space E on which the certain variational associated with (1) is defined and the fast homoclinic solutions are the critical points of the certain functional.

Let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \right] dt < +\infty \right\}, \tag{10}$$

where $Q(t)$ is defined in (2) and for $u, v \in E$, where the inner product is given by

$$\langle u, v \rangle = \int_{\mathbb{R}} e^{Q(t)} \left[(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) \right] dt. \tag{11}$$

Then E is a Hilbert space with the norm given by

$$\|u\| = \left(\int_{\mathbb{R}} e^{Q(t)} \left[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \right] dt \right)^{1/2}. \tag{12}$$

It is obvious that

$$E \subset L^2\left(e^{Q(t)}\right) \tag{13}$$

with the embedding being continuous. Here $L^p(e^{Q(t)})$ ($2 \leq p < +\infty$) denotes the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_p := \left\{ \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p \, dt \right\}^{1/p}. \tag{14}$$

Similar to [16–19], we have the following definition of fast homoclinic solutions.

Definition 1. If (2) holds, a solution of (1) is called a fast homoclinic solution if $u \in E$.

Now, we state our main results.

Theorem 2. *Suppose that q, W satisfy (2) and (W1)–(W3) and K satisfies the following conditions:*

(K1) $K \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and there exist two positive constants a_1 and a_2 such that*

$$a_1(L(t)x, x) \leq K(t,x) \leq a_2(L(t)x, x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{15}$$

where L satisfies (L).

(K2) *There exists a constant $a_3 > 0$ such that*

$$a_3(L(t)x, x) \leq 2K(t,x) - (\nabla K(t,x), x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{16}$$

(K3) *There is a constant $a_4 > 0$ such that*

$$\begin{aligned} &(\nabla K(t,x) - \nabla K(t,y), x - y) \\ &\quad \geq a_4 (L(t)(x-y), x-y), \quad \forall t \in \mathbb{R},\ x, y \in \mathbb{R}^N. \end{aligned} \tag{17}$$

Then problem (1) has at least one nontrivial fast homoclinic solution.

Theorem 3. *Suppose that q, K, and W satisfy (2), (K1)–(K3), (W2), and the following conditions:*

(W1)$'$ $W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, *and*

$$|\nabla W(t,x)| = o(|x|) \quad \text{as } x \longrightarrow 0 \tag{18}$$

uniformly in $t \in \mathbb{R}$.

(W3)′ $W_2(t,0)=0$ *and there exists a constant* $\varrho\in(2,\mu)$ *such that*

$$\left(\nabla W_2(t,x),x\right)\le \varrho W_2(t,x),\quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N. \tag{19}$$

Then problem (1) has at least one nontrivial fast homoclinic solution.

Theorem 4. *Suppose that q, K, and W satisfy (2), (K1)–(K3), (W1)–(W3), and the following condition:*

(W4)′ $K(t,-x)=K(t,x)$, $W(t,-x)=W(t,x)$, $\forall (t,x)\in\mathbb{R}\times\mathbb{R}^N$.

Then problem (1) has an unbounded sequence of fast homoclinic solutions.

Theorem 5. *Suppose that q, K, and W satisfy (2), (K1)–(K3), (W1)′, (W2), (W3)′, and (W4)′. Then problem (1) has an unbounded sequence of fast homoclinic solutions.*

Remark 6. We point out that (K1) is used in [13]. There are functions which can not be written in the form $(1/2)(L(t)x,x)$; then the results we obtain here are different. It is also remarked that the function $K(t,x)$ is not necessarily homogeneous of degree 2 with respect to x and so $\{\int_{\mathbb{R}} e^{Q(t)}[|\dot{x}(t)|^2+K(t,x(t))]dt\}^{1/2}$ may not be a norm in general.

The rest of this paper is organized as follows: in Section 2, some preliminaries are presented. In Section 3, we give the proofs of our results. In Section 4, some examples are given to illustrate our results.

2. Preliminaries

The functional φ corresponding to (1) on E is given by

$$\varphi(u)=\int_{\mathbb{R}} e^{Q(t)}\left[\frac{1}{2}|\dot{u}(t)|^2+K(t,u(t))-W(t,u(t))\right]dt,\quad u\in E. \tag{20}$$

Clearly, it follows from (K1), (W1), or (W1)′ that $\varphi\in C^1(E,\mathbb{R})$ and one can easily check that

$$\left\langle\varphi'(u),v\right\rangle=\int_{\mathbb{R}} e^{Q(t)}\left[(\dot{u}(t),\dot{v}(t))+(\nabla K(t,u(t)),v(t))-(\nabla W(t,u(t)),v(t))\right]dt,\quad u\in E. \tag{21}$$

Furthermore, the critical points of φ in E are classical solutions of (1) with $u(\pm\infty)=0$.

Let E and $\|\cdot\|$ be given in Section 1. The following lemma is important.

Lemma 7 (see [17]). *For any* $u\in E$,

$$\|u\|_\infty\le\frac{1}{\sqrt{2e_0\sqrt{\beta}}}\|u\|=\frac{1}{\sqrt{2e_0\sqrt{\beta}}}\left\{\int_{\mathbb{R}} e^{Q(s)}\left[|\dot{u}(s)|^2+(L(s)u(s),u(s))\right]ds\right\}^{1/2}, \tag{22}$$

$$|u(t)|\le\frac{1}{\sqrt[4]{\beta}}\left\{\int_t^{+\infty} e^{-Q(s)}e^{Q(s)}\left[|\dot{u}(s)|^2+(L(s)u(s),u(s))\right]ds\right\}^{1/2}, \tag{23}$$

$$|u(t)|\le\frac{1}{\sqrt[4]{\beta}}\left\{\int_{-\infty}^{t} e^{-Q(s)}e^{Q(s)}\left[|\dot{u}(s)|^2+(L(s)u(s),u(s))\right]ds\right\}^{1/2}, \tag{24}$$

where $\|u\|_\infty=\operatorname{ess\,sup}_{t\in\mathbb{R}}|u(t)|$, $e_0=e^{\min\{Q(t):t\in\mathbb{R}\}}$.

The following two lemmas are Mountain Pass Theorem and Symmetric Mountain Pass Theorem, which are useful in the proofs of our theorems.

Lemma 8 (see [22]). *Let E be a real Banach space and* $I\in C^1(E,\mathbb{R})$ *satisfying (PS)-condition. Suppose* $I(0)=0$ *and*

(i) *there exist constants* $\rho,\alpha>0$ *such that* $I_{\partial B_\rho(0)}\ge\alpha$;

(ii) *there exists* $e\in E\setminus\overline{B}_\rho(0)$ *such that* $I(e)\le 0$.

Then I possesses critical value $c\ge\alpha$ *which can be characterized as* $c=\inf_{h\in\Phi}\max_{s\in[0,1]}I(h(s))$, *where* $\Phi=\{h\in C([0,1],E)\mid h(0)=0,h(1)=e\}$ *and* $B_\rho(0)$ *is an open ball in E of radius* ρ *centered at* 0.

Lemma 9 (see [22]). *Let E be a real Banach space and* $I\in C^1(E,\mathbb{R})$ *with I being even. Assume that* $I(0)=0$ *and I satisfies (PS)-condition, (i) of Lemma 8, and the following condition:*

(iii) *For each finite dimensional subspace* $E'\subset E$, *there is* $r=r(E')>0$ *such that* $I(u)\le 0$ *for* $u\in E'\setminus B_r(0)$, *where* $B_r(0)$ *is an open ball in E of radius r centered at* 0.

Then I possesses an unbounded sequence of critical values.

Remark 10. Since it is very difficult to check condition (iii) of Lemma 9, only a few results about infinitely many homoclinic solutions can be seen in the literature by using Lemma 9, let alone fast homoclinic solutions. Motivated by the idea of Tang and Lin [10], we use Lemma 9 to obtain infinitely many fast homoclinic solutions for problem (1).

Lemma 11. *Assume that (W2) and (W3) or (W3)′ hold. Then for every* $(t,x)\in\mathbb{R}\times\mathbb{R}^N$,

(i) $s^{-\mu}W_1(t,sx)$ *is nondecreasing on* $(0,+\infty)$;

(ii) $s^{-\varrho}W_2(t,sx)$ *is nonincreasing on* $(0,+\infty)$.

The proof of Lemma 11 is routine and we omit it.

3. Proofs of Theorems

Proof of Theorem 2. Consider the following:

Step 1. The functional φ satisfies the (PS)-condition. Let $\{u_n\} \subset E$ satisfying $\varphi(u_n)$ be bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. Then there exists a constant $C_1 > 0$ such that

$$|\varphi(u_n)| \le C_1, \qquad \|\varphi'(u_n)\|_{E^*} \le \mu C_1. \tag{25}$$

From (20), (21), (25), (K2), (W2), and (W3), we have

$$\begin{aligned}2C_1 + 2C_1\|u_n\| &\ge 2\varphi(u_n) - \frac{2}{\mu}\langle\varphi'(u_n), u_n\rangle \\ &= \frac{\mu-2}{\mu}\|\dot{u}_n\|_2^2 + 2\int_{\mathbb{R}} e^{Q(t)}\Big[K(t,u_n(t)) - \frac{1}{\mu}(\nabla K(t,u_n(t)), u_n(t))\Big]dt \\ &\quad - 2\int_{\mathbb{R}} e^{Q(t)}\Big[W_1(t,u_n(t)) - \frac{1}{\mu}(\nabla W_1(t,u_n(t)), u_n(t))\Big]dt \\ &\quad + 2\int_{\mathbb{R}} e^{Q(t)}\Big[W_2(t,u_n(t)) - \frac{1}{\mu}(\nabla W_2(t,u_n(t)), u_n(t))\Big]dt \ge \frac{\mu-2}{\mu}\|\dot{u}_n\|_2^2 \\ &\quad + \int_{\mathbb{R}} a_3(L(t)u_n(t), u_n(t))\,dt \ge \min\left\{\frac{\mu-2}{\mu}, a_3\right\}\cdot\|u_n\|^2.\end{aligned} \tag{26}$$

The above inequalities imply that there exists a constant $C_2 > 0$ such that

$$\|u_n\| \le C_2, \quad n \in \mathbb{N}. \tag{27}$$

Now we prove that $u_n \to u_0$ in E. Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (W1), we can choose $\xi > 0$ such that

$$|\nabla W(t,x)| \le \varepsilon\beta|x| \quad \text{for } |t| \ge R,\ |x| \le \xi. \tag{28}$$

Since $Q(t) \to \infty$ as $|t| \to \infty$, we can choose $T > R$ such that

$$Q(t) \ge \ln\left(\frac{C_2^2}{\sqrt{\beta}\xi^2}\right) \quad \text{for } |t| \ge T. \tag{29}$$

It follows from (23), (27), and (29) that

$$\begin{aligned}|u_n(t)|^2 &\le \frac{1}{\sqrt{\beta}}\int_t^{+\infty} e^{-Q(s)}e^{Q(s)}\big[|\dot{u}_n(s)|^2 + (L(s)u_n(s), u_n(s))\big]ds \le \frac{\xi^2}{C_2^2}\|u_n\|^2 \le \xi^2 \\ &\quad \text{for } t \ge T,\ n \in \mathbb{N}.\end{aligned} \tag{30}$$

Similarly, by (24), (27), and (29), we have

$$|u_n(t)|^2 \le \xi^2 \quad \text{for } t \le -T,\ n \in \mathbb{N}. \tag{31}$$

Since $u_n \rightharpoonup u_0$ in E, it is easy to verify that $u_n(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{R}$. Hence, it follows from (30) and (31) that

$$|u_0(t)| \le \xi \quad \text{for } t \in (-\infty, -T] \cup [T, +\infty). \tag{32}$$

Since $e^{Q(t)} \ge e_0 > 0$ on $[-T, T] = J$, the operator defined by $S : E \to H^1(J) : u \to u|_J$ is a linear continuous map. So $u_n \to u_0$ in $H^1(J)$. Sobolev theorem implies that $u_n \to u_0$ uniformly on J, so there is $n_0 \in \mathbb{N}$ such that

$$\int_{-T}^{T} e^{Q(t)}|\nabla W(t,u_n(t)) - \nabla W(t,u_0(t))|\cdot|u_n(t) - u_0(t)|\,dt < \varepsilon \quad \text{for } n \ge n_0. \tag{33}$$

From (L), (27), (28), (30), (31), and (32), we have

$$\begin{aligned}&\int_{\mathbb{R}\setminus[-T,T]} e^{Q(t)}|\nabla W(t,u_n(t)) - \nabla W(t,u_0(t))||u_n(t) - u_0(t)|\,dt \le \int_{\mathbb{R}\setminus[-T,T]} e^{Q(t)}(|\nabla W(t,u_n(t))| \\ &\quad + |\nabla W(t,u_0(t))|)(|u_n(t)| + |u_0(t)|)\,dt \\ &\le \varepsilon\int_{\mathbb{R}\setminus[-T,T]} e^{Q(t)}\beta(|u_n(t)| + |u_0(t)|)(|u_n(t)| + |u_0(t)|)\,dt \le 2\varepsilon\int_{\mathbb{R}\setminus[-T,T]} e^{Q(t)}\beta(|u_n(t)|^2 \\ &\quad + |u_0(t)|^2)\,dt \\ &\le 2\varepsilon\int_{\mathbb{R}\setminus[-T,T]} e^{Q(t)}[(L(t)u_n(t), u_n(t)) + (L(t)u_0(t), u_0(t))]\,dt \le 2\varepsilon(\|u_n\|^2 + \|u_0\|^2) \\ &\le 2\varepsilon(C_2^2 + \|u_0\|^2), \quad n \in \mathbb{N}.\end{aligned} \tag{34}$$

It follows from (33) and (34) that

$$\int_{\mathbb{R}} e^{Q(t)}|\nabla W(t,u_n(t)) - \nabla W(t,u_0(t))|\cdot|u_n(t) - u_0(t)|\,dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{35}$$

From (21) and (K3), as $n \to \infty$, we have

$$\begin{aligned} 0 \longleftarrow & \left\langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \right\rangle = \|\dot{u}_n - \dot{u}_0\|_2^2 \\ & + \int_{\mathbb{R}} e^{Q(t)} (\nabla K(t, u_n(t)) - \nabla K(t, u_0(t)), u_n(t) \\ & - u_0(t))\, dt - \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_n(t)) \\ & - \nabla W(t, u_0(t)), u_n(t) - u_0(t))\, dt \geq \|\dot{u}_n - \dot{u}_0\|_2^2 \\ & + a_4 \int_{\mathbb{R}} e^{Q(t)} [(L(t)(u_n(t) - u_0(t)), u_n(t) \\ & - u_0(t))]\, dt - \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_n(t)) \\ & - \nabla W(t, u_0(t)), u_n(t) - u_0(t))\, dt \geq \min\{1, a_4\} \\ & \cdot \|u_n - u_0\|^2 - \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_n(t)) \\ & - \nabla W(t, u_0(t)), u_n(t) - u_0(t))\, dt. \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$\|u_n\| \longrightarrow \|u_0\| \quad \text{as } n \longrightarrow \infty. \tag{37}$$

Hence, $u_n \to u_0$ in E by (37). This shows that φ satisfies (PS)-condition.

Step 2. From (W1), there exists $\delta \in (0,1)$ such that

$$|\nabla W(t,x)| \leq \frac{a_1 \beta}{2} |x| \quad \text{for } |t| \geq R,\ |x| \leq \delta. \tag{38}$$

By (38), we have

$$|W(t,x)| \leq \frac{a_1 \beta}{4} |x|^2 \quad \text{for } |t| \geq R,\ |x| \leq \delta. \tag{39}$$

Let

$$C_3 = \sup\left\{ \frac{W_1(t,x)}{a_1 \beta} \mid t \in [-R, R],\ x \in \mathbb{R}, |x| = 1 \right\}. \tag{40}$$

Set $\sigma = \min\{1/(4C_3+1)^{1/(\mu-2)}, \delta\}$ and $\|u\| = \sqrt{2e_0\sqrt{\beta}}\sigma := \rho$; it follows from Lemma 7 that $|u(t)| \leq \sigma \leq \delta < 1$ for $t \in \mathbb{R}$. From Lemma 11(i), (L), and (40), we have

$$\begin{aligned} & \int_{-R}^{R} e^{Q(t)} W_1(t, u(t))\, dt \\ & \quad \leq \int_{\{t \in [-R,R] : u(t) \neq 0\}} e^{Q(t)} W_1\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\mu}\, dt \\ & \quad \leq C_3 a_1 \beta \int_{-R}^{R} e^{Q(t)} |u(t)|^{\mu}\, dt \\ & \quad \leq C_3 a_1 \sigma^{\mu-2} \int_{-R}^{R} e^{Q(t)} \beta |u(t)|^2\, dt \\ & \quad \leq C_3 a_1 \sigma^{\mu-2} \int_{-R}^{R} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ & \quad \leq \frac{a_1}{4} \int_{-R}^{R} e^{Q(t)} (L(t) u(t), u(t))\, dt. \end{aligned} \tag{41}$$

By (L), (K1), (W3), (39), and (41), we have

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^2\, dt + \int_{\mathbb{R}} e^{Q(t)} K(t, u(t))\, dt \\ & - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t))\, dt \\ \geq & \frac{1}{2} \|\dot{u}\|_2^2 + \int_{\mathbb{R}} e^{Q(t)} K(t, u(t))\, dt \\ & - \int_{\mathbb{R} \setminus [-R,R]} e^{Q(t)} W(t, u(t))\, dt \\ & - \int_{-R}^{R} e^{Q(t)} W_1(t, u(t))\, dt \\ \geq & \frac{1}{2} \|\dot{u}\|_2^2 + a_1 \int_{\mathbb{R}} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ & - \frac{a_1}{4} \int_{\mathbb{R} \setminus [-R,R]} \beta e^{Q(t)} |u(t)|^2\, dt \\ & - \frac{a_1}{4} \int_{-R}^{R} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ \geq & \frac{1}{2} \|\dot{u}\|_2^2 + a_1 \int_{\mathbb{R}} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ & - \frac{a_1}{4} \int_{\mathbb{R} \setminus [-R,R]} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ & - \frac{a_1}{4} \int_{-R}^{R} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ = & \frac{1}{2} \|\dot{u}\|_2^2 + \frac{3a_1}{4} \int_{\mathbb{R}} e^{Q(t)} (L(t) u(t), u(t))\, dt \\ \geq & \min\left\{ \frac{1}{2}, \frac{3a_1}{4} \right\} \|u\|^2. \end{aligned} \tag{42}$$

Therefore, we can choose constant $\alpha > 0$ depending on ρ such that $\varphi(u) \geq \alpha$ for any $u \in E$ with $\|u\| = \rho$.

Step 3. From Lemma 11(ii) and (22), we have for any $u \in E$

$$\begin{aligned} & \int_{-3}^{3} e^{Q(t)} W_2(t, u(t))\, dt \\ & \quad = \int_{\{t \in [-3,3] : |u(t)| > 1\}} e^{Q(t)} W_2(t, u(t))\, dt \end{aligned}$$

$$
\begin{aligned}
&+\int_{\{t\in[-3,3]:|u(t)|\le 1\}} e^{Q(t)} W_2(t,u(t))\,dt \\
&\le \int_{\{t\in[-3,3]:|u(t)|>1\}} e^{Q(t)} W_2\left(t,\frac{u(t)}{|u(t)|}\right)|u(t)|^{\varrho}\,dt \\
&+\int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)\,dt \\
&\le \|u\|_\infty^{\varrho}\int_{-3}^{3} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt \\
&+\int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)\,dt \\
&\le \left(\frac{1}{\sqrt{2e_0\sqrt{\beta}}}\right)^{\varrho}\|u\|^{\varrho}\int_{-3}^{3} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt \\
&+\int_{-3}^{3} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)\,dt = C_4\|u\|^{\varrho}+C_5,
\end{aligned}
\tag{43}
$$

where $C_4 = (1/\sqrt{2e_0\sqrt{\beta}})^{\varrho}\int_{-3}^{3} e^{Q(t)}\max_{|x|=1}W_2(t,x)dt$ and $C_5 = \int_{-3}^{3} e^{Q(t)}\max_{|x|\le 1}W_2(t,x)dt$. Take $\omega \in E$ such that

$$
|\omega(t)| = \begin{cases} 1, & \text{for } |t|\le 1, \\ 0, & \text{for } |t|\ge 3, \end{cases} \tag{44}
$$

and $|\omega(t)| \le 1$ for $|t| \in (1,3]$. For $s > 1$, from Lemma 11(i) and (44), we get

$$
\begin{aligned}
&\int_{-1}^{1} e^{Q(t)} W_1(t,s\omega(t))\,dt \\
&\quad \ge s^{\mu}\int_{-1}^{1} e^{Q(t)} W_1(t,\omega(t))\,dt = C_6 s^{\mu},
\end{aligned}
\tag{45}
$$

where $C_6 = \int_{-1}^{1} e^{Q(t)}W_1(t,\omega(t))dt > 0$. From (W3), (20), (43), (44), and (45), we get for $s > 1$

$$
\begin{aligned}
\varphi(s\omega) &= \frac{s^2}{2}\|\dot{\omega}\|_2^2 + \int_{\mathbb{R}} e^{Q(t)} K(t,s\omega(t))\,dt \\
&+\int_{\mathbb{R}} e^{Q(t)}\left[W_2(t,s\omega(t)) - W_1(t,s\omega(t))\right]dt \\
&\le \frac{s^2}{2}\|\dot{\omega}\|_2^2 + a_2 s^2\int_{\mathbb{R}} (L(t)u(t),u(t))\,dt \\
&+\int_{-3}^{3} e^{Q(t)} W_2(t,s\omega(t))\,dt \\
&-\int_{-1}^{1} e^{Q(t)} W_1(t,s\omega(t))\,dt \\
&\le \max\left\{\frac{1}{2},a_2\right\} s^2\|\omega\|^2 + C_4 s^{\varrho}\|\omega\|^{\varrho} + C_5 \\
&-C_6 s^{\mu}.
\end{aligned}
\tag{46}
$$

Since $\mu > \varrho > 2$ and $C_6 > 0$, it follows from (46) that there exists $s_1 > 1$ such that $\|s_1\omega\| > \rho$ and $\varphi(s_1\omega) < 0$. Set $e = s_1\omega(t)$; then $e \in E$, $\|e\| = \|s_1\omega\| > \rho$, and $\varphi(e) = \varphi(s_1\omega) < 0$. It is easy to see that $\varphi(0) = 0$. By Lemma 8, φ has critical value $c > \alpha$ given by

$$
c = \inf_{g\in\Phi}\max_{s\in[0,1]}\varphi(g(s)), \tag{47}
$$

where

$$
\Phi = \left\{g\in C([0,1],E) : g(0)=0, g(1)=e\right\}. \tag{48}
$$

Hence, there exists $u^* \in E$ such that

$$
\begin{aligned}
\varphi(u^*) &= c, \\
\varphi'(u^*) &= 0.
\end{aligned}
\tag{49}
$$

The function u^* is a desired solution of problem (1). Since $c > 0$, u^* is a nontrivial fast homoclinic solution. The proof is complete. □

Proof of Theorem 3. In the proof of Theorem 2, the condition $W_2(t,x) \ge 0$ in (W3) is only used in the proofs of (27) and Step 2. Therefore, we only need to prove that (27) and Step 2 still hold if we use (W1)′ and (W3)′ instead of (W1) and (W3), respectively. We first prove that (27) holds. From (K2), (W2), (W3)′, (20), (21), and (25), we have

$$
\begin{aligned}
2C_1 + \frac{2C_1\mu}{\varrho}\|u_n\| &\ge 2\varphi(u_n) - \frac{2}{\varrho}\left\langle\varphi'(u_n),u_n\right\rangle \\
&= \frac{(\varrho-2)}{\varrho}\|\dot{u}_n\|_2^2 + 2\int_{\mathbb{R}} e^{Q(t)}\Big[W_2(t,u_n(t)) \\
&-\frac{1}{\varrho}(\nabla W_2(t,u_n(t)),u_n(t))\Big]dt \\
&-2\int_{\mathbb{R}} e^{Q(t)}\Big[W_1(t,u_n(t)) \\
&-\frac{1}{\varrho}(\nabla W_1(t,u_n(t)),u_n(t))\Big]dt \\
&+2\int_{\mathbb{R}} e^{Q(t)}\Big[K(t,u_n(t)) \\
&-\frac{1}{\varrho}(\nabla K(t,u_n(t)),u_n(t))\Big]dt \ge \frac{\varrho-2}{\varrho}\|\dot{u}_n\|_2^2 \\
&+a_3\int_{\mathbb{R}} e^{Q(t)}(L(t)u_n(t),u_n(t))\,dt \\
&\ge \min\left\{\frac{\varrho-2}{\varrho},a_3\right\}\|u_n\|^2,
\end{aligned}
\tag{50}
$$

which implies that there exists a constant $C_2 > 0$ such that (27) holds. Next, we prove that Step 2 still holds. From (W1)′, there exists $\delta \in (0,1)$ such that

$$
|\nabla W(t,x)| \le \frac{1}{2}a_1\beta|x| \quad \text{for } t\in\mathbb{R},\ |x|\le\delta. \tag{51}
$$

By (51), we have

$$|W(t,x)| \le \frac{1}{4} a_1 \beta |x|^2 \quad \text{for } t \in \mathbb{R},\ |x| \le \delta. \tag{52}$$

Let $\|u\| = \sqrt{2e_0\sqrt{\beta}\delta} := \rho$; it follows from Lemma 7 that $|u(t)| \le \delta$. It follows from (L), (K1), (20), and (52) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^2\,dt + \int_{\mathbb{R}} e^{Q(t)} K(t,u(t))\,dt \\
&\quad - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))\,dt \\
&\ge \frac{1}{2}\|\dot{u}\|_2^2 + \int_{\mathbb{R}} a_1 (L(t)u(t),u(t))\,dt \\
&\quad - \frac{a_1}{4}\int_{\mathbb{R}} \beta e^{Q(t)} |u(t)|^2\,dt \\
&\ge \frac{1}{2}\|\dot{u}\|_2^2 + \int_{\mathbb{R}} a_1 (L(t)u(t),u(t))\,dt \\
&\quad - \frac{a_1}{4}\int_{\mathbb{R}} e^{Q(t)} (L(t)u(t),u(t))\,dt \\
&\ge \min\left\{\frac{1}{2}, \frac{3a_1}{4}\right\}\|u\|^2.
\end{aligned} \tag{53}$$

Therefore, we can choose constant $\alpha > 0$ depending on ρ such that $\varphi(u) \ge \alpha$ for any $u \in E$ with $\|u\| = \rho$. The proof of Theorem 3 is complete. □

Proof of Theorem 4. Condition $(W4)'$ shows that φ is even. In view of the proof of Theorem 2, we know that $\varphi \in C^1(E,\mathbb{R})$ and satisfies (PS)-condition and assumption (i) of Lemma 8. Now, we prove (iii) of Lemma 9. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional space are equivalent, there exists $d > 0$ such that

$$\|u\| \le d\|u\|_\infty. \tag{54}$$

Assume that $\dim E' = m$ and $\{u_1, u_2, \ldots, u_m\}$ is a base of E' such that

$$\|u_i\| = d, \quad i = 1,2,\ldots,m. \tag{55}$$

For any $u \in E'$, there exists $\lambda_i \in \mathbb{R}$, $i = 1,2,\ldots,m$, such that

$$u(t) = \sum_{i=1}^{m} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{56}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i| \|u_i\|. \tag{57}$$

It is easy to see that $\|\cdot\|_*$ is a norm of E'. Hence, there exists a constant $d' > 0$ such that $d'\|u\|_* \le \|u\|$. Since $u_i \in E$, by Lemma 7, we can choose $R_1 > R$ such that

$$|u_i(t)| < \frac{d'\delta}{1+d'}, \quad |t| > R_1,\ i = 1,2,\ldots,m, \tag{58}$$

where δ is given in (52). Let

$$\begin{aligned}
\Theta &= \left\{\sum_{i=1}^{m} \lambda_i u_i(t) : \lambda_i \in \mathbb{R},\ i = 1,2,\ldots,m;\ \sum_{i=1}^{m} |\lambda_i| = 1\right\} \\
&= \left\{u \in E' : \|u\|_* = d\right\}.
\end{aligned} \tag{59}$$

Hence, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ such that

$$|u(t_0)| = \|u\|_\infty. \tag{60}$$

Then by (54)–(57), (59), and (60), we have

$$\begin{aligned}
dd' &= dd'\sum_{i=1}^{m}|\lambda_i| = d'\sum_{i=1}^{m}|\lambda_i|\|u_i\| = d'\|u\|_* \le \|u\| \\
&\le d\|u\|_\infty = d|u(t_0)| \le d\sum_{i=1}^{m}|\lambda_i||u_i(t_0)|, \\
&\qquad u \in \Theta.
\end{aligned} \tag{61}$$

This shows that $|u(t_0)| \ge d'$ and there exists $i_0 \in \{1,2,\ldots,m\}$ such that $|u_{i_0}(t_0)| \ge d'$, which, together with (58), implies that $|t_0| \le R_1$. Let $R_2 = R_1 + 1$ and

$$\gamma = \min\left\{e^{Q(t)} W_1(t,x) : -R_2 \le t \le R_2,\ \frac{d'}{\sqrt{2}} \le |x| \le \frac{d}{\sqrt{2e_0\sqrt{\beta}}}\right\}. \tag{62}$$

Since $W_1(t,x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$ and $W_1 \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, it follows that $\gamma > 0$. For any $u \in E$, from Lemma 7 and Lemma 11(i), we have

$$\begin{aligned}
&\int_{-R_2}^{R_2} e^{Q(t)} W_2(t,u(t))\,dt \\
&\quad = \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} e^{Q(t)} W_2(t,u(t))\,dt \\
&\qquad + \int_{\{t\in[-R_2,R_2]:|u(t)|\le1\}} e^{Q(t)} W_2(t,u(t))\,dt \\
&\quad \le \int_{\{t\in[-R_2,R_2]:|u(t)|>1\}} e^{Q(t)} W_2\left(t, \frac{u(t)}{|u(t)|}\right)|u(t)|^{\varrho}\,dt \\
&\qquad + \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|\le1} W_2(t,x)\,dt
\end{aligned}$$

$$\le \|u\|_\infty^\varrho \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt + \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)\,dt \le \left(\frac{1}{\sqrt{2e_0\sqrt{\beta}}}\right)^\varrho \|u\|^\varrho \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|=1} W_2(t,x)\,dt + \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)\,dt = C_7\|u\|^\varrho + C_8, \tag{63}$$

where $C_7 = (1/\sqrt{2e_0\sqrt{\beta}})^\varrho \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|=1} W_2(t,x)dt$ and $C_8 = \int_{-R_2}^{R_2} e^{Q(t)} \max_{|x|\le 1} W_2(t,x)dt$. Since $\dot{u}_i \in L^2(e^{Q(t)})$, $i = 1,2,\ldots,m$, it follows that there exists $\varepsilon \in (0,1)$ such that

$$\int_{t+\varepsilon}^{t-\varepsilon} |\dot{u}_i(s)|\,ds = \int_{t+\varepsilon}^{t-\varepsilon} e^{-Q(s)/2} e^{Q(s)/2} |\dot{u}_i(s)|\,ds \le \frac{1}{\sqrt{e_0}} \int_{t+\varepsilon}^{t-\varepsilon} e^{Q(s)/2} |\dot{u}_i(s)|\,ds \le \frac{1}{\sqrt{e_0}} (2\varepsilon)^{1/2} \left(\int_{t+\varepsilon}^{t-\varepsilon} e^{Q(s)} |\dot{u}_i(s)|^2 ds\right)^{1/2} \le \left(\frac{2\varepsilon}{e_0}\right)^{1/2} \|\dot{u}_i\|_2 \le \frac{d'}{4m} \quad \text{for } t\in\mathbb{R},\ i=1,2,\ldots,m. \tag{64}$$

Then for $u \in \Theta$ with $|u(t_0)| = \|u\|_\infty$ and $t \in [t_0-\varepsilon, t_0+\varepsilon]$, it follows from (56), (59), (60), (61), and (64) that

$$|u(t)|^2 = |u(t_0)|^2 + 2\int_{t_0}^{t} (\dot{u}(s), u(s))\,ds \ge |u(t_0)|^2 - 2\int_{t_0-\varepsilon}^{t_0+\varepsilon} |u(s)||\dot{u}(s)|\,ds \ge |u(t_0)|^2 - 2|u(t_0)| \int_{t_0-\varepsilon}^{t_0+\varepsilon} |\dot{u}(s)|\,ds \ge |u(t_0)|^2 - 2|u(t_0)| \sum_{i=1}^m |\lambda_i| \int_{t_0-\varepsilon}^{t_0+\varepsilon} |\dot{u}_i(s)|\,ds \ge \frac{(d')^2}{2}. \tag{65}$$

On the other hand, since $\|u\| \le d$ for $u \in \Theta$, then

$$|u(t)| \le \|u\|_\infty \le \frac{d}{\sqrt{2e_0\sqrt{\beta}}}, \quad t\in\mathbb{R},\ u\in\Theta. \tag{66}$$

Therefore, from (62), (65), and (66), we have

$$\int_{-R_2}^{R_2} e^{Q(t)} W_1(t,u(t))\,dt \ge \int_{t_0-\varepsilon}^{t_0+\varepsilon} e^{Q(t)} W_1(t,u(t))\,dt \ge 2\varepsilon\gamma \quad \text{for } u\in\Theta. \tag{67}$$

By (58) and (59), we have

$$|u(t)| \le \sum_{i=1}^m |\lambda_i| |u_i(t)| \le \delta \quad \text{for } |t|\ge R_1,\ u\in\Theta. \tag{68}$$

By (L), (K1), (20), (39), (63), (67), (68), and Lemma 11, we have for $u \in \Theta$ and $r > 1$

$$\varphi(ru) = \frac{r^2}{2}\|\dot{u}\|_2^2 + \int_{\mathbb{R}} e^{Q(t)} K(t, ru(t))\,dt + \int_{\mathbb{R}} e^{Q(t)} [W_2(t,ru(t)) - W_1(t,ru(t))]\,dt$$
$$\le \frac{r^2}{2}\|\dot{u}\|_2^2 + a_2 r^2 \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t),u(t))\,dt + r^\varrho \int_{\mathbb{R}} e^{Q(t)} W_2(t,u(t))\,dt - r^\mu \int_{\mathbb{R}} e^{Q(t)} W_1(t,u(t))\,dt$$
$$= \frac{r^2}{2}\|\dot{u}\|_2^2 + a_2 r^2 \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t),u(t))\,dt + r^\varrho \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W_2(t,u(t))\,dt - r^\mu \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W_1(t,u(t))\,dt + r^\varrho \int_{-R_2}^{R_2} e^{Q(t)} W_2(t,u(t))\,dt - r^\mu \int_{-R_2}^{R_2} e^{Q(t)} W_1(t,u(t))\,dt$$
$$\le \frac{r^2}{2}\|\dot{u}\|_2^2 + a_2 r^2 \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t),u(t))\,dt - r^\varrho \int_{\mathbb{R}\setminus(-R_2,R_2)} e^{Q(t)} W(t,u(t))\,dt - r^\mu \int_{-R_2}^{R_2} e^{Q(t)} W_1(t,u(t))\,dt + r^\varrho \int_{-R_2}^{R_2} e^{Q(t)} W_2(t,u(t))\,dt$$
$$\le \frac{r^2}{2}\|\dot{u}\|_2^2 + a_2 r^2 \int_{\mathbb{R}} e^{Q(t)} (L(t)u(t),u(t))\,dt + \frac{r^\varrho a_1}{4} \int_{\mathbb{R}\setminus(-R_2,R_2)} \beta e^{Q(t)} |u(t)|^2\,dt$$

$$
\begin{aligned}
&+r^{\varrho}\left(C_7\|u\|^{\varrho}+C_8\right)-2\varepsilon\gamma r^{\mu}\\
&\le \max\left\{\frac{1}{2},a_2\right\}r^2\|u\|^2+\frac{a_1}{4}r^{\varrho}\|u\|^2\\
&+r^{\varrho}\left(C_7\|u\|^{\varrho}+C_8\right)-2\varepsilon\gamma r^{\mu}\\
&\le \max\left\{\frac{1}{2},a_2\right\}r^2d^2+\frac{a_1r^{\varrho}}{4}d^2+C_7(rd)^{\varrho}\\
&+C_8r^{\varrho}-2\varepsilon\gamma r^{\mu}.
\end{aligned}
\tag{69}
$$

Since $\mu > \varrho > 2$, we deduce that there exists $r_0 = r_0(a_1, a_2, d, d', C_7, C_8, R_1, R_2, \varepsilon, \gamma) = r_0(E') > 1$ such that

$$\varphi(ru) < 0 \quad \text{for } u \in \Theta,\ r \ge r_0. \tag{70}$$

It follows that

$$\varphi(u) < 0 \quad \text{for } u \in E',\ \|u\| \ge dr_0, \tag{71}$$

which shows that (iii) of Lemma 9 holds. By Lemma 9, φ possesses unbounded sequence $\{c_n\}_{n=1}^{\infty}$ of critical values with $c_n = \varphi(u_n)$, where u_n is such that $\varphi'(u_n) = 0$ for $n = 1, 2, \ldots$. If $\{\|u_n\|\}$ is bounded, then there exists $C_9 > 0$ such that

$$\|u_n\| \le C_9 \quad \text{for } n \in \mathbb{N}. \tag{72}$$

By a similar fashion for the proof of (30) and (31), for the given δ in (39), there exists $R_3 > R$ such that

$$|u_n(t)| \le \delta \quad \text{for } |t| \ge R_3,\ n \in \mathbb{N}. \tag{73}$$

Hence, by (L), (20), (22), (39), (72), and (73), we have

$$
\begin{aligned}
&\frac{1}{2}\|\dot{u}_n\|_2^2+\int_{\mathbb{R}}e^{Q(t)}K(t,u_n(t))\,dt\\
&=c_n+\int_{\mathbb{R}}e^{Q(t)}W(t,u_n(t))\,dt\\
&=c_n+\int_{\mathbb{R}\setminus[-R_3,R_3]}e^{Q(t)}W(t,u_n(t))\,dt\\
&+\int_{-R_3}^{R_3}e^{Q(t)}W(t,u_n(t))\,dt\\
&\ge c_n-\frac{a_1}{4}\int_{\mathbb{R}\setminus[-R_3,R_3]}\beta e^{Q(t)}|u_n(t)|^2\,dt\\
&-\int_{-R_3}^{R_3}e^{Q(t)}|W(t,u_n(t))|\,dt\\
&\ge c_n-\frac{a_1}{4}\int_{\mathbb{R}\setminus[-R_3,R_3]}e^{Q(t)}(L(t)u_n(t),u_n(t))\,dt\\
&-\int_{-R_3}^{R_3}e^{Q(t)}|W(t,u_n(t))|\,dt\\
&\ge c_n-\frac{a_1}{4}\|u_n\|^2\\
&-\int_{-R_3}^{R_3}e^{Q(t)}\max_{|x|\le\sqrt{2e_0}\sqrt{\beta}C_9}|W(t,x)|\,dt.
\end{aligned}
\tag{74}
$$

On the other hand, by (K1), we have

$$
\begin{aligned}
&\frac{1}{2}\|\dot{u}_n\|_2^2+\int_{\mathbb{R}}e^{Q(t)}K(t,u_n(t))\,dt\\
&\le\frac{1}{2}\|\dot{u}_n\|_2^2+a_2\int_{\mathbb{R}}e^{Q(t)}(L(t)u_n(t),u_n(t))\,dt\\
&\le\max\left\{\frac{1}{2},a_2\right\}\|u_n\|^2:=b\|u_n\|^2.
\end{aligned}
\tag{75}
$$

It follows from (74) and (75) that

$$
\begin{aligned}
&c_n\le b\|u_n\|^2+\frac{a_1}{4}\|u_n\|^2\\
&+\int_{-R_3}^{R_3}e^{Q(t)}\max_{|x|\le\sqrt{2e_0}\sqrt{\beta}C_9}|W(t,x)|\,dt<+\infty.
\end{aligned}
\tag{76}
$$

This contradicts the fact that $\{c_n\}_{n=1}^{\infty}$ is unbounded, and so $\{\|u_n\|\}$ is unbounded. The proof is complete. □

Proof of Theorem 5. In view of the proofs of Theorems 3 and 4, the conclusion of Theorem 5 holds. The proof is complete. □

4. Examples

Example 1. Consider the following system:

$$\ddot{u}(t)+t\dot{u}(t)-\nabla K(t,u(t))+\nabla W(t,u(t))=0,\quad \text{a.e. } t\in\mathbb{R}, \tag{77}$$

where $q(t) = t$, $t \in \mathbb{R}$, $u \in \mathbb{R}^N$. Let

$$
\begin{aligned}
K(t,x)&=\left(1+t^2+\frac{1+t^2}{\sin\left(1+|x|^2\right)+1}\right)|x|^2,\\
W(t,x)&=\sum_{i=1}^{m}a_i|x|^{\mu_i}-\sum_{j=1}^{n}b_j|x|^{\varrho_j},\\
L(t)&=\operatorname{diag}\left(1+t^2,\ldots,1+t^2\right),
\end{aligned}
\tag{78}
$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 2$ and a_i, $b_j > 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. By an easy computation, we have

$$
\begin{aligned}
&2K(t,x)-(\nabla K(t,x),x)\\
&=\frac{2\left(1+t^2\right)\cos\left(1+|x|^2\right)}{\left[\sin\left(1+|x|^2\right)+1\right]^2}|x|^4,\\
&(\nabla K(t,x)-\nabla K(t,y),x-y)=2\left(1+t^2\right)\Bigg(x-y\\
&+\frac{\left[1+\sin\left(1+|x|^2\right)-\cos\left(1+|x|^2\right)\right]x}{\left[\sin\left(1+|x|^2\right)+1\right]^2}\\
&-\frac{\left[1+\sin\left(1+|y|^2\right)-\cos\left(1+|y|^2\right)\right]y}{\left[\sin\left(1+|y|^2\right)+1\right]^2},x-y\Bigg).
\end{aligned}
\tag{79}
$$

Then K satisfies (K1), (K2), (K3), and $(\mathrm{W4})'$. Let

$$W_1(t,x) = \sum_{i=1}^{m} a_i |x|^{\mu_i}, \qquad W_2(t,x) = \sum_{j=1}^{n} b_j |x|^{\varrho_j}. \tag{80}$$

Then it is easy to check that all the conditions of Theorem 4 are satisfied with $\mu = \mu_m$ and $\varrho = \varrho_1$. Hence, problem (77) has an unbounded sequence of fast homoclinic solutions.

Example 2. Consider the following system:

$$\ddot{u}(t) + \left(t + t^3\right)\dot{u}(t) - \nabla K(t,u(t)) + \nabla W(t,u(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \tag{81}$$

where $q(t) = t + t^3$, $t \in \mathbb{R}$, and $u \in \mathbb{R}^N$. Let K and L be the same in Example 1 and

$$W(t,x) = a_1 |x|^{\mu_1} + a_2 |x|^{\mu_2} - b_1 (\cos t) |x|^{\varrho_1} - b_2 |x|^{\varrho_2}, \tag{82}$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 2$, $a_1, a_2 > 0$, and $b_1, b_2 > 0$. Let

$$W_1(t,x) = a_1 |x|^{\mu_1} + a_2 |x|^{\mu_2}, \qquad W_2(t,x) = b_1 (\cos t) |x|^{\varrho_1} + b_2 |x|^{\varrho_2}. \tag{83}$$

Then it is easy to check that all the conditions of Theorem 5 are satisfied with $\mu = \mu_2$ and $\varrho = \varrho_1$. Hence, by Theorem 5, problem (81) has an unbounded sequence of fast homoclinic solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the NNSF of China (no. 11301108), Guangxi Natural Science Foundation (no. 2013GXNSFBA019004 and no. 2013GXNSFDA019001), the Scientific Research Foundation of Guangxi Education Office (no. 201203YB093), and the Program Sponsor Terms for Innovation in the Construction of Talent Highlands in Guangxi Institutions of Higher Learning ([2011]47).

References

[1] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Gauthier Villars, Paris, France, 1897–1899.

[2] M. Arias, J. Campos, A. M. Robles-Pres, and L. Sanchez, "Fast and heteroclinic solutions for a second order ODE related to Fisher C Kolmogorovs equation," *Calculus of Variations and Partial Differential Equations*, vol. 21, no. 3, pp. 319–334, 2004.

[3] C. N. Chen and S. Y. Tzeng, "Existence and multiplicity results for homoclinic orbits of Hamiltonian systems," *Electronic Journal of Differential Equations*, vol. 1997, paper 7, 19, 1997.

[4] Y. H. Ding, "Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 25, no. 11, pp. 1095–1113, 1995.

[5] M. Izydorek and J. Janczewska, "Homoclinic solutions for a class of the second order Hamiltonian systems," *Journal of Differential Equations*, vol. 219, no. 2, pp. 375–389, 2005.

[6] P. Korman, A. C. Lazer, and Y. Li, "On homoclinic and heteroclinic orbits for Hamiltonian systems," *Differential and Integral Equations*, vol. 10, no. 2, pp. 357–368, 1997.

[7] Y. Lu, C. Li, S. Zhong, and W. Zhang, "Homoclinic orbits for a class of Hamiltonian systems with potentials changing sign," *Annals of Differential Equations*, vol. 21, no. 3, pp. 370–372, 2005.

[8] X. Li, X. Wu, and K. Wu, "On a class of damped vibration problems with super-quadratic potentials," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 1, pp. 135–142, 2010.

[9] P. H. Rabinowitz, "Periodic and Heteroclinic orbits for a periodic Hamiltonian systems," *Annales de l'Institut Henri Poincaré-Analyse Non Linéaire*, vol. 6, no. 5, pp. 331–346, 1989.

[10] X. H. Tang and X. Y. Lin, "Existence of infinitely many homoclinic orbits in Hamiltonian systems," *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, vol. 141, no. 5, pp. 1103–1119, 2011.

[11] D.-L. Wu, X.-P. Wu, and C.-L. Tang, "Homoclinic solutions for a class of nonperiodic and noneven second-order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 1, pp. 154–166, 2010.

[12] Q. F. Zhang and X. H. Tang, "Existence of homoclinic solutions for a class of asymptotically quadratic non-autonomous Hamiltonian systems," *Mathematische Nachrichten*, vol. 285, no. 5-6, pp. 778–789, 2012.

[13] Q. F. Zhang and X. H. Tang, "Existence of homoclinic solutions for a class of second-order non-autonomous hamiltonian systems," *Mathematica Slovaca*, vol. 62, no. 5, pp. 909–920, 2012.

[14] X. Wu and J. Zhou, "On a class of forced vibration problems with obstacles," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1053–1063, 2008.

[15] X. Wu, S. X. Chen, and K. Teng, "On variational methods for a class of damped vibration problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 6, pp. 1432–1441, 2008.

[16] Z. H. Zhang and R. Yuan, "Fast homoclinic solutions for some second order non-autonomous systems," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 51–63, 2011.

[17] P. Chen, X. H. Tang, and R. P. Agarwal, "Fast homoclinic solutions for a class of damped vibration problems," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 6053–6065, 2013.

[18] Q. Zhang, Q.-M. Zhang, and X. Tang, "Existence of fast homoclinic orbits for a class of second-order non-autonomous problems," *Boundary Value Problems*, vol. 2014, article 89, 2014.

[19] Q. F. Zhang, "Existence and multiplicity of fast homoclinic solutions for a class of damped vibration problems with impulsive effects," *Abstract and Applied Analysis*, vol. 2014, Article ID 960276, 12 pages, 2014.

[20] Y. B. Deng and W. Shuai, "Positive solutions for quasilinear Schrödinger equations with critical growth and potential vanishing at infinity," *Communications on Pure and Applied Analysis*, vol. 13, no. 6, pp. 2273–2287, 2014.

[21] M. F. Furtado, E. S. Medeiros, and U. B. Severo, "A Trudinger-Moser inequality in a weighted Sobolev space and applications," *Mathematische Nachrichten*, vol. 287, no. 11-12, pp. 1255–1273, 2014.

[22] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.