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Research Article

Delay-Dependent Stability Analysis for Recurrent Neural Networks with Time-Varying Delays

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This paper concerns the problem of delay-dependent stability criteria for recurrent neural networks with time varying delays. By taking more information of states and activation functions as augmented vectors, a new class of the Lyapunov functional is proposed. Then, some less conservative stability criteria are obtained in terms of linear matrix inequalities (LMIs). Finally, two numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

In the past few decades, the stability analysis for recurrent neural networks has been extensively investigated because of their successful applications in various scientific fields, such as pattern recognition, image processing, associative memories, and fixed-point computations. It is well known that time delay is frequently encountered in neural networks, and it is often a major cause of instability and oscillation. Thus, much more attention has been paid to recurrent delayed neural networks. Many interesting stability conditions, including delay-independent results [1, 2] and delay-dependent results [3–41], have been obtained for neural networks with time delays. Generally speaking, the delay-dependent stability criteria are less conservative than delay-independent ones when the size of time delay is small. For the delay-dependent case, some criteria have been derived by using Lyapunov-Krasovskii functional (LKF). It is well known that the construction of an appropriate LKF is crucial for obtaining less conservative stability conditions. Thus, some new methods have been developed for reducing conservatism, such as free-weighting matrix method [4–8], augmented LKF [9], discretized LKF [10], delay-partitioning method [12–18], and delay-slope-dependent method [19]. Some less conservative stability criteria were proposed in

[6] by considering some useful terms which have been usually neglected in the previous literature and using the free-weighting matrices method. Recently, a novel method was proposed for Hopfield neural networks in [12], which divides the constant time delay interval $[0, h]$ into subintervals with the same size. This method utilizes more information about the delay interval $[0, h]$ to reduce the conservativeness. Very recently, by proposing the idea of dividing the delay interval with the weighted parameters, the weighting-delay-based stability criteria for neural networks with time-varying delay were investigated in [15]. The delay-partitioning method proved to be less conservative than most of the previous results, and the conservatism can be notably reduced by thinning the delay partitioning. However, the above methods suffer two common shortcomings. First, many matrix variables are introduced in the obtained results, which brings a large computational burden. Second, the information of neuron activation function is not adequately considered, which may lead to much conservatism.

In this paper, the problem of delay-dependent stability analysis for neural networks with time-varying delays is investigated. Different from the previous methods of [4–8, 12–18], no delay-partitioning methods or free-weighting matrix methods are utilized. Instead, by taking more information of states and activation functions as augmented vectors, an augmented Lyapunov-Krasovskii functional is proposed. Then, inspired by the results of [19, 42], a less conservative condition is derived to guarantee the asymptotical stablity of the considered systems. Finally, two numerical examples are given to indicate significant improvements over some existing results.

2. Problem Formulation

Consider the following neural networks with time-varying delay:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + \mu, \tag{2.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathcal{R}^n$ is the neuron state vector, $g(x(\cdot)) = [g_1(x_1(\cdot)), g_2(x_2(\cdot)), \ldots, g_n(x_n(\cdot))]^T \in \mathcal{R}^n$ denotes the neuron activation function, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T \in \mathcal{R}^n$ is a constant input vector. $A, B \in \mathcal{R}^{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. $C = \mathrm{diag}(C_1, C_2, \ldots, C_n)$ with $C_i > 0,\ i = 1, 2, \ldots, n$. $\tau(t)$ is a time-varying continuous function that satisfies $0 \leqslant \tau(t) \leqslant h$, $\dot{\tau}(t) \leqslant u$, where h and u are constants. In addition, it is assumed that each neuron activation function in (2.1), $g_i(\cdot),\ i = 1, 2, \ldots, n$, is bounded and satisfies the following condition:

$$k_i^- \leqslant \frac{g_i(x) - g_i(y)}{x - y} \leqslant k_i^+, \quad \forall x, y \in R,\ x \neq y,\ i = 1,\ 2, \ldots, n, \tag{2.2}$$

where $k_i^-, k_i^+,\ i = 1, 2, \ldots, n$ are constants.

Assuming that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is the equilibrium point of (2.1) whose uniqueness has been given in [29] and using the transformation $z(\cdot) = x(\cdot) - x^*$, system (2.1) can be converted to the following system:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t-\tau(t))), \tag{2.3}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T$, $f(z(\cdot)) = [f_1(z_1(\cdot)), f_2(z_2(\cdot)), \ldots, f_n(z_n(\cdot))]^T$ and $f_i(z_i(\cdot)) = g_i(z_i(\cdot) + x_i^*) - g_i(x_i^*)$, $i = 1, 2, \ldots, n$. According to the inequality (2.2), one can obtain that

$$k_i^- \leqslant \frac{f_i(z_i(t))}{z_i(t)} \leqslant k_i^+ f_i(0) = 0, \quad i = 1, 2, \ldots, n. \tag{2.4}$$

Thus, under this assumption, the following inequality holds for any diagonal matrix $Q_4 > 0$,

$$z^T(t) K Q_4 K z(t) - f^T(z(t)) Q_4 f(z(t)) \geqslant 0, \tag{2.5}$$

where $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$, $k_i = \max(|k_i^-|, |k_i^+|)$.

Lemma 2.1 (see[43]). *For any constant matrix $Z \in \mathcal{R}^{n\times n}$, $Z = Z^T > 0$, scalars $h_2 > h_1 > 0$, such that the following integrations are well defined, then*

$$-(h_2 - h_1) \int_{t-h_2}^{t-h_1} x^T(s) Z x(s) ds \leqslant - \int_{t-h_2}^{t-h_1} x^T(s) ds \, Z \int_{t-h_2}^{t-h_1} x(s) ds. \tag{2.6}$$

3. Main Results

In this section, a new Lyapunov functional is constructed and a less conservative delay-dependent stability criterion is obtained.

Theorem 3.1. *For given scalars $h \geqslant 0$, u, diagonal matrices $K_1 = \operatorname{diag}(k_1^-, k_2^-, \ldots, k_n^-)$, $K_2 = \operatorname{diag}(k_1^+, k_2^+, \ldots, k_n^+)$, the system (2.3) is globally asymptotically stable if there exist symmetric positive matrices $P = [\,P_{ij}\,]_{4\times 4}$, $Q = [\,Q_{ij}\,]_{2\times 2}$, $X = [\,X_{ij}\,]_{2\times 2}$, $Y = [\,Y_{ij}\,]_{2\times 2}$, Q_i $(i = 1, 2, 3)$, positive diagonal matrices T_1, T_2, T_3, Q_4, $\Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and any matrices $S_i (i = 1, 2, \ldots, 5)$ with appropriate dimensions, such that the following LMIs hold:*

$$\begin{bmatrix} Y_{11} & Y_{12} & S_1 & S_2 \\ * & Y_{22} & S_3 & S_4 \\ * & * & Y_{11} & Y_{12} \\ * & * & * & Y_{22} \end{bmatrix} > 0, \tag{3.1}$$

$$\begin{bmatrix} Q_1 & S_5 \\ * & Q_1 \end{bmatrix} > 0, \tag{3.2}$$

$$\begin{bmatrix} E & \mathcal{A}^T R \\ * & -R \end{bmatrix} < 0, \tag{3.3}$$

where

$$E=\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & -P_{13} & E_{17} & E_{18} & E_{19} & E_{1,10} & E_{1,11} \\ * & E_{22} & E_{23} & 0 & E_{25} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & E_{33} & 0 & 0 & E_{36} & -P_{22} & -P_{22} & -P_{23} & -P_{23} & -P_{24} \\ * & * & * & E_{44} & E_{45} & 0 & E_{47} & E_{48} & E_{49} & E_{4,10} & E_{4,11} \\ * & * & * & * & E_{55} & 0 & B^TP_{12} & B^TP_{12} & B^TP_{13} & B^TP_{13} & B^TP_{14} \\ * & * & * & * & * & E_{66} & -P_{23}^T & -P_{23}^T & -P_{33} & -P_{33} & -P_{34} \\ * & * & * & * & * & * & E_{77} & E_{78} & E_{7,9} & E_{7,10} & -P_{44} \\ * & * & * & * & * & * & * & E_{88} & E_{89} & E_{8,10} & -P_{44} \\ * & * & * & * & * & * & * & * & -Y_{22} & -S_4 & 0 \\ * & * & * & * & * & * & * & * & * & -Y_{22} & 0 \\ * & * & * & * & * & * & * & * & * & * & -Q_3 \end{bmatrix},$$

$$\mathscr{A}=\begin{bmatrix}-C & 0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$R=h^2Q_1+\frac{h^4}{4}Q_2,$$

$$E_{11}=-P_{11}C-CP_{11}+P_{12}+P_{12}^T+h\left(P_{14}+P_{14}^T\right)+Q_{11}+2K_1\Lambda C$$
$$-2K_2\Delta C+X_{11}+h^2Y_{11}-Q_1-h^2Q_2+KQ_4K-2K_2T_1K_1+\frac{h^4}{4}Q_3,$$

$$E_{12}=Q_1-S_5,$$

$$E_{13}=-P_{12}+S_5,$$

$$E_{14}=P_{11}A+P_{13}+Q_{12}-C\Lambda-K_1\Lambda A+K_2\Delta A+C\Delta+X_{12}+h^2Y_{12}+T_1(K_1+K_2),$$
$$E_{15}=P_{11}B-K_1\Lambda B+K_2\Delta B,$$

$$E_{17}=-CP_{12}+P_{22}+hP_{24}^T-P_{14}+hQ_2,$$

$$E_{18}=-CP_{12}+P_{22}+hP_{24}^T-P_{14}+hQ_2,$$

$$E_{19}=-CP_{13}+P_{23}+hP_{34}^T,$$

$$E_{1,10}=-CP_{13}+P_{23}+hP_{34}^T,$$

$$E_{1,11}=-CP_{14}+P_{24}+hP_{44},$$

$$E_{22}=-(1-u)X_{11}-2Q_1+S_5+S_5^T-(1-u)KQ_4K-2K_2T_2K_1,$$

$$E_{23}=Q_1-S_5,$$

$$E_{25}=-(1-u)X_{12}+T_2(K_1+K_2),$$

$$E_{33}=-Q_{11}-Q_1-2K_2T_3K_1,$$

$$E_{36}=-Q_{12}+T_3(K_1+K_2),$$

$$E_{44}=Q_{22}+\Lambda A+A^T\Lambda-\Delta A-A^T\Delta+X_{22}+h^2Y_{22}-Q_4-2T_1,$$

$$E_{45}=\Lambda B-\Delta B,$$

$$E_{47}=A^TP_{12}+P_{23}^T,$$

$$
\begin{aligned}
E_{48} &= A^T P_{12} + P_{23}^T,\\
E_{49} &= A^T P_{13} + P_{33},\\
E_{4,10} &= A^T P_{13} + P_{33},\\
E_{4,11} &= A^T P_{14} + P_{34},\\
E_{55} &= -(1-u)X_{22} + (1-u)Q_4 - 2T_2,\\
E_{66} &= -Q_{22} - 2T_3,\\
E_{77} &= -P_{24} - P_{24}^T - Y_{11} - Q_2,\\
E_{78} &= -P_{24} - P_{24}^T - S_1 - Q_2,\\
E_{79} &= -P_{34}^T - Y_{12},\\
E_{7,10} &= -P_{34}^T - S_2,\\
E_{88} &= -P_{24} - P_{24}^T - Y_{11} - Q_2,\\
E_{89} &= -P_{34}^T - S_3^T,\\
E_{8,10} &= -P_{34}^T - Y_{12}.
\end{aligned} \tag{3.4}
$$

Proof. Construct a new Lyapunov functional as follow:

$$
V(z_t) = \sum_{i=1}^{8} V_i(z_t), \tag{3.5}
$$

where

$$
\begin{aligned}
V_1(z_t) &= \begin{bmatrix} z(t) \\ \int_{t-h}^{t} z(s)ds \\ \int_{t-h}^{t} f(z(s))ds \\ \int_{-h}^{0}\int_{t+\theta}^{t} z(s)ds\,d\theta \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ * & P_{22} & P_{23} & P_{24} \\ * & * & P_{33} & P_{34} \\ * & * & * & P_{44} \end{bmatrix} \begin{bmatrix} z(t) \\ \int_{t-h}^{t} z(s)ds \\ \int_{t-h}^{t} f(z(s))ds \\ \int_{-h}^{0}\int_{t+\theta}^{t} z(s)ds\,d\theta \end{bmatrix},\\
V_2(z_t) &= \int_{t-h}^{t} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds,\\
V_3(z_t) &= \int_{t-\tau(t)}^{t} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds,\\
V_4(z_t) &= 2\sum_{i=1}^{n} \left\{ \int_0^{z_i(t)} \lambda_i \big(f_i(s) - k_i^- s\big) ds + \int_0^{z_i(t)} \delta_i \big(k_i^+ s - f_i(s)\big) ds \right\},
\end{aligned}
$$

$$\begin{aligned}
V_5(z_t) &= h\int_{-h}^{0}\int_{t+\theta}^{t}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds\, d\theta, \\
V_6(z_t) &= h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}^T(s)Q_1\dot{z}(s)ds\, d\theta, \\
V_7(z_t) &= \frac{h^2}{2}\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}\dot{z}^T(s)Q_2\dot{z}(s)ds\, d\lambda\, d\theta, \\
V_8(z_t) &= \frac{h^2}{2}\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t}z^T(s)Q_3 z(s)ds\, d\lambda\, d\theta, \\
V_9(z_t) &= \int_{t-\tau(t)}^{t}\left[z^T(s)KQ_4Kz(s) - f^T(z(s))Q_4 f(z(s))\right]ds.
\end{aligned} \tag{3.6}$$

□

Remark 3.2. Since the terms $2\sum_{i=1}^{n}\delta_i\int_0^{z_i(t)}(k_i^+ s - f_i(s))ds$ in $V_4(z_t)$ and $\int_{t-\tau(t)}^{t}[z^T(s)KQ_4Kz(s) - f^T(z(s))Q_4f(z(s))]ds$ are taken into account, it is clear that the Lyapunov functional candidate in this paper is more general than that in [5, 6, 8, 9]. So the stability criteria in this paper may be more applicable.

The time derivative of $V(z_t)$ along the trajectory of system (2.4) is given by:

$$\dot{V}(z_t) = \sum_{i=1}^{8}\dot{V}_i(z_t), \tag{3.7}$$

where

$$\dot{V}_1(z_t) = 2\begin{bmatrix} z(t) \\ \int_{t-h}^{t} z(s)ds \\ \int_{t-h}^{t} f(z(s))ds \\ \int_{-h}^{0}\int_{t+\theta}^{t} z(s)ds\, d\theta \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ * & P_{22} & P_{23} & P_{24} \\ * & * & P_{33} & P_{34} \\ * & * & * & P_{44} \end{bmatrix}\begin{bmatrix} \dot{z}(t) \\ z(t) - z(t-h) \\ f(z(t)) - f(z(t-h)) \\ hz(t) - \int_{t-h}^{t} z(s)ds \end{bmatrix}, \tag{3.8}$$

$$\dot{V}_2(z_t) = \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} - \begin{bmatrix} z(t-h) \\ f(z(t-h)) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}\begin{bmatrix} z(t-h) \\ f(z(t-h)) \end{bmatrix}, \tag{3.9}$$

$$\begin{aligned}
\dot{V}_3(z_t) \leqslant{}& \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} \\
&- (1-u)\begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}\begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\dot{V}_4(z_t) &= 2\left[f(z(t)) - K_1z(t)\right]^T\Lambda\dot{z}(t) + 2\left[K_2z(t) - f(z(t))\right]^T\Delta\dot{z}(t), \\
&= 2\left[f(z(t)) - K_1z(t)\right]^T\Lambda\left[-Cz(t) + Af(z(t)) + Bf(z(t-\tau(t)))\right] \\
&\quad + 2\left[K_2z(t) - f(z(t))\right]^T\Delta\left[-Cz(t) + Af(z(t)) + Bf(z(t-\tau(t)))\right].
\end{aligned} \tag{3.11}$$

By the use of Lemma 2.1 and Theorem 3.1 in [42], one can obtain

$$\begin{aligned}
\dot{V}_5(z_t) &= h^2\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} - h\int_{t-h}^{t}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds \\
&= h^2\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} - h\int_{t-\tau(t)}^{t}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds \\
&\quad - h\int_{t-h}^{t-\tau(t)}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds \\
&\leqslant h^2\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} - \frac{h}{\tau(t)}\int_{t-\tau(t)}^{t}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T \\
&\quad \times ds\begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\int_{t-\tau(t)}^{t}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds \\
&\quad - \frac{h}{h-\tau(t)}\int_{t-h}^{t-\tau(t)}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix}^T ds\begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\int_{t-h}^{t-\tau(t)}\begin{bmatrix} z(s) \\ f(z(s)) \end{bmatrix} ds
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\leqslant h^2\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} Y_{11} & Y_{12} \\ * & Y_{22} \end{bmatrix}\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} - \begin{bmatrix} \int_{t-\tau(t)}^{t} z(s)ds \\ \int_{t-\tau(t)}^{t} f(z(s))ds \\ \int_{t-h}^{t-\tau(t)} z(s)ds \\ \int_{t-h}^{t-\tau(t)} f(z(s))ds \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} Y_{11} & Y_{12} & S_1 & S_2 \\ * & Y_{22} & S_3 & S_4 \\ * & * & Y_{11} & Y_{12} \\ * & * & * & Y_{22} \end{bmatrix}\begin{bmatrix} \int_{t-\tau(t)}^{t} z(s)ds \\ \int_{t-\tau(t)}^{t} f(z(s))ds \\ \int_{t-h}^{t-\tau(t)} z(s)ds \\ \int_{t-h}^{t-\tau(t)} f(z(s))ds \end{bmatrix},
\end{aligned} \tag{3.13}$$

where $\begin{bmatrix} Y_{11} & Y_{12} & S_1 & S_2 \\ * & Y_{22} & S_3 & S_4 \\ * & * & Y_{11} & Y_{12} \\ * & * & * & Y_{22} \end{bmatrix} > 0$ should be satisfied. Similar to (3.12), one can obtain

$$\dot{V}_6(z_t) \leqslant h^2\dot{z}^T(t)Q_1\dot{z}(t) - \begin{bmatrix} \int_{t-\tau(t)}^{t} \dot{z}(s)ds \\ \int_{t-h}^{t-\tau(t)} \dot{z}(s)ds \end{bmatrix}^T \begin{bmatrix} Q_1 & S_5 \\ * & Q_1 \end{bmatrix}\begin{bmatrix} \int_{t-\tau(t)}^{t} \dot{z}(s)ds \\ \int_{t-h}^{t-\tau(t)} \dot{z}(s)ds \end{bmatrix}, \tag{3.14}$$

where $\begin{bmatrix} Q_1 & S_5 \\ * & Q_1 \end{bmatrix} > 0$ should be satisfied.

Consider the following:

$$\begin{aligned}
\dot{V}_7(z_t) &= \frac{h^4}{4}\dot{z}^T(t)Q_2\dot{z}(t) - \frac{h^2}{2}\int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}^T(s)Q_2\dot{z}(s)ds\,d\theta \\
&\leqslant \frac{h^4}{4}\dot{z}^T(t)Q_2\dot{z}(t) - \left(\int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\right)^T Q_2\left(\int_{-h}^{0}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\right), \\
&= \frac{h^4}{4}\dot{z}^T(t)Q_2\dot{z}(t) - \left(hz(t) - \int_{t-h}^{t} z(s)ds\right)^T Q_2\left(hz(t) - \int_{t-h}^{t} z(s)ds\right),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\dot{V}_8(z_t) &= \frac{h^4}{4}z^T(t)Q_3z(t) - \frac{h^2}{2}\int_{-h}^{0}\int_{t+\theta}^{t}z^T(s)Q_3z(s)ds\,d\theta \\
&\leqslant \frac{h^4}{4}z^T(t)Q_3z(t) - \left(\int_{-h}^{0}\int_{t+\theta}^{t}z(s)ds\,d\theta\right)^T Q_3\left(\int_{-h}^{0}\int_{t+\theta}^{t}z(s)ds\,d\theta\right)
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\dot{V}_9(z_t) \leqslant{}& z^T(t)KQ_4Kz(t) - f^T(z(t))Q_4f(z(t)) - (1-u)z^T(t-\tau(t))KQ_4Kz(t-\tau(t)) \\
&+ (1-u)f^T(z(t-\tau(t)))Q_4f(z(t-\tau(t))).
\end{aligned} \tag{3.17}$$

Furthermore, there exist positive diagonal matrices T_1, T_2, T_3, such that the following inequalities hold based on (2.4):

$$-2f^T(z(t))T_1f(z(t)) + 2z^T(t)T_1(K_1+K_2)f(z(t)) - 2z^T(t)K_2T_1K_1z(t) \geqslant 0, \tag{3.18}$$

$$\begin{aligned}
&-2f^T(z(t-\tau(t)))T_2f(z(t-\tau(t))) + 2z^T(t-\tau(t))T_2(K_1+K_2)f(z(t-\tau(t))) \\
&\quad -2z^T(t-\tau(t))K_2T_2K_1z(t-\tau(t)) \geqslant 0,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&-2f^T(z(t-h))T_3f(z(t-h)) + 2z^T(t-h)T_3(K_1+K_2)f(z(t-h)), \\
&\quad -2z^T(t-h)K_2T_3K_1z(t-h) \geqslant 0.
\end{aligned} \tag{3.20}$$

From (3.8)–(3.20), one can obtain that

$$\dot{V}(z_t) \leqslant \zeta^T(t)\left(E + \mathscr{A}^T R\mathscr{A}\right)\zeta(t), \tag{3.21}$$

where

$$\zeta^T(t) = \Bigg[z^T(t) \quad z^T(t-\tau(t)) \quad z^T(t-h) \quad f^T(z(t)) \quad f\left(z^T(t-\tau(t))\right) \quad f\left(z^T(t-h)\right)$$
$$\int_{t-\tau(t)}^{t} z^T(s)ds \quad \int_{t-h}^{t-\tau(t)} z^T(s)ds \quad \int_{t-\tau(t)}^{t} f^T(z(s))\mathrm{d}s \quad \int_{t-h}^{t-\tau(t)} f^T(z(s))ds \tag{3.22}$$
$$\int_{-h}^{0}\int_{t+\theta}^{t} z^T(s)ds\, d\theta \Bigg].$$

If $E + \mathcal{A}^T R\mathcal{A} < 0$, then there exists a scalar $\varepsilon > 0$, such that

$$\dot{V}(z_t) \leqslant -\varepsilon\zeta^T(t)\zeta(t) \leqslant -\varepsilon z^T(t)z(t) < 0, \quad \forall z(t) \neq 0. \tag{3.23}$$

Thus, according to [44], system (2.1) is globally asymptotically stable. By Schur complement, $E + \mathcal{A}^T R\mathcal{A} < 0$ is equivalent to (3.3), this completes the proof.

Remark 3.3. By taking the states $\int_{t-\tau(t)}^{t} f^T(z(s))ds, \int_{t-h}^{t-\tau(t)} f^T(z(s))ds$, as augmented variables, the stability condition in Theorem 3.1 utilizes more information about $f(z(t))$ on state variables, which may lead to less conservative results.

Remark 3.4. Recently, the reciprocally convex optimization technique [42] is used to reduce the conservatism of stability criteria for systems with time-varying delays. Motivated by this work, the proposed method of [42] was utilized in (3.12) and (3.14), which have potential to yield less conservative conditions. However, an augmented vector with $\int_{t-\tau(t)}^{t} z^T(s)ds, \int_{t-h}^{t-\tau(t)} z^T(s)ds, \int_{t-\tau(t)}^{t} f^T(z(s))ds, \int_{t-h}^{t-\tau(t)} f^T(z(s))ds, \int_{-h}^{0}\int_{t+\theta}^{t} z^T(s)ds\, d\theta$ was used, which is different from the method of [42].

In many cases, u is unknown. Considering this situation, a rate-independent corollary for the delay $\tau(t)$ satisfying $0 \le \tau(t) \le h$ is derived by setting $X = 0,\ Q_4 = 0$ in the proof of Theorem 3.1.

Corollary 3.5. *For given scalar* $h \geqslant 0$*, diagonal matrices* $K_1 = \mathrm{diag}(k_1^-, k_2^-, \dots, k_n^-)$*,* $K_2 = \mathrm{diag}(k_1^+, k_2^+, \dots, k_n^+)$*, the system* (2.3) *is globally asymptotically stable if there exist symmetric positive matrices* $P = [\,P_{ij}\,]_{4\times4}, Q = [\,Q_{ij}\,]_{2\times2}, Y = [\,Y_{ij}\,]_{2\times2}, Q_i(i = 1,2,3)$*, positive diagonal matrices* $T_1, T_2, T_3, \Delta = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_n), \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$*, and any matrices* $S_i(i = 1,2,\dots,5)$ *with appropriate dimensions, such that* (3.1), (3.2) *and the following LMI hold:*

$$\begin{bmatrix} \overline{E} & \mathcal{A}^T R \\ * & -R \end{bmatrix} < 0, \tag{3.24}$$

Table 1: Allowable upper bound of h for different u.

Method	$u = 0.5$	$u = 0.9$
[3, 4]	2.1502	1.3164
[5]	2.2245	1.5847
[6]	2.5376	2.0853
[11]	2.2261	1.6035
[15]	2.5915	2.1306
[16] ($m = 2$)	2.6438	2.1349
[17] ($m = 2$)	2.4530	1.8593
Theorem 3.1	2.7098	2.2055

where

$$\overline{E} = \begin{pmatrix} \overline{E}_{11} & E_{12} & E_{13} & \overline{E}_{14} & E_{15} & -P_{13} & E_{17} & E_{18} & E_{19} & E_{1,10} & E_{1,11} \\ * & \overline{E}_{22} & E_{23} & 0 & \overline{E}_{25} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & E_{33} & 0 & 0 & E_{36} & -P_{22} & -P_{22} & -P_{23} & -P_{23} & -P_{24} \\ * & * & * & \overline{E}_{44} & E_{45} & 0 & E_{47} & E_{48} & E_{49} & E_{4,10} & E_{4,11} \\ * & * & * & * & -2T_2 & 0 & B^T P_{12} & B^T P_{12} & B^T P_{13} & B^T P_{13} & B^T P_{14} \\ * & * & * & * & * & E_{66} & -P_{23}^T & -P_{23}^T & -P_{33} & -P_{33} & -P_{34} \\ * & * & * & * & * & * & E_{77} & E_{78} & E_{7,9} & E_{7,10} & -P_{44} \\ * & * & * & * & * & * & * & E_{88} & E_{89} & E_{8,10} & -P_{44} \\ * & * & * & * & * & * & * & * & -Y_{22} & -S_4 & 0 \\ * & * & * & * & * & * & * & * & * & -Y_{22} & 0 \\ * & * & * & * & * & * & * & * & * & * & -Q_3 \end{pmatrix},$$

$$\begin{aligned} \overline{E}_{11} &= -P_{11}C - CP_{11} + P_{12} + P_{12}^T + h\left(P_{14} + P_{14}^T\right) + Q_{11} + 2K_1\Lambda C \\ &\quad - 2K_2\Delta C + h^2 Y_{11} - Q_1 - h^2 Q_2 + KQ_4K - 2K_2T_1K_1 + \frac{h^4}{4}Q_3, \\ \overline{E}_{14} &= P_{11}A + P_{13} + Q_{12} - C\Lambda - K_1\Lambda A + K_2\Delta A + C\Delta + h^2 Y_{12} + T_1(K_1 + K_2), \\ \overline{E}_{22} &= -2Q_1 + S_5 + S_5^T - 2K_2T_2K_1, \\ \overline{E}_{25} &= T_2(K_1 + K_2), \\ \overline{E}_{44} &= Q_{22} + \Lambda A + A^T\Lambda - \Delta A - A^T\Delta + h^2 Y_{22} - Q_4 - 2T_1. \end{aligned} \tag{3.25}$$

The other E_{ij} is defined in Theorem 3.1.

4. Numerical Examples

In this section, two numerical examples are given to demonstrate the effectiveness of the proposed method.

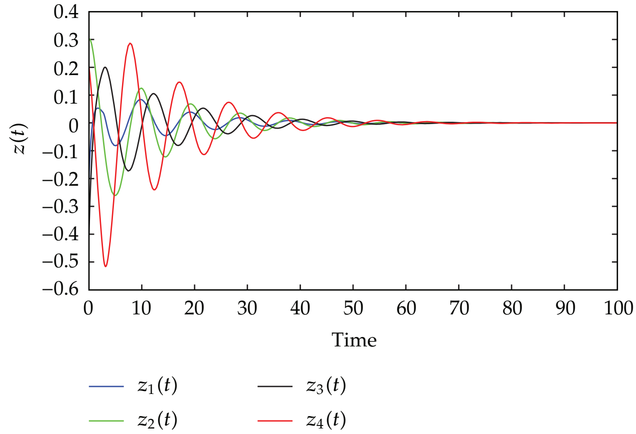

Figure 1: The dynamical behavior of $z(t)$.

Example 4.1. Consider the system (2.3) with the following parameters:

$$C = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \qquad A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}, \qquad K_1 = \operatorname{diag}\{0, 0, 0, 0\},$$
$$K_2 = \operatorname{diag}\{0.1137, 0.1279, 0.7994, 0.2368\},$$
$$f_1(s) = 0.05685(|s+1| - |s-1|), \qquad f_2(s) = 0.06395(|s+1| - |s-1|),$$
$$f_3(s) = 0.3997(|s+1| - |s-1|),$$
$$f_4(s) = 0.1184(|s+1| - |s-1|). \tag{4.1}$$

The upper bounds of h for different u are derived by Theorem 3.1 in our paper and the results in [13–16] are listed in Table 1. According to Table 1, this example shows that the stability condition in this paper gives much less conservative results than those in the literature. For $h = 2.7098$, the global asymptotic stability with the initial state$(-0.2, 0.3, -0.4, 0.2)^T$ is shown in Figure 1.

Table 2: Allowable upper bound of h for different u.

Method	$u = 0.8$	$u = 0.9$	Unknown u
[3, 4]	1.2281	0.8636	0.8298
[5]	1.6831	1.1493	1.0880
[6]	2.3534	1.6050	1.5103
[11]	1.5948	1.1323	1.0743
[15]	2.5406	1.7273	—
[16] $(m = 2)$	2.2495	1.5966	1.4902
[17] $(m = 2)$	2.1150	1.4286	1.3126
Our results	2.3731	1.6219	1.5103

Example 4.2. Consider the system (2.3) with the following parameters:

$$C = \operatorname{diag}(2,2), \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \tag{4.2}$$

$$K_1 \operatorname{diag}\{0,0\}, \qquad K_2 = \operatorname{diag}\{0.4, 0.8\}. \tag{4.3}$$

Our purpose is to estimate the allowable upper bounds delay h under different u such that the system (2.3) is globally asymptotically stable. According to the Table 2, this example is given to indicate significant improvements over some existing results.

5. Conclusions

In this paper, a new Lyapunov functional was proposed to investigate the stability of neural networks with time-varying delays. Some improved generalized delay-dependent stability criteria have been established. The obtained criteria are less conservative because a convex optimization approach is considered. Finally, two numerical examples have shown that these new stability criteria are less conservative than some existing ones in the literature.

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